

**User's Guide**  
**to**  
**the PARI library**

**(version 2.11.4)**

The PARI Group

Institut de Mathématiques de Bordeaux, UMR 5251 du CNRS.  
Université de Bordeaux, 351 Cours de la Libération  
F-33405 TALENCE Cedex, FRANCE  
e-mail: `pari@math.u-bordeaux.fr`

**Home Page:**  
`http://pari.math.u-bordeaux.fr/`

Copyright © 2000–2018 The PARI Group

Permission is granted to make and distribute verbatim copies of this manual provided the copyright notice and this permission notice are preserved on all copies.

Permission is granted to copy and distribute modified versions, or translations, of this manual under the conditions for verbatim copying, provided also that the entire resulting derived work is distributed under the terms of a permission notice identical to this one.

PARI/GP is Copyright © 2000–2018 The PARI Group

PARI/GP is free software; you can redistribute it and/or modify it under the terms of the GNU General Public License as published by the Free Software Foundation. It is distributed in the hope that it will be useful, but WITHOUT ANY WARRANTY WHATSOEVER.

## Table of Contents

<b>Chapter 4: Programming PARI in Library Mode</b>	<b>13</b>
4.1 Introduction: initializations, universal objects	13
4.2 Important technical notes	14
4.2.1 Backward compatibility	14
4.2.2 Types	14
4.2.3 Type recursivity	15
4.2.4 Variations on basic functions	15
4.2.5 Portability: 32-bit / 64-bit architectures	16
4.2.6 Using <code>malloc</code> / <code>free</code>	17
4.3 Garbage collection	17
4.3.1 Why and how	17
4.3.2 Variants	20
4.3.3 Examples	20
4.3.4 Comments	24
4.4 Creation of PARI objects, assignments, conversions	24
4.4.1 Creation of PARI objects	24
4.4.2 Sizes	26
4.4.3 Assignments	26
4.4.4 Copy	27
4.4.5 Clones	27
4.4.6 Conversions	28
4.5 Implementation of the PARI types	28
4.5.1 Type <code>t_INT</code> (integer)	29
4.5.2 Type <code>t_REAL</code> (real number)	30
4.5.3 Type <code>t_INTMOD</code>	31
4.5.4 Type <code>t_FRAC</code> (rational number)	31
4.5.5 Type <code>t_FFELT</code> (finite field element)	31
4.5.6 Type <code>t_COMPLEX</code> (complex number)	31
4.5.7 Type <code>t_PADIC</code> ( $p$ -adic numbers)	31
4.5.8 Type <code>t_QUAD</code> (quadratic number)	32
4.5.9 Type <code>t_POLMOD</code> (polmod)	32
4.5.10 Type <code>t_POL</code> (polynomial)	32
4.5.11 Type <code>t_SER</code> (power series)	33
4.5.12 Type <code>t_RFRAC</code> (rational function)	33
4.5.13 Type <code>t_QFR</code> (indefinite binary quadratic form)	33
4.5.14 Type <code>t_QFI</code> (definite binary quadratic form)	34
4.5.15 Type <code>t_VEC</code> and <code>t_COL</code> (vector)	34
4.5.16 Type <code>t_MAT</code> (matrix)	34
4.5.17 Type <code>t_VECSMALL</code> (vector of small integers)	34
4.5.18 Type <code>t_STR</code> (character string)	34
4.5.19 Type <code>t_ERROR</code> (error context)	34
4.5.20 Type <code>t_CLOSURE</code> (closure)	34
4.5.21 Type <code>t_INFINITY</code> (infinity)	34
4.5.22 Type <code>t_LIST</code> (list)	34
4.6 PARI variables	35
4.6.1 Multivariate objects	35

4.6.2	Creating variables	35
4.6.3	Comparing variables	37
4.7	Input and output	37
4.7.1	Input	37
4.7.2	Output to screen or file, output to string	38
4.7.3	Errors	39
4.7.4	Warnings	40
4.7.5	Debugging output	40
4.7.6	Timers and timing output	41
4.8	Iterators, Numerical integration, Sums, Products	42
4.8.1	Iterators	42
4.8.2	Iterating over primes	43
4.8.3	Numerical analysis	44
4.9	Catching exceptions	45
4.9.1	Basic use	45
4.9.2	Advanced use	45
4.10	A complete program	46
<b>Chapter 5:</b>	<b>Technical Reference Guide: the basics</b>	<b>49</b>
5.1	Initializing the library	49
5.1.1	General purpose	49
5.1.2	Technical functions	50
5.1.3	Notions specific to the GP interpreter	52
5.1.4	Public callbacks	53
5.1.5	Configuration variables	54
5.1.6	Utility functions	54
5.1.7	Saving and restoring the GP context	55
5.1.8	GP history	55
5.2	Handling GENs	55
5.2.1	Allocation	56
5.2.2	Length conversions	57
5.2.3	Read type-dependent information	58
5.2.4	Eval type-dependent information	59
5.2.5	Set type-dependent information	60
5.2.6	Type groups	61
5.2.7	Accessors and components	61
5.3	Global numerical constants	62
5.3.1	Constants related to word size	62
5.3.2	Masks used to implement the GEN type	62
5.3.3	$\log 2$ , $\pi$	63
5.4	Iterating over small primes, low-level interface	63
5.5	Handling the PARI stack	64
5.5.1	Allocating memory on the stack	64
5.5.2	Stack-independent binary objects	65
5.5.3	Garbage collection	66
5.5.4	Garbage collection: advanced use	67
5.5.5	Debugging the PARI stack	68
5.5.6	Copies	68
5.5.7	Simplify	68
5.6	The PARI heap	69

5.6.1	Introduction	69
5.6.2	Public interface	69
5.6.3	Implementation note	69
5.7	Handling user and temp variables	70
5.7.1	Low-level	70
5.7.2	User variables	70
5.7.3	Temporary variables	70
5.8	Adding functions to PARI	71
5.8.1	Nota Bene	71
5.8.2	Coding guidelines	71
5.8.3	GP prototypes, parser codes	72
5.8.4	Integration with <b>gp</b> as a shared module	74
5.8.5	Library interface for <b>install</b>	75
5.8.6	Integration by patching <b>gp</b>	75
5.9	Globals related to PARI configuration	76
5.9.1	PARI version numbers	76
5.9.2	Miscellaneous	76
<b>Chapter 6:</b>	<b>Arithmetic kernel: Level 0 and 1</b>	<b>77</b>
6.1	Level 0 kernel (operations on ulongs)	77
6.1.1	Micro-kernel	77
6.1.2	Modular kernel	78
6.1.3	Modular kernel with “precomputed inverse”	79
6.1.4	Switching between Fl_xxx and standard operators	80
6.2	Level 1 kernel (operations on longs, integers and reals)	81
6.2.1	Creation	81
6.2.2	Assignment	82
6.2.3	Copy	83
6.2.4	Conversions	83
6.2.5	Integer parts	84
6.2.6	2-adic valuations and shifts	85
6.2.7	From <b>t_INT</b> to bits or digits in base $2^k$ and back	86
6.2.8	Integer valuation	86
6.2.9	Generic unary operators	87
6.2.10	Comparison operators	88
6.2.11	Generic binary operators	89
6.2.12	Exact division and divisibility	92
6.2.13	Division with integral operands and <b>t_REAL</b> result	92
6.2.14	Division with remainder	93
6.2.15	Modulo to longs	94
6.2.16	Powering, Square root	95
6.2.17	GCD, extended GCD and LCM	96
6.2.18	Continued fractions and convergents	96
6.2.19	Pseudo-random integers	96
6.2.20	Modular operations	97
6.2.21	Extending functions to vector inputs	99
6.2.22	Miscellaneous arithmetic functions	100
<b>Chapter 7:</b>	<b>Level 2 kernel</b>	<b>103</b>
7.1	Naming scheme	103
7.2	Coefficient ring	105

7.3 Modular arithmetic . . . . .	106
7.3.1 FpC / FpV, FpM . . . . .	106
7.3.2 Flc / Flv, Flm . . . . .	110
7.3.3 F2c / F2v, F2m . . . . .	112
7.3.4 FlxqV, FlxqC, FlxqM . . . . .	114
7.3.5 FpX . . . . .	114
7.3.6 FpXQ, Fq . . . . .	118
7.3.7 FpXQ . . . . .	120
7.3.8 Fq . . . . .	120
7.3.9 FpXn . . . . .	122
7.3.10 FpXC, FpXM . . . . .	122
7.3.11 FpXX, FpXY . . . . .	123
7.3.12 FpXQX, FqX . . . . .	123
7.3.13 FpXQXn, FqXn . . . . .	126
7.3.14 FpXQXQ, FqXQ . . . . .	126
7.3.15 Flx . . . . .	129
7.3.16 FlxV . . . . .	133
7.3.17 FlxM . . . . .	134
7.3.18 FlxT . . . . .	134
7.3.19 Flxq . . . . .	134
7.3.20 FlxX . . . . .	135
7.3.21 FlxqX . . . . .	136
7.3.22 FlxqXQ . . . . .	138
7.3.23 F2x . . . . .	139
7.3.24 F2xq . . . . .	141
7.3.25 F2xqV, F2xqM . . . . .	142
7.3.26 F2xX . . . . .	142
7.3.27 F2xXV/F2xXC . . . . .	143
7.3.28 F2xqX . . . . .	143
7.3.29 F2xqXQ . . . . .	144
7.3.30 Functions returning objects with <code>t_INTMOD</code> coefficients . . . . .	144
7.3.31 Slow Chinese remainder theorem over $\mathbf{Z}$ . . . . .	145
7.3.32 Fast remainders . . . . .	147
7.3.33 Fast Chinese remainder theorem over $\mathbf{Z}$ . . . . .	148
7.3.34 Rational reconstruction . . . . .	149
7.3.35 Zp . . . . .	149
7.3.36 ZpX . . . . .	150
7.3.37 ZpXQ . . . . .	151
7.3.38 Zq . . . . .	151
7.3.39 ZpXQM . . . . .	151
7.3.40 ZpXQX . . . . .	152
7.3.41 ZqX . . . . .	152
7.3.42 Other $p$ -adic functions . . . . .	152
7.3.43 Conversions involving single precision objects . . . . .	154
7.4 Higher arithmetic over $\mathbf{Z}$ : primes, factorization . . . . .	157
7.4.1 Pure powers . . . . .	157
7.4.2 Factorization . . . . .	158
7.4.3 Coprime factorization . . . . .	160
7.4.4 Checks attached to arithmetic functions . . . . .	160

7.4.5 Incremental integer factorization . . . . .	161
7.4.6 Integer core, squarefree factorization . . . . .	162
7.4.7 Primes, primality and compositeness tests . . . . .	162
7.4.8 Iterators over primes . . . . .	164
<b>7.5 Integral, rational and generic linear algebra . . . . .</b>	<b>164</b>
7.5.1 <b>ZC / ZV, ZM</b> . . . . .	164
7.5.2 <b>QM</b> . . . . .	168
7.5.3 <b>Qevproj</b> . . . . .	168
7.5.4 <b>zv, zm</b> . . . . .	169
7.5.5 <b>ZMV / zmV</b> (vectors of <b>ZM/zm</b> ) . . . . .	169
7.5.6 <b>QC / QV, QM</b> . . . . .	170
7.5.7 <b>RgC / RgV, RgM</b> . . . . .	170
7.5.8 <b>ZG</b> . . . . .	174
7.5.9 Blackbox linear algebra . . . . .	175
7.5.10 Obsolete functions . . . . .	176
<b>7.6 Integral, rational and generic polynomial arithmetic . . . . .</b>	<b>176</b>
7.6.1 <b>ZX</b> . . . . .	176
7.6.2 Resultants . . . . .	179
7.6.3 <b>ZXV</b> . . . . .	179
7.6.4 <b>ZXT</b> . . . . .	179
7.6.5 <b>ZXQ</b> . . . . .	180
7.6.6 <b>ZXn</b> . . . . .	180
7.6.7 <b>ZXQM</b> . . . . .	180
7.6.8 <b>ZXQX</b> . . . . .	180
7.6.9 <b>ZXX</b> . . . . .	180
7.6.10 <b>QX</b> . . . . .	181
7.6.11 <b>QXQ</b> . . . . .	181
7.6.12 <b>QXQM</b> . . . . .	182
7.6.13 <b>zx</b> . . . . .	183
7.6.14 <b>RgX</b> . . . . .	183
7.6.15 <b>RgXn</b> . . . . .	188
7.6.16 <b>RgXnV</b> . . . . .	188
7.6.17 <b>RgXQ</b> . . . . .	189
7.6.18 <b>RgXQV, RgXQC</b> . . . . .	189
7.6.19 <b>RgXQM</b> . . . . .	190
7.6.20 <b>RgXQX</b> . . . . .	190
<b>Chapter 8: Black box algebraic structures . . . . .</b>	<b>191</b>
8.1 Black box groups . . . . .	191
8.1.1 Black box groups with pairing . . . . .	193
8.1.2 Functions returning black box groups . . . . .	193
8.2 Black box fields . . . . .	194
8.2.1 Functions returning black box fields . . . . .	195
8.3 Black box algebra . . . . .	195
8.3.1 Functions returning black box algebras . . . . .	196
8.4 Black box ring . . . . .	196
8.5 Black box free <b>Z<sub>p</sub></b> -modules . . . . .	197
<b>Chapter 9: Operations on general PARI objects . . . . .</b>	<b>199</b>
9.1 Assignment . . . . .	199
9.2 Conversions . . . . .	199

9.2.1 Scalars . . . . .	199
9.2.2 Modular objects / lifts . . . . .	201
9.2.3 Between polynomials and coefficient arrays . . . . .	201
9.3 Constructors . . . . .	203
9.3.1 Clean constructors . . . . .	203
9.3.2 Unclean constructors . . . . .	206
9.3.3 From roots to polynomials . . . . .	209
9.4 Integer parts . . . . .	209
9.5 Valuation and shift . . . . .	210
9.6 Comparison operators . . . . .	210
9.6.1 Generic . . . . .	210
9.6.2 Comparison with a small integer . . . . .	211
9.7 Miscellaneous Boolean functions . . . . .	212
9.7.1 Obsolete . . . . .	212
9.8 Sorting . . . . .	213
9.8.1 Basic sort . . . . .	213
9.8.2 Indirect sorting . . . . .	213
9.8.3 Generic sort and search . . . . .	213
9.8.4 Further useful comparison functions . . . . .	214
9.9 Divisibility, Euclidean division . . . . .	215
9.10 GCD, content and primitive part . . . . .	216
9.10.1 Generic . . . . .	216
9.10.2 Over the rationals . . . . .	216
9.11 Generic arithmetic operators . . . . .	217
9.11.1 Unary operators . . . . .	217
9.11.2 Binary operators . . . . .	218
9.12 Generic operators: product, powering, factorback . . . . .	219
9.13 Matrix and polynomial norms . . . . .	220
9.14 Substitution and evaluation . . . . .	221
<b>Chapter 10: Miscellaneous mathematical functions . . . . .</b>	<b>223</b>
10.1 Fractions . . . . .	223
10.2 Binomials . . . . .	223
10.3 Real numbers . . . . .	223
10.4 Complex numbers . . . . .	224
10.5 Quadratic numbers and binary quadratic forms . . . . .	224
10.6 Polynomials . . . . .	225
10.7 Power series . . . . .	226
10.8 Functions to handle <code>t_FFELT</code> . . . . .	226
10.8.1 FFX . . . . .	229
10.8.2 FFM . . . . .	230
10.8.3 FFXQ . . . . .	230
10.9 Transcendental functions . . . . .	231
10.9.1 Transcendental functions with <code>t_REAL</code> arguments . . . . .	231
10.9.2 Other complex transcendental functions . . . . .	232
10.9.3 Modular functions . . . . .	233
10.9.4 Transcendental functions with <code>t_PADIC</code> arguments . . . . .	233
10.9.5 Cached constants . . . . .	233
10.10 Permutations . . . . .	234
10.11 Small groups . . . . .	235



<b>Chapter 11: Standard data structures</b>	<b>239</b>
11.1 Character strings	239
11.1.1 Functions returning a <code>char *</code>	239
11.1.2 Functions returning a <code>t_STR</code>	240
11.1.3 Dynamic strings	240
11.2 Output	241
11.2.1 Output contexts	241
11.2.2 Default output context	241
11.2.3 PARI colors	242
11.2.4 Obsolete output functions	242
11.3 Files	243
11.3.1 <code>pariFILE</code>	243
11.3.2 Temporary files	244
11.4 Errors	244
11.4.1 Internal errors, “system” errors	244
11.4.2 Syntax errors, type errors	245
11.4.3 Overflows	246
11.4.4 Errors triggered intentionally	247
11.4.5 Mathematical errors	248
11.4.6 Miscellaneous functions	249
11.5 Hashtables	249
11.6 Dynamic arrays	251
11.6.1 Initialization	251
11.6.2 Adding elements	251
11.6.3 Accessing elements	252
11.6.4 Stack of stacks	252
11.6.5 Public interface	252
11.7 Vectors and Matrices	253
11.7.1 Access and extract	253
11.7.2 Componentwise operations	254
11.7.3 Low-level vectors and columns functions	255
11.8 Vectors of small integers	255
11.8.1 <code>t_VECSMALL</code>	255
11.8.2 Vectors of <code>t_VECSMALL</code>	257
<b>Chapter 12: Functions related to the GP interpreter</b>	<b>259</b>
12.1 Handling closures	259
12.1.1 Functions to evaluate <code>t_CLOSURE</code>	259
12.1.2 Functions to handle control flow changes	260
12.1.3 Functions to deal with lexical local variables	260
12.1.4 Functions returning new closures	260
12.1.5 Functions used by the gp debugger (break loop)	261
12.1.6 Standard wrappers for iterators	261
12.2 Defaults	262
12.3 Records and Lazy vectors	265
<b>Chapter 13: Algebraic Number Theory</b>	<b>267</b>
13.1 General Number Fields	267
13.1.1 Number field types	267
13.1.2 Extracting info from a <code>nf</code> structure	269
13.1.3 Extracting info from a <code>bnf</code> structure	270

13.1.4	Extracting info from a <b>bnr</b> structure	270
13.1.5	Extracting info from an <b>rnf</b> structure	271
13.1.6	Extracting info from a <b>bid</b> structure	271
13.1.7	Extracting info from a <b>znstar</b> structure	272
13.1.8	Inserting info in a number field structure	273
13.1.9	Increasing accuracy	273
13.1.10	Number field arithmetic	274
13.1.11	Elements in factored form	277
13.1.12	Ideal arithmetic	278
13.1.13	Maximal ideals	281
13.1.14	Decomposition group	282
13.1.15	Reducing modulo maximal ideals	282
13.1.16	Valuations	284
13.1.17	Signatures	284
13.1.18	Maximal order and discriminant, conversion to <b>nf</b> structure	285
13.1.19	Computing in the class group	287
13.1.20	Floating point embeddings, the $T_2$ quadratic form	288
13.1.21	Ideal reduction, low level	289
13.1.22	Ideal reduction, high level	290
13.1.23	Class field theory	291
13.1.24	Grunwald–Wang theorem	292
13.1.25	Relative equations, Galois conjugates	292
13.1.26	Cyclotomics units	294
13.1.27	Obsolete routines	294
13.2	Galois extensions of <b>Q</b>	295
13.2.1	Extracting info from a <b>gal</b> structure	295
13.2.2	Miscellaneous functions	295
13.3	Quadratic number fields and quadratic forms	296
13.3.1	Checks	296
13.3.2	Class number	296
13.3.3	<b>t_QFI</b> , <b>t_QFR</b>	296
13.3.4	Efficient real quadratic forms	298
13.4	Linear algebra over <b>Z</b>	299
13.4.1	Hermite and Smith Normal Forms	299
13.4.2	The LLL algorithm	303
13.4.3	Linear dependencies	304
13.4.4	Reduction modulo matrices	305
13.5	Finite abelian groups and characters	306
13.5.1	Abstract groups	306
13.5.2	Dirichlet characters	307
13.6	Central simple algebras	308
13.6.1	Initialization	308
13.6.2	Type checks	308
13.6.3	Shallow accessors	309
13.6.4	Other low-level functions	309
<b>Chapter 14: Elliptic curves and arithmetic geometry</b>		<b>311</b>
14.1	Elliptic curves	311
14.1.1	Types of elliptic curves	311
14.1.2	Type checking	311

14.1.3	Extracting info from an <code>ell</code> structure . . . . .	312
14.1.4	Points . . . . .	316
14.1.5	Change of variables . . . . .	316
14.1.6	Generic helper functions . . . . .	316
14.1.7	Functions to handle elliptic curves over finite fields . . . . .	317
14.2	Arithmetic on elliptic curve over a finite field in simple form . . . . .	317
14.2.1	Helper functions . . . . .	317
14.2.2	Elliptic curves over $\mathbf{F}_p$ , $p > 3$ . . . . .	318
14.2.3	<code>FpE</code> . . . . .	318
14.2.4	<code>Fle</code> . . . . .	319
14.2.5	<code>FpJ</code> . . . . .	320
14.2.6	<code>F1j</code> . . . . .	320
14.2.7	Elliptic curves over $\mathbf{F}_{2^n}$ . . . . .	321
14.2.8	<code>F2xqE</code> . . . . .	321
14.2.9	Elliptic curves over $\mathbf{F}_q$ , small characteristic $p > 2$ . . . . .	322
14.2.10	<code>F1xqE</code> . . . . .	322
14.2.11	Elliptic curves over $\mathbf{F}_q$ , large characteristic . . . . .	323
14.2.12	<code>FpXQE</code> . . . . .	323
14.3	Functions related to modular polynomials . . . . .	324
14.3.1	Functions related to modular invariants . . . . .	324
14.4	Other curves . . . . .	325
<b>Chapter 15:</b>	<b><i>L</i>-functions . . . . .</b>	<b>327</b>
15.1	Accessors . . . . .	327
15.2	Conversions and constructors . . . . .	328
15.3	Variants of GP functions . . . . .	328
15.4	Inverse Mellin transforms of Gamma products . . . . .	328
<b>Chapter 16:</b>	<b>Modular symbols . . . . .</b>	<b>329</b>
<b>Chapter 17:</b>	<b>Modular forms . . . . .</b>	<b>329</b>
17.1	Implementation of public data structures . . . . .	329
17.1.1	Accessors for modular form spaces . . . . .	329
17.1.2	Accessors for individual modular forms . . . . .	330
17.1.3	Nebentypus . . . . .	331
17.1.4	Miscellaneous functions . . . . .	331
<b>Chapter 18:</b>	<b>Plots . . . . .</b>	<b>333</b>
18.0.5	Highlevel function . . . . .	333
18.0.6	Function . . . . .	333
18.0.7	Obsolete functions . . . . .	334
18.0.8	Dump rectwindows to a PostScript or SVG file . . . . .	334
18.0.9	Technical functions exported for convenience . . . . .	334
<b>Appendix A:</b>	<b>A Sample program and Makefile . . . . .</b>	<b>335</b>
<b>Appendix B:</b>	<b>PARI and threads . . . . .</b>	<b>337</b>
<b>Index</b>	<b>. . . . .</b>	<b>340</b>



## Chapter 4:

# Programming PARI in Library Mode

The *User's Guide to Pari/GP* gives in three chapters a general presentation of the system, of the `gp` calculator, and detailed explanation of high level PARI routines available through the calculator. The present manual assumes general familiarity with the contents of these chapters and the basics of ANSI C programming, and focuses on the usage of the PARI library. In this chapter, we introduce the general concepts of PARI programming and describe useful general purpose functions; the following chapters describes all public low or high-level functions, underlying or extending the GP functions seen in Chapter 3 of the User's guide.

### 4.1 Introduction: initializations, universal objects.

To use PARI in library mode, you must write a C program and link it to the PARI library. See the installation guide or the Appendix to the *User's Guide to Pari/GP* on how to create and install the library and include files. A sample Makefile is presented in Appendix A, and a more elaborate one in `examples/Makefile`. The best way to understand how programming is done is to work through a complete example. We will write such a program in Section 4.10. Before doing this, a few explanations are in order.

First, one must explain to the outside world what kind of objects and routines we are going to use. This is done\* with the directive

```
#include <pari/pari.h>
```

In particular, this defines the fundamental type for all PARI objects: the type **GEN**, which is simply a pointer to `long`.

Before any PARI routine is called, one must initialize the system, and in particular the PARI stack which is both a scratchboard and a repository for computed objects. This is done with a call to the function

```
void pari_init(size_t size, ulong maxprime)
```

The first argument is the number of bytes given to PARI to work with, and the second is the upper limit on a precomputed prime number table; `size` should not reasonably be taken below 500000 but you may set `maxprime = 0`, although the system still needs to precompute all primes up to about  $2^{16}$ . For lower-level variants allowing finer control, e.g. preventing PARI from installing its own error or signal handlers, see Section 5.1.2.

We have now at our disposal:

- a PARI *stack* containing nothing. This is a big connected chunk of `size` bytes of memory, where all computations take place. In large computations, intermediate results quickly clutter up memory so some kind of garbage collecting is needed. Most systems do garbage collecting when the memory is getting scarce, and this slows down the performance. PARI takes a different approach,

---

\* This assumes that PARI headers are installed in a directory which belongs to your compiler's search path for header files. You might need to add flags like `-I/usr/local/include` or modify `C_INCLUDE_PATH`.

admittedly more demanding on the programmer: you must do your own cleaning up when the intermediate results are not needed anymore. We will see later how (and when) this is done.

- the following *universal objects* (by definition, objects which do not belong to the stack): the integers 0, 1, -1, 2 and -2 (respectively called `gen_0`, `gen_1`, `gen_m1`, `gen_2` and `gen_m2`), the fraction  $\frac{1}{2}$  (`ghalf`). All of these are of type `GEN`.

- a *heap* which is just a linked list of permanent universal objects. For now, it contains exactly the ones listed above. You will probably very rarely use the heap yourself; and if so, only as a collection of copies of objects taken from the stack (called clones in the sequel). Thus you need not bother with its internal structure, which may change as PARI evolves. Some complex PARI functions create clones for special garbage collecting purposes, usually destroying them when returning.

- a table of primes (in fact of *differences* between consecutive primes), called `diffptr`, of type `byteptr` (pointer to `unsigned char`). Its use is described in Section 5.4 later. Using it directly is deprecated, high-level iterators provide a cleaner and more flexible interface, see Section 4.8.2 (such iterators use the private prime table, but extend it dynamically).

- access to all the built-in functions of the PARI library. These are declared to the outside world when you include `pari.h`, but need the above things to function properly. So if you forget the call to `pari_init`, you will get a fatal error when running your program.

## 4.2 Important technical notes.

**4.2.1 Backward compatibility.** The PARI function names evolved over time, and deprecated functions are eventually deleted. The file `pariold.h` contains macros implementing a weak form of backward compatibility. In particular, whenever the name of a documented function changes, a `#define` is added to this file so that the old name expands to the new one (provided the prototype didn't change also).

This file is included by `pari.h`, but a large section is commented out by default. Define `PARI_OLD_NAMES` before including `pari.h` to pollute your namespace with lots of obsolete names like `un*`: that might enable you to compile old programs without having to modify them. The preferred way to do that is to add `-DPARI_OLD_NAMES` to your compiler `CFLAGS`, so that you don't need to modify the program files themselves.

Of course, it's better to fix the program if you can!

### 4.2.2 Types.

Although PARI objects all have the C type `GEN`, we will freely use the word **type** to refer to PARI dynamic subtypes: `t_INT`, `t_REAL`, etc. The declaration

```
GEN x;
```

declares a C variable of type `GEN`, but its “value” will be said to have type `t_INT`, `t_REAL`, etc. The meaning should always be clear from the context.

---

\* For `(long)gen_1`. Since 2004 and version 2.2.9, typecasts are completely unnecessary in PARI programs.

### 4.2.3 Type recursivity.

Conceptually, most PARI types are recursive. But the **GEN** type is a pointer to **long**, not to **GEN**. So special macros must be used to access **GEN**'s components. The simplest one is **gel**(*V*, *i*), where **el** stands for **e**lement, to access component number *i* of the **GEN** *V*. This is a valid **lvalue** (may be put on the left side of an assignment), and the following two constructions are exceedingly frequent

```
gel(V, i) = x;
x = gel(V, i);
```

where **x** and **V** are **GEN**s. This macro accesses and modifies directly the components of *V* and do not create a copy of the coefficient, contrary to all the library *functions*.

More generally, to retrieve the values of elements of lists of ... of lists of vectors we have the **gmael** macros (for **m**ultidimensional **a**rray **e**lement). The syntax is **gmael***n*(*V*, *a*<sub>1</sub>, ..., *a*<sub>*n*</sub>), where *V* is a **GEN**, the *a*<sub>*i*</sub> are indexes, and *n* is an integer between 1 and 5. This stands for *x*[*a*<sub>1</sub>][*a*<sub>2</sub>] ... [*a*<sub>*n*</sub>], and returns a **GEN**. The macros **gel** (resp. **gmael**) are synonyms for **gmael1** (resp. **gmael12**).

Finally, the macro **gcoeff**(*M*, *i*, *j*) has exactly the meaning of *M*[*i*, *j*] in GP when *M* is a matrix. Note that due to the implementation of **t\_MAT**s as horizontal lists of vertical vectors, **gcoeff**(*x*, *y*) is actually equivalent to **gmael**(*y*, *x*). One should use **gcoeff** in matrix context, and **gmael** otherwise.

**4.2.4 Variations on basic functions.** In the library syntax descriptions in Chapter 3, we have only given the basic names of the functions. For example **gadd**(*x*, *y*) assumes that *x* and *y* are **GEN**s, and *creates* the result *x*+*y* on the PARI stack. For most of the basic operators and functions, many other variants are available. We give some examples for **gadd**, but the same is true for all the basic operators, as well as for some simple common functions (a complete list is given in Chapter 6):

```
GEN gaddgs(GEN x, long y)
```

```
GEN gaddsg(long x, GEN y)
```

In the following one, **z** is a preexisting **GEN** and the result of the corresponding operation is put into **z**. The size of the PARI stack does not change:

```
void gaddz(GEN x, GEN y, GEN z)
```

(This last form is inefficient in general and deprecated outside of PARI kernel programming.) Low level kernel functions implement these operators for specialized arguments and are also available: Level 0 deals with operations at the word level (**longs** and **ulongs**), Level 1 with **t\_INT** and **t\_REAL** and Level 2 with the rest (modular arithmetic, polynomial arithmetic and linear algebra). Here are some examples of Level 1 functions:

**GEN addii**(**GEN** *x*, **GEN** *y*): here *x* and *y* are **GEN**s of type **t\_INT** (this is not checked).

**GEN addrr**(**GEN** *x*, **GEN** *y*): here *x* and *y* are **GEN**s of type **t\_REAL** (this is not checked).

There also exist functions **addir**, **addri**, **mpadd** (whose two arguments can be of type **t\_INT** or **t\_REAL**), **addis** (to add a **t\_INT** and a **long**) and so on.

The Level 1 names are self-explanatory once you know that **i** stands for a **t\_INT**, **r** for a **t\_REAL**, **mp** for **i** or **r**, **s** for a signed C long integer, **u** for an unsigned C long integer; finally the suffix **z** means that the result is not created on the PARI stack but assigned to a preexisting **GEN** object passed as an extra argument. Chapter 6 gives a description of these low-level functions.

Level 2 names are more complicated, see Section 7.1 for all the gory details, and we content ourselves with a simple example used to implement `t_INTMOD` arithmetic:

`GEN Fp_add(GEN x, GEN y, GEN m)`: returns the sum of  $x$  and  $y$  modulo  $m$ . Here  $x, y, m$  are `t_INTs` (this is not checked). The operation is more efficient if the inputs  $x, y$  are reduced modulo  $m$ , but this is not a necessary condition.

**Important Note.** These specialized functions are of course more efficient than the generic ones, but note the hidden danger here: the types of the objects involved (which is not checked) must be severely controlled, e.g. using `addii` on a `t_FRAC` argument will cause disasters. Type mismatches may corrupt the PARI stack, though in most cases they will just immediately overflow the stack. Because of this, the PARI philosophy of giving a result which is as exact as possible, enforced for generic functions like `gadd` or `gmul`, is dropped in kernel routines of Level 1, where it is replaced by the much simpler rule: the result is a `t_INT` if and only if all arguments are integer types (`t_INT` but also `C long` and `ulong`) and a `t_REAL` otherwise. For instance, multiplying a `t_REAL` by a `t_INT` always yields a `t_REAL` if you use `mulir`, where `gmul` returns the `t_INT` `gen_0` if the integer is 0.

#### 4.2.5 Portability: 32-bit / 64-bit architectures.

PARI supports both 32-bit and 64-bit based machines, but not simultaneously! The library is compiled assuming a given architecture, and some of the header files you include (through `pari.h`) will have been modified to match the library.

Portable macros are defined to bypass most machine dependencies. If you want your programs to run identically on 32-bit and 64-bit machines, you have to use these, and not the corresponding numeric values, whenever the precise size of your `long` integers might matter. Here are the most important ones:

	64-bit	32-bit	
<code>BITS_IN_LONG</code>	64	32	
<code>LONG_IS_64BIT</code>	defined	undefined	
<code>DEFAULTPREC</code>	3	4	( $\approx 19$ decimal digits, see formula below)
<code>MEDDEFAULTPREC</code>	4	6	( $\approx 38$ decimal digits)
<code>BIGDEFAULTPREC</code>	5	8	( $\approx 57$ decimal digits)

For instance, suppose you call a transcendental function, such as

`GEN gexp(GEN x, long prec)`.

The last argument `prec` is an integer  $\geq 3$ , corresponding to the default floating point precision required. It is *only* used if `x` is an exact object, otherwise the relative precision is determined by the precision of `x`. Since the parameter `prec` sets the size of the inexact result counted in (`long`) *words* (including codewords), the same value of `prec` will yield different results on 32-bit and 64-bit machines. Real numbers have two codewords (see Section 4.5), so the formula for computing the bit accuracy is

$$\text{bit\_accuracy}(\text{prec}) = (\text{prec} - 2) * \text{BITS\_IN\_LONG}$$

(this is actually the definition of an inline function). The corresponding accuracy expressed in decimal digits would be

$$\text{bit\_accuracy}(\text{prec}) * \log(2) / \log(10).$$

For example if the value of `prec` is 5, the corresponding accuracy for 32-bit machines is  $(5 - 2) * \log(2^{32}) / \log(10) \approx 28$  decimal digits, while for 64-bit machines it is  $(5 - 2) * \log(2^{64}) / \log(10) \approx 57$  decimal digits.



Thus, you must take care to change the `prec` parameter you are supplying according to the bit size, either using the default precisions given by the various `DEFAULTPREC`s, or by using conditional constructs of the form:

```
#ifndef LONG_IS_64BIT
    prec = 4;
#else
    prec = 6;
#endif
```

which is in this case equivalent to the statement `prec = MEDDEFAULTPREC;`.

Note that for parity reasons, half the accuracies available on 32-bit architectures (the odd ones) have no precise equivalents on 64-bit machines.

**4.2.6 Using `malloc` / `free`.** You should make use of the PARI stack as much as possible, and avoid allocating objects using the customary functions. If you do, you should use, or at least have a very close look at, the following wrappers:

`void* pari_malloc(size_t size)` calls `malloc` to allocate `size` bytes and returns a pointer to the allocated memory. If the request fails, an error is raised. The `SIGINT` signal is blocked until `malloc` returns, to avoid leaving the system stack in an inconsistent state.

`void* pari_realloc(void* ptr, size_t size)` as `pari_malloc` but calls `realloc` instead of `malloc`.

`void* pari_calloc(size_t size)` as `pari_malloc`, setting the memory to zero.

`void pari_free(void* ptr)` calls `free` to liberate the memory space pointed to by `ptr`, which must have been allocated by `malloc` (`pari_malloc`) or `realloc` (`pari_realloc`). The `SIGINT` signal is blocked until `free` returns.

If you use the standard `libc` functions instead of our wrappers, then your functions will be subtly incompatible with the `gp` calculator: when the user tries to interrupt a computation, the calculator may crash (if a system call is interrupted at the wrong time).

## 4.3 Garbage collection.

### 4.3.1 Why and how.

As we have seen, `pari_init` allocates a big range of addresses, the *stack*, that are going to be used throughout. Recall that all PARI objects are pointers. Except for a few universal objects, they all point at some part of the stack.

The stack starts at the address `bot` and ends just before `top`. This means that the quantity

$$(\text{top} - \text{bot}) / \text{sizeof}(\text{long})$$

is (roughly) equal to the `size` argument of `pari_init`. The PARI stack also has a “current stack pointer” called `avma`, which stands for *a*vailable *m*emory *a*ddress. These three variables are global (declared by `pari.h`). They are of type `pari_sp`, which means *pari stack pointer*.

The stack is oriented upside-down: the more recent an object, the closer to `bot`. Accordingly, initially `avma = top`, and `avma` gets *decremented* as new objects are created. As its name indicates,

`avma` always points just *after* the first free address on the stack, and `(GEN)avma` is always (a pointer to) the latest created object. When `avma` reaches `bot`, the stack overflows, aborting all computations, and an error message is issued. To avoid this *you* need to clean up the stack from time to time, when intermediate objects are not needed anymore. This is called “*garbage collecting*.”

We are now going to describe briefly how this is done. We will see many concrete examples in the next subsection.

- First, PARI routines do their own garbage collecting, which means that whenever a documented function from the library returns, only its result(s) have been added to the stack, possibly up to a very small overhead (non-documented ones may not do this). In particular, a PARI function that does not return a `GEN` does not clutter the stack. Thus, if your computation is small enough (e.g. you call few PARI routines, or most of them return `long` integers), then you do not need to do any garbage collecting. This is probably the case in many of your subroutines. Of course the objects that were on the stack *before* the function call are left alone. Except for the ones listed below, PARI functions only collect their own garbage.

- It may happen that all objects that were created after a certain point can be deleted — for instance, if the final result you need is not a `GEN`, or if some search proved futile. Then, it is enough to record the value of `avma` just *before* the first garbage is created, and restore it upon exit:

```
pari_sp av = avma; /* record initial avma */
garbage ...
avma = av; /* restore it */
```

All objects created in the `garbage` zone will eventually be overwritten: they should no longer be accessed after `avma` has been restored.

- If you want to destroy (i.e. give back the memory occupied by) the *latest* PARI object on the stack (e.g. the latest one obtained from a function call), you can use the function

```
void cgiv(GEN z)
```

where `z` is the object you want to give back. This is equivalent to the above where the initial `av` is computed from `z`.

- Unfortunately life is not so simple, and sometimes you will want to give back accumulated garbage *during* a computation without losing recent data. We shall start with the lowest level function to get a feel for the underlying mechanisms, we shall describe simpler variants later:

`GEN gerepile(pari_sp ltop, pari_sp lbot, GEN q)`. This function cleans up the stack between `ltop` and `lbot`, where `lbot < ltop`, and returns the updated object `q`. This means:

1) we translate (copy) all the objects in the interval `[avma, lbot[`, so that its right extremity abuts the address `ltop`. Graphically

```

      bot          avma  lbot          ltop    top
End of stack  |-----[+++++[---/--/--/--/--|+++++] Start
              free memory          garbage
```

becomes:

```

      bot          avma  ltop    top
End of stack  |-----[+++++[+++++] Start
              free memory
```

where ++ denote significant objects, -- the unused part of the stack, and -/- the garbage we remove.

2) The function then inspects all the PARI objects between `avma` and `lbot` (i.e. the ones that we want to keep and that have been translated) and looks at every component of such an object which is not a codeword. Each such component is a pointer to an object whose address is either

- between `avma` and `lbot`, in which case it is suitably updated,
- larger than or equal to `ltop`, in which case it does not change, or
- between `lbot` and `ltop` in which case `gerepile` raises an error (“significant pointers lost in `gerepile`”).

3) `avma` is updated (we add `ltop - lbot` to the old value).

4) We return the (possibly updated) object `q`: if `q` initially pointed between `avma` and `lbot`, we return the updated address, as in 2). If not, the original address is still valid, and is returned!

As stated above, no component of the remaining objects (in particular `q`) should belong to the erased segment `[lbot, ltop[`, and this is checked within `gerepile`. But beware as well that the addresses of the objects in the translated zone change after a call to `gerepile`, so you must not access any pointer which previously pointed into the zone below `ltop`. If you need to recover more than one object, use the `gerepileall` function below.

**Remark.** As a consequence of the preceding explanation, if a PARI object is to be relocated by `gerepile` then, apart from universal objects, the chunks of memory used by its components should be in consecutive memory locations. All GENs created by documented PARI functions are guaranteed to satisfy this. This is because the `gerepile` function knows only about *two connected zones*: the garbage that is erased (between `lbot` and `ltop`) and the significant pointers that are copied and updated. If there is garbage interspersed with your objects, disaster occurs when we try to update them and consider the corresponding “pointers”. In most cases of course the said garbage is in fact a bunch of other GENs, in which case we simply waste time copying and updating them for nothing. But be wary when you allow objects to become disconnected.

In practice this is achieved by the following programming idiom:

```
ltop = avma; garbage(); lbot = avma; q = anything();
return gerepile(ltop, lbot, q); /* returns the updated q */
```

or directly

```
ltop = avma; garbage(); lbot = avma;
return gerepile(ltop, lbot, anything());
```

Beware that

```
ltop = avma; garbage();
return gerepile(ltop, avma, anything())
```

might work, but should be frowned upon. We cannot predict whether `avma` is evaluated after or before the call to `anything()`: it depends on the compiler. If we are out of luck, it is *after* the call, so the result belongs to the garbage zone and the `gerepile` statement becomes equivalent to `avma = ltop`. Thus we return a pointer to random garbage.

### 4.3.2 Variants.

GEN `gerepileupto(pari_sp ltop, GEN q)`. Cleans the stack between `ltop` and the *connected* object `q` and returns `q` updated. For this to work, `q` must have been created *before* all its components, otherwise they would belong to the garbage zone! Unless mentioned otherwise, documented PARI functions guarantee this.

GEN `gerepilecopy(pari_sp ltop, GEN x)`. Functionally equivalent to, but more efficient than

```
gerepileupto(ltop, gcopy(x))
```

In this case, the GEN parameter `x` need not satisfy any property before the garbage collection: it may be disconnected, components created before the root, and so on. Of course, this is about twice slower than either `gerepileupto` or `gerepile`, because `x` has to be copied to a clean stack zone first. This function is a special case of `gerepileall` below, where  $n = 1$ .

void `gerepileall(pari_sp ltop, int n, ...)`. To cope with complicated cases where many objects have to be preserved. The routine expects  $n$  further arguments, which are the *addresses* of the GENs you want to preserve:

```
pari_sp ltop = avma;  
...; y = ...; ... x = ...; ...;  
gerepileall(ltop, 2, &x, &y);
```

It cleans up the most recent part of the stack (between `ltop` and `avma`), updating all the GENs added to the argument list. A copy is done just before the cleaning to preserve them, so they do not need to be connected before the call. With `gerepilecopy`, this is the most robust of the `gerepile` functions (the less prone to user error), hence the slowest.

void `gerepileallsp(pari_sp ltop, pari_sp lbot, int n, ...)`. More efficient, but trickier than `gerepileall`. Cleans the stack between `lbot` and `ltop` and updates the GENs pointed at by the elements of `gptr` without any further copying. This is subject to the same restrictions as `gerepile`, the only difference being that more than one address gets updated.

### 4.3.3 Examples.

#### 4.3.3.1 gerepile.

Let `x` and `y` be two preexisting PARI objects and suppose that we want to compute  $x^2 + y^2$ . This is done using the following program:

```
GEN x2 = gsqr(x);  
GEN y2 = gsqr(y), z = gadd(x2,y2);
```

The GEN `z` indeed points at the desired quantity. However, consider the stack: it contains as unnecessary garbage `x2` and `y2`. More precisely it contains (in this order) `z`, `y2`, `x2`. (Recall that, since the stack grows downward from the top, the most recent object comes first.)

It is not possible to get rid of `x2`, `y2` before `z` is computed, since they are used in the final operation. We cannot record `avma` before `x2` is computed and restore it later, since this would destroy `z` as well. It is not possible either to use the function `cgiv` since `x2` and `y2` are not at the bottom of the stack and we do not want to give back `z`.

But using `gerepile`, we can give back the memory locations corresponding to `x2`, `y2`, and move the object `z` upwards so that no space is lost. Specifically:

```
pari_sp ltop = avma; /* remember the current top of the stack */
```

```

GEN x2 = gsqr(x);
GEN y2 = gsqr(y);
pari_sp lbot = avma; /* the bottom of the garbage pile */
GEN z = gadd(x2, y2); /* z is now the last object on the stack */
z = gerepile(ltop, lbot, z);

```

Of course, the last two instructions could also have been written more simply:

```
z = gerepile(ltop, lbot, gadd(x2,y2));
```

In fact `gerepileupto` is even simpler to use, because the result of `gadd` is the last object on the stack and `gadd` is guaranteed to return an object suitable for `gerepileupto`:

```
ltop = avma;
z = gerepileupto(ltop, gadd(gsqr(x), gsqr(y)));
```

Make sure you understand exactly what has happened before you go on!

**Remark on assignments and `gerepile`.** When the tree structure and the size of the PARI objects which will appear in a computation are under control, one may allocate sufficiently large objects at the beginning, use assignment statements, then simply restore `avma`. Coming back to the above example, note that *if* we know that `x` and `y` are of type real fitting into `DEFAULTPREC` words, we can program without using `gerepile` at all:

```
z = cgetr(DEFAULTPREC); ltop = avma;
gaffect(gadd(gsqr(x), gsqr(y)), z);
avma = ltop;
```

This is often *slower* than a craftily used `gerepile` though, and certainly more cumbersome to use. As a rule, assignment statements should generally be avoided.

**Variations on a theme.** it is often necessary to do several `gerepiles` during a computation. However, the fewer the better. The only condition for `gerepile` to work is that the garbage be connected. If the computation can be arranged so that there is a minimal number of connected pieces of garbage, then it should be done that way.

For example suppose we want to write a function of two GEN variables `x` and `y` which creates the vector  $[x^2 + y, y^2 + x]$ . Without garbage collecting, one would write:

```
p1 = gsqr(x); p2 = gadd(p1, y);
p3 = gsqr(y); p4 = gadd(p3, x);
z = mkvec2(p2, p4); /* not suitable for gerepileupto! */
```

This leaves a dirty stack containing (in this order) `z`, `p4`, `p3`, `p2`, `p1`. The garbage here consists of `p1` and `p3`, which are separated by `p2`. But if we compute `p3` *before* `p2` then the garbage becomes connected, and we get the following program with garbage collecting:

```
ltop = avma; p1 = gsqr(x); p3 = gsqr(y);
lbot = avma; z = cgetg(3, t_VEC);
gel(z, 1) = gadd(p1,y);
gel(z, 2) = gadd(p3,x); z = gerepile(ltop,lbot,z);
```

Finishing by `z = gerepileupto(ltop, z)` would be ok as well. Beware that

```
ltop = avma; p1 = gadd(gsqr(x), y); p3 = gadd(gsqr(y), x);
z = cgetg(3, t_VEC);
```

```

gel(z, 1) = p1;
gel(z, 2) = p3; z = gerepileupto(ltop,z); /* WRONG */

```

is a disaster since `p1` and `p3` are created before `z`, so the call to `gerepileupto` overwrites them, leaving `gel(z, 1)` and `gel(z, 2)` pointing at random data! The following does work:

```

ltop = avma; p1 = gsqr(x); p3 = gsqr(y);
lbot = avma; z = mkvec2(gadd(p1,y), gadd(p3,x));
z = gerepile(ltop,lbot,z);

```

but is very subtly wrong in the sense that `z = gerepileupto(ltop, z)` would *not* work. The reason being that `mkvec2` creates the root `z` of the vector *after* its arguments have been evaluated, creating the components of `z` too early; `gerepile` does not care, but the created `z` is a time bomb which will explode on any later `gerepileupto`. On the other hand

```

ltop = avma; z = cgetg(3, t_VEC);
gel(z, 1) = gadd(gsqr(x), y);
gel(z, 2) = gadd(gsqr(y), x); z = gerepileupto(ltop,z); /* INEFFICIENT */

```

leaves the results of `gsqr(x)` and `gsqr(y)` on the stack (and lets `gerepileupto` update them for naught). Finally, the most elegant and efficient version (with respect to time and memory use) is as follows

```

z = cgetg(3, t_VEC);
ltop = avma; gel(z, 1) = gerepileupto(ltop, gadd(gsqr(x), y));
ltop = avma; gel(z, 2) = gerepileupto(ltop, gadd(gsqr(y), x));

```

which avoids updating the container `z` and cleans up its components individually, as soon as they are computed.

**One last example.** Let us compute the product of two complex numbers  $x$  and  $y$ , using the  $3M$  method which requires 3 multiplications instead of the obvious 4. Let  $z = x*y$ , and set  $x = x_r + i*x_i$  and similarly for  $y$  and  $z$ . We compute  $p_1 = x_r * y_r$ ,  $p_2 = x_i * y_i$ ,  $p_3 = (x_r + x_i) * (y_r + y_i)$ , and then we have  $z_r = p_1 - p_2$ ,  $z_i = p_3 - (p_1 + p_2)$ . The program is as follows:

```

ltop = avma;
p1 = gmul(gel(x,1), gel(y,1));
p2 = gmul(gel(x,2), gel(y,2));
p3 = gmul(gadd(gel(x,1), gel(x,2)), gadd(gel(y,1), gel(y,2)));
p4 = gadd(p1,p2);
lbot = avma; z = cgetg(3, t_COMPLEX);
gel(z, 1) = gsub(p1,p2);
gel(z, 2) = gsub(p3,p4); z = gerepile(ltop,lbot,z);

```

**Exercise.** Write a function which multiplies a matrix by a column vector. Hint: start with a `cgetg` of the result, and use `gerepile` whenever a coefficient of the result vector is computed. You can look at the answer in `src/basemath/RgV.c:RgM_RgC_mul()`.

#### 4.3.3.2 gerepileall.

Let us now see why we may need the `gerepileall` variants. Although it is not an infrequent occurrence, we do not give a specific example but a general one: suppose that we want to do a computation (usually inside a larger function) producing more than one PARI object as a result, say two for instance. Then even if we set up the work properly, before cleaning up we have a stack which has the desired results `z1`, `z2` (say), and then connected garbage from `lbot` to `ltop`. If we write

```
z1 = gerepile(ltop, lbot, z1);
```

then the stack is cleaned, the pointers fixed up, but we have lost the address of `z2`. This is where we need the `gerepileall` function:

```
gerepileall(ltop, 2, &z1, &z2)
```

copies `z1` and `z2` to new locations, cleans the stack from `ltop` to the old `avma`, and updates the pointers `z1` and `z2`. Here we do not assume anything about the stack: the garbage can be disconnected and `z1`, `z2` need not be at the bottom of the stack. If all of these assumptions are in fact satisfied, then we can call `gerepilemanysp` instead, which is usually faster since we do not need the initial copy (on the other hand, it is less cache friendly).

A most important usage is “random” garbage collection during loops whose size requirements we cannot (or do not bother to) control in advance:

```
pari_sp av = avma;
GEN x, y;
while (...)
{
    garbage(); x = anything();
    garbage(); y = anything(); garbage();
    if (gc_needed(av,1)) /* memory is running low (half spent since entry) */
        gerepileall(av, 2, &x, &y);
}
```

Here we assume that only `x` and `y` are needed from one iteration to the next. As it would be costly to call `gerepile` once for each iteration, we only do it when it seems to have become necessary.

More precisely, the macro `stack_lim(av,n)` denotes an address where  $2^{n-1}/(2^{n-1}+1)$  of the remaining stack space since reference point `av` is exhausted (1/2 for  $n = 1$ , 2/3 for  $n = 2$ ). The test `gc_needed(av,n)` becomes true whenever `avma` drops below that address.

#### 4.3.4 Comments.

First, `gerepile` has turned out to be a flexible and fast garbage collector for number-theoretic computations, which compares favorably with more sophisticated methods used in other systems. Our benchmarks indicate that the price paid for using `gerepile` and `gerepile`-related copies, when properly used, is usually less than 1% of the total running time, which is quite acceptable!

Second, it is of course harder on the programmer, and quite error-prone if you do not stick to a consistent PARI programming style. If all seems lost, just use `gerepilecopy` (or `gerepileall`) to fix up the stack for you. You can always optimize later when you have sorted out exactly which routines are crucial and what objects need to be preserved and their usual sizes.

If you followed us this far, congratulations, and rejoice: the rest is much easier.

### 4.4 Creation of PARI objects, assignments, conversions.

**4.4.1 Creation of PARI objects.** The basic function which creates a PARI object is

`GEN cgetg(long l, long t)`  $l$  specifies the number of longwords to be allocated to the object, and  $t$  is the type of the object, in symbolic form (see Section 4.5 for the list of these). The precise effect of this function is as follows: it first creates on the PARI *stack* a chunk of memory of size `length` longwords, and saves the address of the chunk which it will in the end return. If the stack has been used up, a message to the effect that “the PARI stack overflows” is printed, and an error raised. Otherwise, it sets the type and length of the PARI object. In effect, it fills its first codeword (`z[0]`). Many PARI objects also have a second codeword (types `t_INT`, `t_REAL`, `t_PADIC`, `t_POL`, and `t_SER`). In case you want to produce one of those from scratch, which should be exceedingly rare, *it is your responsibility to fill this second codeword*, either explicitly (using the macros described in Section 4.5), or implicitly using an assignment statement (using `gaffect`).

Note that the length argument  $l$  is predetermined for a number of types: 3 for types `t_INTMOD`, `t_FRAC`, `t_COMPLEX`, `t_POLMOD`, `t_RFRAC`, 4 for type `t_QUAD` and `t_QFI`, and 5 for type `t_PADIC` and `t_QFR`. However for the sake of efficiency, `cgetg` does not check this: disasters will occur if you give an incorrect length for those types.

**Notes.** 1) The main use of this function is create efficiently a constant object, or to prepare for later assignments (see Section 4.4.3). Most of the time you will use `GEN` objects as they are created and returned by PARI functions. In this case you do not need to use `cgetg` to create space to hold them.

2) For the creation of leaves, i.e. `t_INT` or `t_REAL`,

`GEN cgeti(long length)`

`GEN cgetr(long length)`

should be used instead of `cgetg(length, t_INT)` and `cgetg(length, t_REAL)` respectively. Finally

`GEN cgetc(long prec)`

creates a `t_COMPLEX` whose real and imaginary part are `t_REALs` allocated by `cgetr(prec)`.



**Examples.** 1) Both `z = cgeti(DEFAULTPREC)` and `cgetg(DEFAULTPREC, t_INT)` create a `t_INT` whose “precision” is `bit_accuracy(DEFAULTPREC) = 64`. This means `z` can hold rational integers of absolute value less than  $2^{64}$ . Note that in both cases, the second codeword is *not* filled. Of course we could use numerical values, e.g. `cgeti(4)`, but this would have different meanings on different machines as `bit_accuracy(4)` equals 64 on 32-bit machines, but 128 on 64-bit machines.

2) The following creates a *complex number* whose real and imaginary parts can hold real numbers of precision `bit_accuracy(MEDDEFAULTPREC) = 96` bits:

```
z = cgetg(3, t_COMPLEX);
gel(z, 1) = cgetr(MEDDEFAULTPREC);
gel(z, 2) = cgetr(MEDDEFAULTPREC);
```

or simply `z = cgetc(MEDDEFAULTPREC)`.

3) To create a matrix object for  $4 \times 3$  matrices:

```
z = cgetg(4, t_MAT);
for(i=1; i<4; i++) gel(z, i) = cgetg(5, t_COL);
```

or simply `z = zeromatcopy(4, 3)`, which further initializes all entries to `gen_0`.

These last two examples illustrate the fact that since PARI types are recursive, all the branches of the tree must be created. The function `cgetg` creates only the “root”, and other calls to `cgetg` must be made to produce the whole tree. For matrices, a common mistake is to think that `z = cgetg(4, t_MAT)` (for example) creates the root of the matrix: one needs also to create the column vectors of the matrix (obviously, since we specified only one dimension in the first `cgetg`!). This is because a matrix is really just a row vector of column vectors (hence a priori not a basic type), but it has been given a special type number so that operations with matrices become possible.

Finally, to facilitate input of constant objects when speed is not paramount, there are four `varargs` functions:

`GEN mkintn(long n, ...)` returns the non-negative `t_INT` whose development in base  $2^{32}$  is given by the following  $n$  32bit-words (`unsigned int`).

```
mkintn(3, a2, a1, a0);
```

returns  $a_2 2^{64} + a_1 2^{32} + a_0$ .

`GEN mkpoln(long n, ...)` Returns the `t_POL` whose  $n$  coefficients (`GEN`) follow, in order of decreasing degree.

```
mkpoln(3, gen_1, gen_2, gen_0);
```

returns the polynomial  $X^2 + 2X$  (in variable 0, use `setvarn` if you want other variable numbers). Beware that  $n$  is the number of coefficients, hence *one more* than the degree.

`GEN mkvecn(long n, ...)` returns the `t_VEC` whose  $n$  coefficients (`GEN`) follow.

`GEN mkcoln(long n, ...)` returns the `t_COL` whose  $n$  coefficients (`GEN`) follow.

**Warning.** Contrary to the policy of general PARI functions, the latter three functions do *not* copy their arguments, nor do they produce an object a priori suitable for `gerepileupto`. For instance

```
/* gerepile-safe: components are universal objects */
z = mkvecn(3, gen_1, gen_0, gen_2);
/* not OK for gerepileupto: stoi(3) creates component before root */
z = mkvecn(3, stoi(3), gen_0, gen_2);
/* NO! First vector component x is destroyed */
x = gclone(gen_1);
z = mkvecn(3, x, gen_0, gen_2);
guncclone(x);
```

The following function is also available as a special case of `mkintn`:

`GEN uu32toi(ulong a, ulong b)`

Returns the `GEN` equal to  $2^{32}a + b$ , *assuming* that  $a, b < 2^{32}$ . This does not depend on `sizeof(long)`: the behavior is as above on both 32 and 64-bit machines.

#### 4.4.2 Sizes.

`long gsizeword(GEN x)` returns the total number of `BITS_IN_LONG`-bit words occupied by the tree representing `x`.

`long gsizebyte(GEN x)` returns the total number of bytes occupied by the tree representing `x`, i.e. `gsizeword(x)` multiplied by `sizeof(long)`. This is normally useless since PARI functions use a number of *words* as input for lengths and precisions.

**4.4.3 Assignments.** Firstly, if `x` and `y` are both declared as `GEN` (i.e. pointers to something), the ordinary C assignment `y = x` makes perfect sense: we are just moving a pointer around. However, physically modifying either `x` or `y` (for instance, `x[1] = 0`) also changes the other one, which is usually not desirable.

**Very important note.** Using the functions described in this paragraph is inefficient and often awkward: one of the `gerepile` functions (see Section 4.3) should be preferred. See the paragraph end for one exception to this rule.

The general PARI assignment function is the function `gaffect` with the following syntax:

```
void gaffect(GEN x, GEN y)
```

Its effect is to assign the PARI object `x` into the *preexisting* object `y`. Both `x` and `y` must be *scalar* types. For convenience, vector or matrices of scalar types are also allowed.

This copies the whole structure of `x` into `y` so many conditions must be met for the assignment to be possible. For instance it is allowed to assign a `t_INT` into a `t_REAL`, but the converse is forbidden. For that, you must use the truncation or rounding function of your choice, e.g. `mpfloor`.

It can also happen that `y` is not large enough or does not have the proper tree structure to receive the object `x`. For instance, let `y` the zero integer with length equal to 2; then `y` is too small to accommodate any non-zero `t_INT`. In general common sense tells you what is possible, keeping in mind the PARI philosophy which says that if it makes sense it is valid. For instance, the assignment of an imprecise object into a precise one does *not* make sense. However, a change in precision of imprecise objects is allowed, even if it *increases* its accuracy: we complement the

“mantissa” with infinitely many 0 digits in this case. (Mantissa between quotes, because this is not restricted to `t_REALs`, it also applies for  $p$ -adics for instance.)

All functions ending in “z” such as **gaddz** (see Section 4.2.4) implicitly use this function. In fact what they exactly do is record **avma** (see Section 4.3), perform the required operation, **gaffect** the result to the last operand, then restore the initial **avma**.

You can assign ordinary C long integers into a PARI object (not necessarily of type `t_INT`) using

```
void gaffsg(long s, GEN y)
```

**Note.** Due to the requirements mentioned above, it is usually a bad idea to use **gaffect** statements. There is one exception: for simple objects (e.g. leaves) whose size is controlled, they can be easier to use than **gerepile**, and about as efficient.

**Coercion.** It is often useful to coerce an inexact object to a given precision. For instance at the beginning of a routine where precision can be kept to a minimum; otherwise the precision of the input is used in all subsequent computations, which is inefficient if the latter is known to thousands of digits. One may use the **gaffect** function for this, but it is easier and more efficient to call

`GEN gtofp(GEN x, long prec)` converts the complex number `x` (`t_INT`, `t_REAL`, `t_FRAC`, `t_QUAD` or `t_COMPLEX`) to either a `t_REAL` or `t_COMPLEX` whose components are `t_REAL` of length `prec`.

**4.4.4 Copy.** It is also very useful to copy a PARI object, not just by moving around a pointer as in the `y = x` example, but by creating a copy of the whole tree structure, without pre-allocating a possibly complicated `y` to use with **gaffect**. The function which does this is called **gcopy**. Its syntax is:

```
GEN gcopy(GEN x)
```

and the effect is to create a new copy of `x` on the PARI stack.

Sometimes, on the contrary, a quick copy of the skeleton of `x` is enough, leaving pointers to the original data in `x` for the sake of speed instead of making a full recursive copy. Use `GEN shallowcopy(GEN x)` for this. Note that the result is not suitable for **gerepileupto** !

Make sure at this point that you understand the difference between `y = x`, `y = gcopy(x)`, `y = shallowcopy(x)` and **gaffect**(`x,y`).

**4.4.5 Clones.** Sometimes, it is more efficient to create a *persistent* copy of a PARI object. This is not created on the stack but on the heap, hence unaffected by **gerepile** and friends. The function which does this is called **gclone**. Its syntax is:

```
GEN gclone(GEN x)
```

A clone can be removed from the heap (thus destroyed) using

```
void gunclone(GEN x)
```

No PARI object should keep references to a clone which has been destroyed!

**4.4.6 Conversions.** The following functions convert C objects to PARI objects (creating them on the stack as usual):

`GEN stoi(long s):` C long integer (“small”) to `t_INT`.

`GEN dbltor(double s):` C double to `t_REAL`. The accuracy of the result is 19 decimal digits, i.e. a type `t_REAL` of length `DEFAULTPREC`, although on 32-bit machines only 16 of them are significant.

We also have the converse functions:

`long itos(GEN x):` `x` must be of type `t_INT`,

`double rtodbl(GEN x):` `x` must be of type `t_REAL`,

as well as the more general ones:

`long gtolong(GEN x),`

`double gtodouble(GEN x).`

## 4.5 Implementation of the PARI types.

We now go through each type and explain its implementation. Let `z` be a `GEN`, pointing at a PARI object. In the following paragraphs, we will constantly mix two points of view: on the one hand, `z` is treated as the C pointer it is, on the other, as PARI’s handle on some mathematical entity, so we will shamelessly write `z ≠ 0` to indicate that the *value* thus represented is nonzero (in which case the *pointer* `z` is certainly non-NULL). We offer no apologies for this style. In fact, you had better feel comfortable juggling both views simultaneously in your mind if you want to write correct PARI programs.

Common to all the types is the first codeword `z[0]`, which we do not have to worry about since this is taken care of by `cgetg`. Its precise structure depends on the machine you are using, but it always contains the following data: the *internal type number* attached to the symbolic type name, the *length* of the root in longwords, and a technical bit which indicates whether the object is a clone or not (see Section 4.4.5). This last one is used by `gp` for internal garbage collecting, you will not have to worry about it.

Some types have a second codeword, different for each type, which we will soon describe as we will shortly consider each of them in turn.

The first codeword is handled through the following *macros*:

`long typ(GEN z)` returns the type number of `z`.

`void settyp(GEN z, long n)` sets the type number of `z` to `n` (you should not have to use this function if you use `cgetg`).

`long lg(GEN z)` returns the length (in longwords) of the root of `z`.

`long setlg(GEN z, long l)` sets the length of `z` to `l`; you should not have to use this function if you use `cgetg`.

`void lg_increase(GEN z)` increase the length of `z` by 1; you should not have to use this function if you use `cgetg`.

`long isclone(GEN z)` is `z` a clone?

`void setisclone(GEN z)` sets the *clone* bit.

`void unsetisclone(GEN z)` clears the *clone* bit.

**Important remark.** For the sake of efficiency, none of the codeword-handling macros check the types of their arguments even when there are stringent restrictions on their use. It is trivial to create invalid objects, or corrupt one of the “universal constants” (e.g. setting the sign of `gen_0` to 1), and they usually provide negligible savings. Use higher level functions whenever possible.

**Remark.** The clone bit is there so that `gunclone` can check it is deleting an object which was allocated by `gclone`. Miscellaneous vector entries are often cloned by `gp` so that a GP statement like `v[1] = x` does not involve copying the whole of `v`: the component `v[1]` is deleted if its clone bit is set, and is replaced by a clone of `x`. Don’t set/unset yourself the clone bit unless you know what you are doing: in particular *never* set the clone bit of a vector component when the said vector is scheduled to be uncloned. Hackish code may abuse the clone bit to tag objects for reasons unrelated to the above instead of using proper data structures. Don’t do that.

**4.5.1 Type `t_INT` (integer).** this type has a second codeword `z[1]` which contains the following information:

the sign of `z`: coded as 1, 0 or  $-1$  if  $z > 0$ ,  $z = 0$ ,  $z < 0$  respectively.

the *effective length* of `z`, i.e. the total number of significant longwords. This means the following: apart from the integer 0, every integer is “normalized”, meaning that the most significant mantissa longword is non-zero. However, the integer may have been created with a longer length. Hence the “length” which is in `z[0]` can be larger than the “effective length” which is in `z[1]`.

This information is handled using the following macros:

`long signe(GEN z)` returns the sign of `z`.

`void setsigne(GEN z, long s)` sets the sign of `z` to `s`.

`long lgefint(GEN z)` returns the effective length of `z`.

`void setlgefint(GEN z, long l)` sets the effective length of `z` to `l`.

The integer 0 can be recognized either by its sign being 0, or by its effective length being equal to 2. Now assume that  $z \neq 0$ , and let

$$|z| = \sum_{i=0}^n z_i B^i, \quad \text{where } z_n \neq 0 \text{ and } B = 2^{\text{BITS\_IN\_LONG}}.$$

With these notations,  $n$  is `lgefint(z) - 3`, and the mantissa of `z` may be manipulated via the following interface:

`GEN int_MSW(GEN z)` returns a pointer to the most significant word of `z`,  $z_n$ .

`GEN int_LSW(GEN z)` returns a pointer to the least significant word of `z`,  $z_0$ .

`GEN int_W(GEN z, long i)` returns the  $i$ -th significant word of `z`,  $z_i$ . Accessing the  $i$ -th significant word for  $i > n$  yields unpredictable results.

`GEN int_W_lg(GEN z, long i, long lz)` returns the  $i$ -th significant word of `z`,  $z_i$ , assuming `lgefint(z)` is `lz` ( $= n + 3$ ). Accessing the  $i$ -th significant word for  $i > n$  yields unpredictable results.

`GEN int_precW(GEN z)` returns the previous (less significant) word of `z`,  $z_{i-1}$  assuming `z` points to  $z_i$ .

`GEN int_nextW(GEN z)` returns the next (more significant) word of  $z$ ,  $z_{i+1}$  assuming  $z$  points to  $z_i$ .

Unnormalized integers, such that  $z_n$  is possibly 0, are explicitly forbidden. To enforce this, one may write an arbitrary mantissa then call

```
void int_normalize(GEN z, long known0)
```

normalizes in place a non-negative integer (such that  $z_n$  is possibly 0), assuming at least the first `known0` words are zero.

For instance a binary `and` could be implemented in the following way:

```
GEN AND(GEN x, GEN y) {
    long i, lx, ly, lout;
    long *xp, *yp, *outp; /* mantissa pointers */
    GEN out;

    if (!signe(x) || !signe(y)) return gen_0;
    lx = lgefint(x); xp = int_LSW(x);
    ly = lgefint(y); yp = int_LSW(y); lout = min(lx, ly); /* > 2 */
    out = cgeti(lout); out[1] = evalsigne(1) | evallgefint(lout);
    outp = int_LSW(out);
    for (i=2; i < lout; i++)
    {
        *outp = (*xp) & (*yp);
        outp = int_nextW(outp);
        xp = int_nextW(xp);
        yp = int_nextW(yp);
    }
    if (!*int_MSW(out)) out = int_normalize(out, 1);
    return out;
}
```

This low-level interface is mandatory in order to write portable code since PARI can be compiled using various multiprecision kernels, for instance the native one or GNU MP, with incompatible internal structures (for one thing, the mantissa is oriented in different directions).

**4.5.2 Type `t_REAL` (real number).** this type has a second codeword  $z[1]$  which also encodes its sign, obtained or set using the same functions as for a `t_INT`, and a binary exponent. This exponent is handled using the following macros:

`long expo(GEN z)` returns the exponent of  $z$ . This is defined even when  $z$  is equal to zero.

`void setexpo(GEN z, long e)` sets the exponent of  $z$  to  $e$ .

Note the functions:

`long gexpo(GEN z)` which tries to return an exponent for  $z$ , even if  $z$  is not a real number.

`long gsigne(GEN z)` which returns a sign for  $z$ , even when  $z$  is neither real nor integer (a rational number for instance).

The real zero is characterized by having its sign equal to 0. If  $z$  is not equal to 0, then it is represented as  $2^e M$ , where  $e$  is the exponent, and  $M \in [1, 2[$  is the mantissa of  $z$ , whose digits are stored in  $z[2], \dots, z[\lg(z) - 1]$ .

More precisely, let  $m$  be the integer  $(z[2], \dots, z[\lg(z)-1])$  in base  $2^{\text{BITS\_IN\_LONG}}$ ; here,  $z[2]$  is the most significant longword and is normalized, i.e. its most significant bit is 1. Then we have  $M := m/2^{\text{bit\_accuracy}(\lg(z))-1-\text{expo}(z)}$ .

`GEN mantissa_real(GEN z, long *e)` returns the mantissa  $m$  of  $z$ , and sets  $*e$  to the exponent  $\text{bit\_accuracy}(\lg(z)) - 1 - \text{expo}(z)$ , so that  $z = m/2^e$ .

Thus, the real number 3.5 to accuracy  $\text{bit\_accuracy}(\lg(z))$  is represented as  $z[0]$  (encoding  $\text{type} = \text{t\_REAL}$ ,  $\lg(z)$ ),  $z[1]$  (encoding  $\text{sign} = 1$ ,  $\text{expo} = 1$ ),  $z[2] = 0\text{x}e0000000$ ,  $z[3] = \dots = z[\lg(z) - 1] = 0\text{x}0$ .

**4.5.3 Type `t_INTMOD`.**  $z[1]$  points to the modulus, and  $z[2]$  at the number representing the class  $z$ . Both are separate GEN objects, and both must be `t_INTs`, satisfying the inequality  $0 \leq z[2] < z[1]$ .

**4.5.4 Type `t_FRAC` (rational number).**  $z[1]$  points to the numerator  $n$ , and  $z[2]$  to the denominator  $d$ . Both must be of type `t_INT` such that  $n \neq 0$ ,  $d > 0$  and  $(n, d) = 1$ .

**4.5.5 Type `t_FFELT` (finite field element).** (Experimental)

Components of this type should normally not be accessed directly. Instead, finite field elements should be created using `ffgen`.

The second codeword  $z[1]$  determines the storage format of the element, among

- `t_FF_FpXq`:  $A=z[2]$  and  $T=z[3]$  are `FpX`,  $p=z[4]$  is a `t_INT`, where  $p$  is a prime number,  $T$  is irreducible modulo  $p$ , and  $\deg A < \deg T$ . This represents the element  $A \pmod{T}$  in  $\mathbf{F}_p[X]/T$ .

- `t_FF_Flxq`:  $A=z[2]$  and  $T=z[3]$  are `Flx`,  $l=z[4]$  is a `t_INT`, where  $l$  is a prime number,  $T$  is irreducible modulo  $l$ , and  $\deg A < \deg T$ . This represents the element  $A \pmod{T}$  in  $\mathbf{F}_l[X]/T$ .

- `t_FF_F2xq`:  $A=z[2]$  and  $T=z[3]$  are `F2x`,  $l=z[4]$  is the `t_INT` 2,  $T$  is irreducible modulo 2, and  $\deg A < \deg T$ . This represents the element  $A \pmod{T}$  in  $\mathbf{F}_2[X]/T$ .

**4.5.6 Type `t_COMPLEX` (complex number).**  $z[1]$  points to the real part, and  $z[2]$  to the imaginary part. The components  $z[1]$  and  $z[2]$  must be of type `t_INT`, `t_REAL` or `t_FRAC`. For historical reasons `t_INTMOD` and `t_PADIC` are also allowed (the latter for  $p = 2$  or congruent to 3 mod 4 only), but one should rather use the more general `t_POLMOD` construction.

**4.5.7 Type `t_PADIC` ( $p$ -adic numbers).** this type has a second codeword  $z[1]$  which contains the following information: the  $p$ -adic precision (the exponent of  $p$  modulo which the  $p$ -adic unit corresponding to  $z$  is defined if  $z$  is not 0), i.e. one less than the number of significant  $p$ -adic digits, and the exponent of  $z$ . This information can be handled using the following functions:

`long precp(GEN z)` returns the  $p$ -adic precision of  $z$ . This is 0 if  $z = 0$ .

`void setprecp(GEN z, long l)` sets the  $p$ -adic precision of  $z$  to  $l$ .

`long valp(GEN z)` returns the  $p$ -adic valuation of  $z$  (i.e. the exponent). This is defined even if  $z$  is equal to 0.

`void setvalp(GEN z, long e)` sets the  $p$ -adic valuation of  $z$  to  $e$ .

In addition to this codeword,  $z[2]$  points to the prime  $p$ ,  $z[3]$  points to  $p^{\text{precp}(z)}$ , and  $z[4]$  points to `at_INT` representing the  $p$ -adic unit attached to  $z$  modulo  $z[3]$  (and to zero if  $z$  is zero). To summarize, if  $z \neq 0$ , we have the equality:

$$z = p^{\text{valp}(z)} * (z[4] + O(z[3])), \quad \text{where} \quad z[3] = O(p^{\text{precp}(z)}).$$

**4.5.8 Type `t_QUAD` (quadratic number).** `z[1]` points to the canonical polynomial  $P$  defining the quadratic field (as output by `quadpoly`), `z[2]` to the “real part” and `z[3]` to the “imaginary part”. The latter are of type `t_INT`, `t_FRAC`, `t_INTMOD`, or `t_PADIC` and are to be taken as the coefficients of  $z$  with respect to the canonical basis  $(1, X)$  of  $\mathbf{Q}[X]/(P(X))$ . Exact complex numbers may be implemented as quadratics, but `t_COMPLEX` is in general more versatile (`t_REAL` components are allowed) and more efficient.

Operations involving a `t_QUAD` and `t_COMPLEX` are implemented by converting the `t_QUAD` to a `t_REAL` (or `t_COMPLEX` with `t_REAL` components) to the accuracy of the `t_COMPLEX`. As a consequence, operations between `t_QUAD` and *exact* `t_COMPLEX`s are not allowed.

**4.5.9 Type `t_POLMOD` (polmod).** as for `t_INTMOD`s, `z[1]` points to the modulus, and `z[2]` to a polynomial representing the class of  $z$ . Both must be of type `t_POL` in the same variable, satisfying the inequality  $\deg z[2] < \deg z[1]$ . However, `z[2]` is allowed to be a simplification of such a polynomial, e.g. a scalar. This is tricky considering the hierarchical structure of the variables; in particular, a polynomial in variable of *lesser* priority (see Section 4.6) than the modulus variable is valid, since it is considered as the constant term of a polynomial of degree 0 in the correct variable. On the other hand a variable of *greater* priority is not acceptable.

**4.5.10 Type `t_POL` (polynomial).** this type has a second codeword. It contains a “*sign*”: 0 if the polynomial is equal to 0, and 1 if not (see however the important remark below) and a *variable number* (e.g. 0 for  $x$ , 1 for  $y$ , etc. . .).

These data can be handled with the following macros: **`signe`** and **`setsigne`** as for `t_INT` and `t_REAL`,

**`long varn(GEN z)`** returns the variable number of the object  $z$ ,

**`void setvarn(GEN z, long v)`** sets the variable number of  $z$  to  $v$ .

The variable numbers encode the relative priorities of variables, we will give more details in Section 4.6. Note also the function **`long gvar(GEN z)`** which tries to return a variable number for  $z$ , even if  $z$  is not a polynomial or power series. The variable number of a scalar type is set by definition equal to `NO_VARIABLE`, which has lower priority than any other variable number.

The components `z[2]`, `z[3]`, . . . `z[lg(z)-1]` point to the coefficients of the polynomial *in ascending order*, with `z[2]` being the constant term and so on.

For a `t_POL` of non-zero sign, **`degpol`**, **`leading_coeff`**, **`constant_coeff`**, return its degree, and a pointer to the leading, resp. constant, coefficient with respect to the main variable. Note that no copy is made on the PARI stack so the returned value is not safe for a basic **`gerepile`** call. Applied to any other type than `t_POL`, the result is unspecified. Those three functions are still defined when the sign is 0, see Section 5.2.7 and Section 10.6.

**`long degree(GEN x)`** returns the degree of  $x$  with respect to its main variable even when  $x$  is not a polynomial (a rational function for instance). By convention, the degree of a zero polynomial is  $-1$ .



**Important remark.** The leading coefficient of a `t_POL` may be equal to zero:

- it is not allowed to be an exact rational 0, such as `gen_0`;
- an exact non-rational 0, like `Mod(0,2)`, is possible for constant polynomials, i.e. of length 3 and no other coefficient: this carries information about the base ring for the polynomial;
- an inexact 0, like `0.E-38` or `0(3^5)`, is always possible. Inexact zeroes do not correspond to an actual 0, but to a very small coefficient according to some metric; we keep them to give information on how much cancellation occurred in previous computations.

A polynomial disobeying any of these rules is an invalid *unnormalized* object. We advise *not* to use low-level constructions to build a `t_POL` coefficient by coefficient, such as

```
GEN T = cgetg(4, t_POL);
T[1] = evalvarn(0);
gel(T, 2) = x;
gel(T, 3) = y;
```

But if you do and it is not clear whether the result will be normalized, call

`GEN normalizepol(GEN x)` applied to an unnormalized `t_POL` `x` (with all coefficients correctly set except that `leading_term(x)` might be zero), normalizes `x` correctly in place and returns `x`. This function sets `signe` (to 0 or 1) properly.

**Caveat.** A consequence of the remark above is that zero polynomials are characterized by the fact that their sign is 0. It is in general incorrect to check whether `lg(x)` is 2 or `degpol(x) < 0`, although both tests are valid when the coefficient types are under control: for instance, when they are guaranteed to be `t_INTs` or `t_FRACs`. The same remark applies to `t_SERs`.

**4.5.11 Type `t_SER` (power series).** This type also has a second codeword, which encodes a “*sign*”, i.e. 0 if the power series is 0, and 1 if not, a *variable number* as for polynomials, and an *exponent*. This information can be handled with the following functions: `signe`, `setsigne`, `varn`, `setvarn` as for polynomials, and `valp`, `setvalp` for the exponent as for  $p$ -adic numbers. Beware: do *not* use `expo` and `setexpo` on power series.

The coefficients `z[2]`, `z[3]`, ... `z[lg(z)-1]` point to the coefficients of `z` in ascending order. As for polynomials (see remark there), the sign of a `t_SER` is 0 if and only if all its coefficients are equal to 0. (The leading coefficient cannot be an integer 0.) A series whose coefficients are integers equal to zero is represented as  $O(x^n)$  (`zerozer(vx, n)`). A series whose coefficients are exact zeroes, but not all of them integers (e.g. an `t_INTMOD` such as `Mod(0,2)`) is represented as  $z * x^{n-1} + O(x^n)$ , where  $z$  is the 0 of the base ring, as per `Rg_get_0`.

Note that the exponent of a power series can be negative, i.e. we are then dealing with a Laurent series (with a finite number of negative terms).

**4.5.12 Type `t_RFRAC` (rational function).** `z[1]` points to the numerator  $n$ , and `z[2]` on the denominator  $d$ . The denominator must be of type `t_POL`, with variable of higher priority than the numerator. The numerator  $n$  is not an exact 0 and  $(n, d) = 1$  (see `gred_rfac2`).

**4.5.13 Type `t_QFR` (indefinite binary quadratic form).** `z[1]`, `z[2]`, `z[3]` point to the three coefficients of the form and are of type `t_INT`. `z[4]` is Shanks’s distance function, and must be of type `t_REAL`.

**4.5.14 Type `t_QFI` (definite binary quadratic form).** `z[1]`, `z[2]`, `z[3]` point to the three coefficients of the form. All three are of type `t_INT`.

**4.5.15 Type `t_VEC` and `t_COL` (vector).** `z[1]`, `z[2]`, ..., `z[lg(z)-1]` point to the components of the vector.

**4.5.16 Type `t_MAT` (matrix).** `z[1]`, `z[2]`, ..., `z[lg(z)-1]` point to the column vectors of `z`, i.e. they must be of type `t_COL` and of the same length.

**4.5.17 Type `t_VECSMALL` (vector of small integers).** `z[1]`, `z[2]`, ..., `z[lg(z)-1]` are ordinary signed long integers. This type is used instead of a `t_VEC` of `t_INT`s for efficiency reasons, for instance to implement efficiently permutations, polynomial arithmetic and linear algebra over small finite fields, etc.

**4.5.18 Type `t_STR` (character string).**

`char * GSTR(z) (= (z+1))` points to the first character of the (NULL-terminated) string.

**4.5.19 Type `t_ERROR` (error context).** This type holds error messages, as well as details about the error, as returned by the exception handling system. The second codeword `z[1]` contains the error type (an `int`, as passed to `pari_err`). The subsequent words `z[2]`, ..., `z[lg(z)-1]` are `GEN`s containing additional data, depending on the error type.

**4.5.20 Type `t_CLOSURE` (closure).** This type holds GP functions and closures, in compiled form. The internal detail of this type is subject to change each time the GP language evolves. Hence we do not describe it here and refer to the Developer's Guide. However functions to create or to evaluate `t_CLOSURE`s are documented in Section 12.1.

`long closure_arity(GEN C)` returns the arity of the `t_CLOSURE`.

`long closure_is_variadic(GEN C)` returns 1 if the closure `C` is variadic, 0 else.

**4.5.21 Type `t_INFINITY` (infinity).**

This type has a single `t_INT` component, which is either 1 or  $-1$ , corresponding to  $+\infty$  and  $-\infty$  respectively.

`GEN mkmoo()` returns  $-\infty$

`GEN mkoo()` returns  $\infty$

`long inf_get_sign(GEN x)` returns 1 if  $x$  is  $+\infty$ , and  $-1$  if  $x$  is  $-\infty$ .

**4.5.22 Type `t_LIST` (list).** this type was introduced for specific `gp` use and is rather inefficient compared to a straightforward linked list implementation (it requires more memory, as well as many unnecessary copies). Hence we do not describe it here and refer to the Developer's Guide.

**Implementation note.** For the types including an exponent (or a valuation), we actually store a biased non-negative exponent (bit-ORing the biased exponent to the codeword), obtained by adding a constant to the true exponent: either `HIGHEXPOBIT` (for `t_REAL`) or `HIGHVALPBIT` (for `t_PADIC` and `t_SER`). Of course, this is encapsulated by the exponent/valuation-handling macros and needs not concern the library user.

## 4.6 PARI variables.

### 4.6.1 Multivariate objects.

We now consider variables and formal computations. As we have seen in Section 4.5, the codewords for types `t_POL` and `t_SER` encode a “variable number”. This is an integer, ranging from 0 to `MAXVARN`. Relative priorities may be ascertained using

```
int varncmp(long v, long w)
```

which is  $> 0$ ,  $= 0$ ,  $< 0$  whenever  $v$  has lower, resp. same, resp. higher priority than  $w$ .

The way an object is considered in formal computations depends entirely on its “principal variable number” which is given by the function

```
long gvar(GEN z)
```

which returns a variable number for  $z$ , even if  $z$  is not a polynomial or power series. The variable number of a scalar type is set by definition equal to `NO_VARIABLE` which has lower priority than any valid variable number. The variable number of a recursive type which is not a polynomial or power series is the variable number with highest priority among its components. But for polynomials and power series only the “outermost” number counts (we directly access `varn(x)` in the codewords): the representation is not symmetrical at all.

Under `gp`, one needs not worry too much since the interpreter defines the variables as it sees them\* and do the right thing with the polynomials produced.

But in library mode, they are tricky objects if you intend to build polynomials yourself (and not just let PARI functions produce them, which is less efficient). For instance, it does not make sense to have a variable number occur in the components of a polynomial whose main variable has a lower priority, even though PARI cannot prevent you from doing it.

**4.6.2 Creating variables.** A basic difficulty is to “create” a variable. Some initializations are needed before you can use a given integer  $v$  as a variable number.

Initially, this is done for 0 and 1 (the variables `x` and `y` under `gp`), and  $2, \dots, 9$  (printed as `t2`,  $\dots$  `t9`), with decreasing priority.

**4.6.2.1 User variables.** When the program starts, `x` (number 0) and `y` (number 1) are the only available variables, numbers 2 to 9 (decreasing priority) are reserved for building polynomials with predictable priorities.

To define further ones, you may use

```
GEN varhigher(const char *s)
```

```
GEN varlower(const char *s)
```

to recover a monomial of degree 1 in a new variable, which is guaranteed to have higher (resp. lower) priority than all existing ones at the time of the function call. The variable is printed as  $s$ , but is not part of GP’s interpreter: it is not a symbol bound to a value.

---

\* The first time a given identifier is read by the GP parser a new variable is created, and it is assigned a strictly lower priority than any variable in use at this point. On startup, before any user input has taken place, ‘`x`’ is defined in this way and has initially maximal priority (and variable number 0).

On the other hand

`long fetch_user_var(char *s)`: inspects the user variable whose name is the string pointed to by `s`, creating it if needed, and returns its variable number.

```
long v = fetch_user_var("y");
GEN gy = pol_x(v);
```

The function raises an exception if the name is already in use for an `installed` or built-in function, or an alias. This function is mostly useless since it returns a variable with unpredictable priority. Don't use it to create new variables.

**Caveat.** You can use `gp_read_str` (see Section 4.7.1) to execute a GP command and create GP variables on the fly as needed:

```
GEN gy = gp_read_str("'y"); /* returns pol_x(v), for some v */
long v = varn(gy);
```

But please note the quote `'y` in the above. Using `gp_read_str("y")` might work, but is dangerous, especially when programming functions to be used under `gp`. The latter reads the value of `y`, as *currently* known by the `gp` interpreter, possibly creating it in the process. But if `y` has been modified by previous `gp` commands (e.g. `y = 1`), then the value of `gy` is not what you expected it to be and corresponds instead to the current value of the `gp` variable (e.g. `gen_1`).

`GEN fetch_var_value(long v)` returns a shallow copy of the current value of the variable numbered `v`. Returns `NULL` if that variable number is unknown to the interpreter, e.g. it is a user variable. Note that this may not be the same as `pol_x(v)` if assignments have been performed in the interpreter.

**4.6.2.2 Temporary variables.** You can create temporary variables using

`long fetch_var()` returns a new variable with *lower* priority than any variable currently in use.

`long fetch_var_higher()` returns a new variable with *higher* priority than any variable currently in use.

After the statement `v = fetch_var()`, you can use `pol_1(v)` and `pol_x(v)`. The variables created in this way have no identifier assigned to them though, and are printed as `tnumber`. You can assign a name to a temporary variable, after creating it, by calling the function

```
void name_var(long n, char *s)
```

after which the output machinery will use the name `s` to represent the variable number `n`. The GP parser will *not* recognize it by that name, however, and calling this on a variable known to `gp` raises an error. Temporary variables are meant to be used as free variables to build polynomials and power series, and you should never assign values or functions to them as you would do with variables under `gp`. For that, you need a user variable.

All objects created by `fetch_var` are on the heap and not on the stack, thus they are not subject to standard garbage collecting (they are not destroyed by a `gerepile` or `avma = ltop` statement). When you do not need a variable number anymore, you can delete it using

```
long delete_var()
```

which deletes the *latest* temporary variable created and returns the variable number of the previous one (or simply returns 0 if none remain). Of course you should make sure that the deleted variable does not appear anywhere in the objects you use later on. Here is an example:

```

long first = fetch_var();
long n1 = fetch_var();
long n2 = fetch_var(); /* prepare three variables for internal use */
...
/* delete all variables before leaving */
do { num = delete_var(); } while (num && num <= first);

```

The (dangerous) statement

```
while (delete_var()) /* empty */;
```

removes all temporary variables in use.

#### 4.6.3 Comparing variables.

Let us go back to `varncmp`. There is an interesting corner case, when one of the compared variables (from `gvar`, say) is `NO_VARIABLE`. In this case, `varncmp` declares it has lower priority than any other variable; of course, comparing `NO_VARIABLE` with itself yields 0 (same priority);

In addition to `varncmp` we have

`long varnmax(long v, long w)` given two variable numbers (possibly `NO_VARIABLE`), returns the variable with the highest priority. This function always returns a valid variable number unless it is comparing `NO_VARIABLE` to itself.

`long varnmin(long x, long y)` given two variable numbers (possibly `NO_VARIABLE`), returns the variable with the lowest priority. Note that when comparing a true variable with `NO_VARIABLE`, this function returns `NO_VARIABLE`, which is not a valid variable number.

### 4.7 Input and output.

Two important aspects have not yet been explained which are specific to library mode: input and output of PARI objects.

#### 4.7.1 Input.

For input, PARI provides several powerful high level functions which enable you to input your objects as if you were under `gp`. In fact, it *is* essentially the GP syntactical parser.

There are two similar functions available to parse a string:

```
GEN gp_read_str(const char *s)
```

```
GEN gp_read_str_multiline(const char *s, char *last)
```

Both functions read the whole string `s`. The function `gp_read_str` ignores newlines: it assumes that the input is one expression and returns the result of this expression.

The function `gp_read_str_multiline` processes the text in the same way as the GP command `read`: newlines are significant and can be used to separate expressions. The return value is that of the last non-empty expression evaluated.

In `gp_read_str_multiline`, if `last` is non-NULL, then `*last` receives the last character from the *filtered* input: this can be used to check if the last character was a semi-colon (to hide the output in interactive usage). If (and only if) the input contains no statements, then `*last` is set to 0.

For both functions, `gp`'s metacommands *are* recognized.

**Note.** The obsolete form

```
GEN readseq(char *t)
```

still exists for backward compatibility (assumes filtered input, without spaces or comments). Don't use it.

To read a GEN from a file, you can use the simpler interface

```
GEN gp_read_stream(FILE *file)
```

which reads a character string of arbitrary length from the stream `file` (up to the first complete expression sequence), applies `gp_read_str` to it, and returns the resulting GEN. This way, you do not have to worry about allocating buffers to hold the string. To interactively input an expression, use `gp_read_stream(stdin)`.

Finally, you can read in a whole file, as in GP's `read` statement

```
GEN gp_read_file(char *name)
```

As usual, the return value is that of the last non-empty expression evaluated. There is one technical exception: if `name` is a *binary* file (from `writebin`) containing more than one object, a `t_VEC` containing them all is returned. This is because binary objects bypass the parser, hence reading them has no useful side effect.

#### 4.7.2 Output to screen or file, output to string.

General output functions return nothing but print a character string as a side effect. Low level routines are available to write on PARI output stream `pari_outfile` (`stdout` by default):

`void pari_putc(char c)`: write character `c` to the output stream.

`void pari_puts(char *s)`: write `s` to the output stream.

`void pari_flush()`: flush output stream; most streams are buffered by default, this command makes sure that all characters output so are actually written.

`void pari_printf(const char *fmt, ...)`: the most versatile such function. `fmt` is a character string similar to the one `printf` uses. In there, `%` characters have a special meaning, and describe how to print the remaining operands. In addition to the standard format types (see the GP function `printf`), you can use the *length modifier* `P` (for PARI of course!) to specify that an argument is a GEN. For instance, the following are valid conversions for a GEN argument

<code>%Ps</code>	<i>convert to char* (will print an arbitrary GEN)</i>
<code>%P.10s</code>	<i>convert to char*, truncated to 10 chars</i>
<code>%P.2f</code>	<i>convert to floating point format with 2 decimals</i>
<code>%P4d</code>	<i>convert to integer, field width at least 4</i>

```
pari_printf("x[%d] = %Ps is not invertible!\n", i, gel(x,i));
```

Here `i` is an `int`, `x` a GEN which is not a leaf (presumably a vector, or a polynomial) and this would insert the value of its *i*-th GEN component: `gel(x,i)`.

Simple but useful variants to `pari_printf` are

`void output(GEN x)` prints `x` in raw format, followed by a newline and a buffer flush. This is more or less equivalent to

```
pari_printf("%Ps\n", x);
```

```
pari_flush();
```

`void outmat(GEN x)` as above except if  $x$  is a `t_MAT`, in which case a multi-line display is used to display the matrix. This is prettier for small dimensions, but quickly becomes unreadable and cannot be pasted and reused for input. If all entries of  $x$  are small integers, you may use the recursive features of `%Pd` and obtain the same (or better) effect with

```
pari_printf("%Pd\n", x);
pari_flush();
```

A variant like `%5Pd` would improve alignment by imposing 5 chars for each coefficient. Similarly if all entries are to be converted to floats, a format like `%5.1Pf` could be useful.

These functions write on (PARI's idea of) standard output, and must be used if you want your functions to interact nicely with `gp`. In most programs, this is not a concern and it is more flexible to write to an explicit `FILE*`, or to recover a character string:

`void pari_fprintf(FILE *file, const char *fmt, ...)` writes the remaining arguments to stream `file` according to the format specification `fmt`.

`char* pari_sprintf(const char *fmt, ...)` produces a string from the remaining arguments, according to the PARI format `fmt` (see `printf`). This is the `libpari` equivalent of `Strprintf`, and returns a `malloc`'ed string, which must be freed by the caller. Note that contrary to the analogous `sprintf` in the `libc` you do not provide a buffer (leading to all kinds of buffer overflow concerns); the function provided is actually closer to the GNU extension `asprintf`, although the latter has a different interface.

Simple variants of `pari_sprintf` convert a `GEN` to a `malloc`'ed ASCII string, which you must still `free` after use:

`char* GENtostr(GEN x)`, using the current default output format (`prettymat` by default).

`char* GENtoTeXstr(GEN x)`, suitable for inclusion in a `TeX` file.

Note that we have `va_list` analogs of the functions of `printf` type seen so far:

```
void pari_vprintf(const char *fmt, va_list ap)
```

```
void pari_vfprintf(FILE *file, const char *fmt, va_list ap)
```

```
char* pari_vsprintf(const char *fmt, va_list ap)
```

### 4.7.3 Errors.

If you want your functions to issue error messages, you can use the general error handling routine `pari_err`. The basic syntax is

```
pari_err(e_MISC, "error message");
```

This prints the corresponding error message and exit the program (in library mode; go back to the `gp` prompt otherwise). You can also use it in the more versatile guise

```
pari_err(e_MISC, format, ...);
```

where `format` describes the format to use to write the remaining operands, as in the `pari_printf` function. For instance:

```
pari_err(e_MISC, "x[%d] = %Ps is not invertible!", i, gel(x,i));
```

The simple syntax seen above is just a special case with a constant format and no remaining arguments. The general syntax is

```
void pari_err(numerr, ...)
```

where `numerr` is a codeword which specifies the error class and what to do with the remaining arguments and what message to print. For instance, if  $x$  is a GEN with internal type `t_STR`, say, `pari_err(e_TYPE, "extgcd", x)` prints the message:

```
*** incorrect type in extgcd (t_STR),
```

See Section 11.4 for details. In the libpari code itself, the general-purpose `e_MISC` is used sparingly: it is so flexible that the corresponding error contexts (`t_ERROR`) become hard to use reliably. Other more rigid error types are generally more useful: for instance the error context attached to the `e_TYPE` exception above is precisely documented and contains `"extgcd"` and  $x$  (not only its type) as readily available components.

#### 4.7.4 Warnings.

To issue a warning, use

`void pari_warn(warnerr, ...)` In that case, of course, we do *not* abort the computation, just print the requested message and go on. The basic example is

```
pari_warn(warner, "Strategy 1 failed. Trying strategy 2")
```

which is the exact equivalent of `pari_err(e_MISC, ...)` except that you certainly do not want to stop the program at this point, just inform the user that something important has occurred; in particular, this output would be suitably highlighted under `gp`, whereas a simple `printf` would not.

The valid *warning* keywords are `warner` (general), `warnprec` (increasing precision), `warnmem` (garbage collecting) and `warnfile` (error in file operation), used as follows:

```
pari_warn(warnprec, "bnfinit", newprec);
pari_warn(warnmem, "bnfinit");
pari_warn(warnfile, "close", "afile"); /* error when closing "afile" */
```

#### 4.7.5 Debugging output.

For debugging output, you can use the standard output functions, `output` and `pari_printf` mainly. Corresponding to the `gp` metacommand `\x`, you can also output the hexadecimal tree attached to an object:

`void dbgGEN(GEN x, long nb = -1)`, displays the recursive structure of  $x$ . If `nb = -1`, the full structure is printed, otherwise the leaves (non-recursive components) are truncated to `nb` words.

The function `output` is vital under debuggers, since none of them knows how to print PARI objects by default. Seasoned PARI developers add the following `gdb` macro to their `.gdbinit`:

```
define i
  call output((GEN)$arg0)
end
```

Typing `i x` at a breakpoint in `gdb` then prints the value of the GEN  $x$  (provided the optimizer has not put it into a register, but it is rarely a good idea to debug optimized code).



The global variables **DEBUGLEVEL** and **DEBUGMEM** (corresponding to the default **debug** and **debugmem**) are used throughout the PARI code to govern the amount of diagnostic and debugging output, depending on their values. You can use them to debug your own functions, especially if you install the latter under **gp**.

`void dbg_pari_heap(void)` print debugging statements about the PARI stack, heap, and number of variables used. Corresponds to `\s` under **gp**.

#### 4.7.6 Timers and timing output.

To handle timings in a reentrant way, PARI defines a dedicated data type, `pari_timer`, together with the following methods:

`void timer_start(pari_timer *T)` start (or reset) a timer.

`long timer_delay(pari_timer *T)` returns the number of milliseconds elapsed since the timer was last reset. Resets the timer as a side effect.

`long timer_get(pari_timer *T)` returns the number of milliseconds elapsed since the timer was last reset. Does *not* reset the timer.

`long timer_printf(pari_timer *T, char *format, ...)` This diagnostics function is equivalent to the following code

```
err_printf("Time ")
... prints remaining arguments according to format ...
err_printf(": %ld", timer_delay(T));
```

Resets the timer as a side effect.

They are used as follows:

```
pari_timer T;
timer_start(&T); /* initialize timer */
...
printf("Total time: %ldms\n", timer_delay(&T));
```

or

```
pari_timer T;
timer_start(&T);
for (i = 1; i < 10; i++) {
    ...
    timer_printf(&T, "for i = %ld (L[i] = %Ps)", i, gel(L,i));
}
```

The following functions provided the same functionality, in a non-reentrant way, and are now deprecated.

`long timer(void)`

`long timer2(void)`

`void msgtimer(const char *format, ...)`

The following function implements **gp**'s timer and should not be used in libpari programs: `long gettime(void)` equivalent to `timer_delay(T)` attached to a private timer *T*.

## 4.8 Iterators, Numerical integration, Sums, Products.

**4.8.1 Iterators.** Since it is easier to program directly simple loops in library mode, some GP iterators are mainly useful for GP programming. Here are the others:

- **fordiv** is a trivial iteration over a list produced by **divisors**.
- **forell** and **forsubgroup** are currently not implemented as an iterator but as a procedure with callbacks.

**void forell(void \*E, long fun(void\*, GEN), GEN a, GEN b)** goes through the same curves as **forell(e11,a,b,)**, calling **fun(E, e11)** for each curve **e11**, stopping if **fun** returns a non-zero value.

**void forsubgroup(void \*E, long fun(void\*, GEN), GEN G, GEN B)** goes through the same subgroups as **forsubgroup(H = G, B,)**, calling **fun(E, H)** for each subgroup **H**, stopping if **fun** returns a non-zero value.

- **forprime**, for which we refer you to the next subsection.
- **forcomposite**, we provide an iterator over composite integers:

**int forcomposite(forcomposite\_t \*T, GEN a, GEN b)** initialize an iterator  $T$  over composite integers in  $[a, b]$ ; over composites  $\geq a$  if  $b = \text{NULL}$ . Return 0 if the range is known to be empty from the start (as if  $b < a$  or  $b < 0$ ), and return 1 otherwise.

**GEN forcomposite\_next(forcomposite\_t \*T)** returns the next composite in the range, assuming that  $T$  was initialized by **forcomposite\_init**.

- **forvec**, for which we provide a convenient iterator. To initialize the analog of **forvec(X = v, ..., flag)**, call

**int forvec\_init(forvec\_t \*T, GEN v, long flag)** initialize an iterator  $T$  over the vectors generated by **forvec(X = v, ..., flag)**. This returns 0 if this vector list is empty, and 1 otherwise.

**GEN forvec\_next(forvec\_t \*T)** returns the next element in the **forvec** sequence, or **NULL** if we are done. The return value must be used immediately or copied since the next call to the iterator destroys it: the relevant vector is updated in place. The iterator works hard to not use up PARI stack, and is more efficient when all lower bounds in the initialization vector  $v$  are integers. In that case, the cost is linear in the number of tuples enumerated, and you can expect to run over more than  $10^9$  tuples per minute. If speed is critical and all integers involved would fit in  $C$  longs, write a simple direct backtracking algorithm yourself.

- **forpart** is a variant of **forvec** which iterates over partitions. See the documentation of the **forpart** GP function for details. This function is available as a loop with callbacks:

**void forpart(void \*data, long (\*call)(void\*, GEN), long k, GEN a, GEN n)**

It is also available as an iterator:

**void forpart\_init(forpart\_t \*T, long k, GEN a, GEN n)** initializes an iterator over the partitions of  $k$ , with length restricted by  $n$ , and components restricted by  $a$ , either of which can be set to **NULL** to run without restriction.

**GEN forpart\_next(forpart\_t \*T)** returns the next partition, or **NULL** when all partitions have been exhausted.

GEN `forpart_prev(forpart_t *T)` returns the previous partition, or NULL when all partitions have been exhausted.

In both cases, the partition must be used or copied before the next call since it is returned from a state array which will be modified in place. You may *not* mix calls to `forpart_next` and `forpart_prev`: the first one called determines the ordering used to iterate over the partitions; you can not go back since the `forpart_t` structure is used in incompatible ways.

- `forperm` to loop over permutations of  $k$ . See the documentation of the `forperm` GP function for details. This function is available as an iterator:

`void forperm_init(forperm_t *T, GEN k)` initializes an iterator over the permutations of  $k$  (`t_INT`, `t_VEC` or `t_VECSMALL`).

GEN `forperm_next(forperm_t *T)` returns the next permutation as a `t_VECSMALL` or NULL when all permutations have been exhausted. The permutation must be used or copied before the next call since it is returned from a state array which will be modified in place.

- `forsubset` to loop over subsets. See the documentation of the `forsubset` GP function for details. This function is available as two iterators:

`void forallsubset_init(forsubset_t *T, long n)`

`void forksubset_init(forsubset_t *T, long n, long k)`

It is also available in generic form:

`void forsubset_init(forsubset_t *T, GEN nk)` where `nk` is either a `t_INT`  $n$  or a `t_VEC` with two integral components  $[n, k]$ .

In all three cases, GEN `forsubset_next(forsubset_t *T)` returns the next subset as a `t_VEC-SMALL` or NULL when all subsets have been exhausted.

#### 4.8.2 Iterating over primes.

The library provides a high-level iterator, which stores its (private) data in a `struct forprime_t` and runs over arbitrary ranges of primes, without ever overflowing.

The iterator has two flavors, one providing the successive primes as `ulongs`, the other as GEN. They are initialized as follows, where we expect to run over primes  $\geq a$  and  $\leq b$ :

`int forprime_init(forprime_t *T, GEN a, GEN b)` for the GEN variant, where  $b = \text{NULL}$  means  $+\infty$ .

`int u_forprime_init(forprime_t *T, ulong a, ulong b)` for the `ulong` variant, where  $b = \text{ULONG\_MAX}$  means we will run through all primes representable in a `ulong` type.

Both variant return 1 on success, and 0 if the iterator would run over an empty interval (if  $a > b$ , for instance). They allocate the `forprime_t` data structure on the PARI stack.

The successive primes are then obtained using

GEN `forprime_next(forprime_t *T)`, returns NULL if no more primes are available in the interval.

`ulong u_forprime_next(forprime_t *T)`, returns 0 if no more primes are available in the interval.

These two functions leave alone the PARI stack, and write their state information in the preallocated `forprime_t` struct. The typical usage is thus:

```

forprime_t T;
GEN p;
pari_sp av = avma, av2;
forprime_init(&T, gen_2, stoi(1000));
av2 = avma;
while ( (p = forprime_next(&T)) )
{
    ...
    if ( prime_is_OK(p) ) break;
    avma = av2; /* delete garbage accumulated in this iteration */
}
avma = av; /* delete all */

```

Of course, the final `avma = av` could be replaced by a `gerepile` call. Beware that swapping the `av2 = avma` and `forprime_init` call would be incorrect: the first `avma = av2` would delete the `forprime_t` structure!

#### 4.8.3 Numerical analysis.

Numerical routines code a function (to be integrated, summed, zeroed, etc.) with two parameters named

```

void *E;
GEN (*eval)(void*, GEN)

```

The second is meant to contain all auxiliary data needed by your function. The first is such that `eval(x, E)` returns your function evaluated at `x`. For instance, one may code the family of functions  $f_t : x \rightarrow (x + t)^2$  via

```

GEN fun(void *t, GEN x) { return gsqr(gadd(x, (GEN)t)); }

```

One can then integrate  $f_1$  between  $a$  and  $b$  with the call

```

intnum((void*)stoi(1), &fun, a, b, NULL, prec);

```

Since you can set `E` to a pointer to any `struct` (typecast to `void*`) the above mechanism handles arbitrary functions. For simple functions without extra parameters, you may set `E = NULL` and ignore that argument in your function definition.

## 4.9 Catching exceptions.

### 4.9.1 Basic use.

PARI provides a mechanism to trap exceptions generated via `pari_err` using the `pari_CATCH` construction. The basic usage is as follows

```
pari_CATCH(err_code) {  
    recovery branch  
}  
pari_TRY {  
    main branch  
}  
pari_ENDCATCH
```

This fragment executes the main branch, then the recovery branch *if* exception `err_code` is thrown, e.g. `e_TYPE`. See Section 11.4 for the description of all error classes. The special error code `CATCH_ALL` is available to catch all errors.

One can replace the `pari_TRY` keyword by `pari_RETRY`, in which case once the recovery branch is run, we run the main branch again, still catching the same exceptions.

#### Restrictions.

- Such constructs can be nested without adverse effect, the innermost handler catching the exception.
- It is *valid* to leave either branch using `pari_err`.
- It is *invalid* to use C flow control instructions (`break`, `continue`, `return`) to directly leave either branch without seeing the `pari_ENDCATCH` keyword. This would leave an invalid structure in the exception handler stack, and the next exception would crash.
- In order to leave using `break`, `continue` or `return`, one must precede the keyword by a call to

`void pari_CATCH_reset()` disable the current handler, allowing to leave without adverse effect.

### 4.9.2 Advanced use.

In the recovery branch, the exception context can be examined via the following helper routines:

`GEN pari_err_last()` returns the exception context, as a `t_ERROR`. The exception *E* returned by `pari_err_last` can be rethrown, using

```
pari_err(0, E);
```

`long err_get_num(GEN E)` returns the error symbolic name. E.g `e_TYPE`.

`GEN err_get_compo(GEN E, long i)` error *i*-th component, as documented in Section 11.4.

For instance

```
pari_CATCH(CATCH_ALL) { /* catch everything */  
    GEN x, E = pari_err_last();  
    long code = err_get_num(E);  
    if (code != e_INV) pari_err(0, E); /* unexpected error, rethrow */
```

```

    x = err_get_compo(E, 2);
    /* e_INV has two components, 1: function name 2: non-invertible x */
    if (typ(x) != t_INTMOD) pari_err(0, E); /* unexpected type, rethrow */
    pari_CATCH_reset();
    return x; /* leave ! */
    ...
} pari_TRY {
    main branch
}
pari_ENDCATCH

```

## 4.10 A complete program.

Now that the preliminaries are out of the way, the best way to learn how to use the library mode is to study a detailed example. We want to write a program which computes the gcd of two integers, together with the Bezout coefficients. We shall use the standard quadratic algorithm which is not optimal but is not too far from the one used in the PARI function **bezout**.

Let  $x, y$  two integers and initially  $\begin{pmatrix} s_x & s_y \\ t_x & t_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ , so that

$$\begin{pmatrix} s_x & s_y \\ t_x & t_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

To apply the ordinary Euclidean algorithm to the right hand side, multiply the system from the left by  $\begin{pmatrix} 0 & 1 \\ 1 & -q \end{pmatrix}$ , with  $q = \text{floor}(x/y)$ . Iterate until  $y = 0$  in the right hand side, then the first line of the system reads

$$s_x x + s_y y = \gcd(x, y).$$

In practice, there is no need to update  $s_y$  and  $t_y$  since  $\gcd(x, y)$  and  $s_x$  are enough to recover  $s_y$ . The following program is now straightforward. A couple of new functions appear in there, whose description can be found in the technical reference manual in Chapter 5, but whose meaning should be clear from their name and the context.

This program can be found in `examples/extgcd.c` together with a proper `Makefile`. You may ignore the first comment

```

/*
GP;install("extgcd", "GG&&", "gcdex", "./libextgcd.so");
*/

```

which instruments the program so that `gp2c-run extgcd.c` can import the `extgcd()` routine into an instance of the `gp` interpreter (under the name `gcdex`). See the `gp2c` manual for details.

```

#include <pari/pari.h>
/*
GP;install("extgcd", "GG&&", "gcdex", "./libextgcd.so");
*/
/* return d = gcd(a,b), sets u, v such that au + bv = gcd(a,b) */
GEN
extgcd(GEN A, GEN B, GEN *U, GEN *V)
{
    pari_sp av = avma;
    GEN ux = gen_1, vx = gen_0, a = A, b = B;
    if (typ(a) != t_INT) pari_err_TYPE("extgcd",a);
    if (typ(b) != t_INT) pari_err_TYPE("extgcd",b);
    if (signe(a) < 0) { a = negi(a); ux = negi(ux); }
    while (!gequal0(b))
    {
        GEN r, q = dvmdii(a, b, &r), v = vx;
        vx = subii(ux, mulii(q, vx));
        ux = v; a = b; b = r;
    }
    *U = ux;
    *V = diviiexact( subii(a, mulii(A,ux)), B );
    gerepileall(av, 3, &a, U, V); return a;
}

int
main()
{
    GEN x, y, d, u, v;
    pari_init(1000000,2);
    printf("x = "); x = gp_read_stream(stdin);
    printf("y = "); y = gp_read_stream(stdin);
    d = extgcd(x, y, &u, &v);
    pari_printf("gcd = %Ps\nu = %Ps\nv = %Ps\n", d, u, v);
    pari_close();
    return 0;
}

```

For simplicity, the inner loop does not include any garbage collection, hence memory use is quadratic in the size of the inputs instead of linear. Here is a better version of that loop:

```

    pari_sp av = avma;
    ...
    while (!gequal0(b))
    {
        GEN r, q = dvmdii(a, b, &r), v = vx;
        vx = subii(ux, mulii(q, vx));
        ux = v; a = b; b = r;
        if (gc_needed(av,1))
            gerepileall(av, 4, &a, &b, &ux, &vx);
    }

```

}



## Chapter 5:

### Technical Reference Guide: the basics

In the following chapters, we describe all public low-level functions of the PARI library. These include specialized functions for handling all the PARI types. Simple higher level functions, such as arithmetic or transcendental functions, are described in Chapter 3 of the GP user's manual; we will eventually see more general or flexible versions in the chapters to come. A general introduction to the major concepts of PARI programming can be found in Chapter 4, which you should really read first.

We shall now study specialized functions, more efficient than the library wrappers, but sloppier on argument checking and damage control; besides speed, their main advantage is to give finer control about the inner workings of generic routines, offering more options to the programmer.

**Important advice.** Generic routines eventually call lower level functions. Optimize your algorithms first, not overhead and conversion costs between PARI routines. For generic operations, use generic routines first; do not waste time looking for the most specialized one available unless you identify a genuine bottleneck, or you need some special behavior the generic routine does not offer. The PARI source code is part of the documentation; look for inspiration there.

The type `long` denotes a `BITS_IN_LONG`-bit signed long integer (32 or 64 bits). The type `ulong` is defined as `unsigned long`. The word *stack* always refer to the PARI stack, allocated through an initial `pari_init` call. Refer to Chapters 1–2 and 4 for general background.

We shall often refer to the notion of *shallow* function, which means that some components of the result may point to components of the input, which is more efficient than a *deep* copy (full recursive copy of the object tree). Such outputs are not suitable for `gerepileupto` and particular care must be taken when garbage collecting objects which have been input to shallow functions: corresponding outputs also become invalid and should no longer be accessed.

A function is *not stack clean* if it leaves intermediate data on the stack besides its output, for efficiency reasons.

### 5.1 Initializing the library.

The following functions enable you to start using the PARI functions in a program, and cleanup without exiting the whole program.

#### 5.1.1 General purpose.

`void pari_init(size_t size, ulong maxprime)` initialize the library, with a stack of `size` bytes and a prime table up to the maximum of `maxprime` and  $2^{16}$ . Unless otherwise mentioned, no PARI function will function properly before such an initialization.

`void pari_close(void)` stop using the library (assuming it was initialized with `pari_init`) and frees all allocated objects.

### 5.1.2 Technical functions.

`void pari_init_opts(size_t size, ulong maxprime, ulong opts)` as `pari_init`, more flexible. `opts` is a mask of flags among the following:

`INIT_JMPm`: install PARI error handler. When an exception is raised, the program is terminated with `exit(1)`.

`INIT_SIGm`: install PARI signal handler.

`INIT_DFTm`: initialize the `GP_DATA` environment structure. This one *must* be enabled once. If you close `pari`, then restart it, you need not reinitialize `GP_DATA`; if you do not, then old values are restored.

`INIT_noPRIMEm`: do not compute the prime table (ignore the `maxprime` argument). The user *must* call `pari_init_primes` later.

`INIT_noIMTm`: (technical, see `pari_mt_init` in the Developer's Guide for detail). Do not call `pari_mt_init` to initialize the multi-thread engine. If this flag is set, `pari_mt_init()` will need to be called manually. See `examples/pari-mt.c` for an example.

`INIT_noINTGMPm`: do not install PARI-specific GMP memory functions. This option is ignored when the GMP library is not in use. You may install PARI-specific GMP memory functions later by calling

```
void pari_kernel_init(void)
```

and restore the previous values using

```
void pari_kernel_close(void)
```

This option should not be used without a thorough understanding of the problem you are trying to solve. The GMP memory functions are global variables used by the GMP library. If your program is linked with two libraries that require these variables to be set to different values, conflict ensues. To avoid a conflict, the proper solution is to record their values with `mp_get_memory_functions` and to call `mp_set_memory_functions` to restore the expected values each time the code switches from using one library to the other. Here is an example:

```
void>(*pari_alloc_ptr)(size_t);
void>(*pari_realloc_ptr)(void *, size_t, size_t);
void(*pari_free_ptr)(void *, size_t);
void(*otherlib_alloc_ptr)(size_t);
void(*otherlib_realloc_ptr)(void *, size_t, size_t);
void(*otherlib_free_ptr)(void *, size_t);

void init(void)
{
    pari_init(8000000, 500000);
    mp_get_memory_functions(&pari_alloc_ptr,&pari_realloc_ptr,
                           &pari_free_ptr);

    otherlib_init();
    mp_get_memory_functions(&otherlib_alloc_ptr,&otherlib_realloc_ptr,
                           &otherlib_free_ptr);
}

void function_that_use_pari(void)
{
```

```

    mp_set_memory_functions(pari_alloc_ptr, pari_realloc_ptr,
                           pari_free_ptr);
    /*use PARI functions*/
}
void function_that_use_otherlib(void)
{
    mp_set_memory_functions(otherlib_alloc_ptr, otherlib_realloc_ptr,
                           otherlib_free_ptr);
    /*use OTHERLIB functions*/
}

```

`void pari_close_opts(ulong init_opts)` as `pari_close`, for a library initialized with a mask of options using `pari_init_opts`. `opts` is a mask of flags among

`INIT.SIGm`: restore `SIG_DFL` default action for signals tampered with by PARI signal handler.

`INIT.DFTm`: frees the `GP_DATA` environment structure.

`INIT.noIMTm`: (technical, see `pari_mt_init` in the Developer's Guide for detail). Do not call `pari_mt_close` to close the multi-thread engine. `INIT.noINTGMPm`: do not restore GMP memory functions.

`void pari_sig_init(void (*f)(int))` install the signal handler `f` (see `signal(2)`): the signals `SIGBUS`, `SIGFPE`, `SIGINT`, `SIGBREAK`, `SIGPIPE` and `SIGSEGV` are concerned.

`void pari_init_primes(ulong maxprime)` Initialize the PARI primes. This function is called by `pari_init(..., maxprime)`. It is provided for users calling `pari_init_opts` with the flag `INIT.noPRIMEm`.

`void pari_sighandler(int signum)` the actual signal handler that PARI uses. This can be used as argument to `pari_sig_init` or `signal(2)`.

`void pari_stackcheck_init(void *stackbase)` controls the system stack exhaustion checking code in the GP interpreter. This should be used when the system stack base address change or when the address seen by `pari_init` is too far from the base address. If `stackbase` is `NULL`, disable the check, else set the base address to `stackbase`. It is normally used this way

```

int thread_start (...)
{
    long first_item_on_the_stack;
    ...
    pari_stackcheck_init(&first_item_on_the_stack);
}

```

`int pari_daemon(void)` forks a PARI daemon, detaching from the main process group. The function returns 1 in the parent, and 0 in the forked son.

`void paristack_setsize(size_t rsize, size_t vsize)` sets the default `parisize` to `rsize` and the default `parisizemax` to `vsize`, and reallocate the stack to match these value, destroying its content. Generally used just after `pari_init`.

`void paristack_resize(ulong newsize)` changes the current stack size to `newsize` (double it if `newsize` is 0). The new size is clipped to be at least the current stack size and at most `parisizemax`. The stack content is not affected by this operation.

`void parivstack_reset(void)` resets the current stack to its default size `parisize`. This is used to recover memory after a computation that enlarged the stack. This function destroys the content of the enlarged stack (between the old and the new bottom of the stack). Before calling this function, you must ensure that `avma` lies within the new smaller stack.

`void paristack_newsize(ulong newsize)` (*does not return*). Library version of  
`default(parisize, "newsize")`

Set the default `parisize` to `newsize`, or double `parisize` if `newsize` is equal to 0, then call `cb_pari_err_recover(-1)`.

`void parivstack_resize(ulong newsize)` (*does not return*). Library version of  
`default(parisizemax, "newsize")`

Set the default `parisizemax` to `newsize` and call `cb_pari_err_recover(-1)`.

### 5.1.3 Notions specific to the GP interpreter.

An **entree** is the generic object attached to an identifier (a name) in GP's interpreter, be it a built-in or user function, or a variable. For a function, it has at least the following fields:

`char *name`: the name under which the interpreter knows us.  
`void *value`: a pointer to the C function to call.  
`long menu`: a small integer  $\geq 1$  (to which group of function help do we belong, for the `?n` help menu).  
`char *code`: the prototype code.  
`char *help`: the help text for the function.

A routine in GP is described to the analyzer by an **entree** structure. Built-in PARI routines are grouped in *modules*, which are arrays of **entree** structs, the last of which satisfy `name = NULL` (sentinel). There are currently four modules in PARI/GP:

- general functions (`functions_basic`, known to `libpari`),
- gp-specific functions (`functions_gp`),

and two modules of obsolete functions. The function `pari_init` initializes the interpreter and declares all symbols in `functions_basic`. You may declare further functions on a case by case basis or as a whole module using

`void pari_add_function(entree *ep)` adds a single routine to the table of symbols in the interpreter. It assumes `pari_init` has been called.

`void pari_add_module(entree *mod)` adds all the routines in module `mod` to the table of symbols in the interpreter. It assumes `pari_init` has been called.

For instance, `gp` implements a number of private routines, which it adds to the default set via the calls

```
pari_add_module(functions_gp);
```

A GP **default** is likewise attached to a helper routine, that is run when the value is consulted, or changed by `default0` or `setdefault`. Such routines are grouped in the module `functions_default`.

`void pari_add_defaults_module(entree *mod)` adds all the defaults in module `mod` to the interpreter. It assumes that `pari_init` has been called. From this point on, all defaults in module `mod` are known to `setdefault` and friends.

#### 5.1.4 Public callbacks.

The `gp` calculator associates elaborate functions (for instance the break loop handler) to the following callbacks, and so can you:

`void (*cb_pari_ask_confirm)(const char *s)` initialized to `NULL`. Called with argument `s` whenever PARI wants confirmation for action `s`, for instance in `secure` mode.

`void (*cb_pari_init_histfile)(void)` initialized to `NULL`. Called when the `histfile` default is changed. The intent is for that callback to read the file content, append it to history in memory, then dump the expanded history to the new `histfile`.

`int (*cb_pari_is_interactive)(void)`; initialized to `NULL`.

`void (*cb_pari_quit)(long)` initialized to a no-op. Called when `gp` must evaluate the `quit` command.

`void (*cb_pari_start_output)(void)` initialized to `NULL`.

`int (*cb_pari_handle_exception)(long)` initialized to `NULL`. If not `NULL`, this routine is called with argument `-1` on `SIGINT`, and argument `err` on error `err`. If it returns a non-zero value, the error or signal handler returns, in effect further ignoring the error or signal, otherwise it raises a fatal error. A possible simple-minded handler, used by the `gp` interpreter, is

`int gp_handle_exception(long err)` if the `breakloop` default is enabled (set to 1) and `cb_pari_break_loop` is not `NULL`, we call this routine with `err` argument and return the result.

`int (*cb_pari_err_handle)(GEN)` If not `NULL`, this routine is called with a `t_ERROR` argument from `pari_err`. If it returns a non-zero value, the error returns, in effect further ignoring the error, otherwise it raises a fatal error.

The default behavior is to print a descriptive error message (display the error), then return 0, thereby raising a fatal error. This differs from `cb_pari_handle_exception` in that the function is not called on `SIGINT` (which do not generate a `t_ERROR`), only from `pari_err`. Use `cb_pari_sigint` if you need to handle `SIGINT` as well.

The following function can be used by `cb_pari_err_handle` to display the error message.

`const char* closure_func_err()` return a statically allocated string holding the name of the function that triggered the error. Return `NULL` if the error was not caused by a function.

`int (*cb_pari_break_loop)(int)` initialized to `NULL`.

`void (*cb_pari_sigint)(void)`. Function called when we receive `SIGINT`. By default, raises

```
pari_err(e_MISC, "user interrupt");
```

A possible simple-minded variant, used by the `gp` interpreter, is

```
void gp_sigint_fun(void)
```

`void (*cb_pari_pre_recover)(long)` initialized to `NULL`. If not `NULL`, this routine is called just before PARI cleans up from an error. It is not required to return. The error number is passed as argument, unless the PARI stack has been destroyed (`allocatemem`), in which case `-1` is passed.

`void (*cb_pari_err_recover)(long)` initialized to `pari_exit()`. This callback must not return. It is called after PARI has cleaned-up from an error. The error number is passed as argument, unless the PARI stack has been destroyed, in which case it is called with argument `-1`.

`int (*cb_pari_whatnow)(PariOUT *out, const char *s, int flag)` initialized to `NULL`. If not `NULL`, must check whether `s` existed in older versions of `pari` (the `gp` callback checks against `pari-1.39.15`). All output must be done via `out` methods.

- `flag = 0`: should print verbosely the answer, including help text if available.
- `flag = 1`: must return 0 if the function did not change, and a non-0 result otherwise. May print a help message.

### 5.1.5 Configuration variables.

`pari_library_path`: If set, It should be a path to the `libpari` library. It is used by the function `gpinstall` to locate the PARI library when searching for symbols. This should only be useful on Windows.

### 5.1.6 Utility functions.

`void pari_ask_confirm(const char *s)` raise an error if the callback `cb_pari_ask_confirm` is `NULL`. Otherwise calls

```
cb_pari_ask_confirm(s);
```

`char* gp_filter(const char *s)` pre-processor for the GP parser: filter out whitespace and GP comments from `s`.

`GEN pari_compile_str(const char *s)` low-level form of `compile_str`: assumes that `s` does not contain spaces or GP comments and returns the closure attached to the GP expression `s`. Note that GP metacommands are not recognized.

`int gp_meta(const char *s, int ismain)` low-level component of `gp_read_str`: assumes that `s` does not contain spaces or GP comments and try to interpret `s` as a GP metacommand (e.g. starting by `\` or `?`). If successful, execute the metacommand and return 1; otherwise return 0. The `ismain` parameter modifies the way `\r` commands are handled: if non-zero, act as if the file contents were entered via standard input (i.e. call `switchin` and divert `pari_infile`); otherwise, simply call `gp_read_file`.

`void pari_hit_return(void)` wait for the use to enter `\n` via standard input.

`void gp_load_gprc(void)` read and execute the user's GPRC file.

`void pari_center(const char *s)` print `s`, centered.

`void pari_print_version(void)` print verbose version information.

`long pari_community(void)` return the index of the support section in the help.

`const char* gp_format_time(long t)` format a delay of `t` ms suitable for `gp` output, with `timer` set.

`const char* gp_format_prompt(const char *p)` format a prompt `p` suitable for `gp` prompting (includes colors and protecting ANSI escape sequences for readline).

`void pari_alarm(long s)` set an alarm after `s` seconds (raise an `e_ALARM` exception).

`void gp_help(const char *s, long flag)` print help for *s*, depending on the value of *flag*:

- `h_REGULAR`, basic help (?);
- `h_LONG`, extended help (??);
- `h_APROPOS`, a propos help (??).

`const char ** gp_help_keyword_list(void)` return a NULL-terminated array of strings, containing keywords known to `gp_help` besides GP functions (e.g. `modulus` or `operator`). Used by the online help system and the contextual completion engine.

`void gp_echo_and_log(const char *p, const char *s)` given a prompt *p* and attached input command *s*, update logfile and possibly print on standard output if `echo` is set and we are not in interactive mode. The callback `cb_pari_is_interactive` must be set to a sensible value.

`void gp_alarm_handler(int sig)` the SIGALRM handler set by the `gp` interpreter.

`void print_fun_list(char **list, long n)` print all elements of *list* in columns, pausing (hit return) every *n* lines. *list* is NULL terminated.

### 5.1.7 Saving and restoring the GP context.

`void gp_context_save(struct gp_context* rec)` save the current GP context.

`void gp_context_restore(struct gp_context* rec)` restore a GP context. The new context must be an ancestor of the current context.

### 5.1.8 GP history.

These functions allow to control the GP history (the `%` operator).

`void pari_add_hist(GEN x, long t)` adds *x* as the last history entry; *t* is the time we used to compute it.

`GEN pari_get_hist(long p)`, if *p* > 0 returns entry of index *p* (i.e. `%p`), else returns entry of index *n* + *p* where *n* is the index of the last entry (used for `%`, `%'`, `%''`, etc.).

`long pari_get_histtime(long p)` as `pari_get_hist`, returning the time used to compute the history entry, instead of the entry itself.

`ulong pari_nb_hist(void)` return the index of the last entry.

## 5.2 Handling GENs.

Almost all these functions are either macros or inlined. Unless mentioned otherwise, they do not evaluate their arguments twice. Most of them are specific to a set of types, although no consistency checks are made: e.g. one may access the `sign` of a `t_PADIC`, but the result is meaningless.

### 5.2.1 Allocation.

`GEN cgetg(long l, long t)` allocates (the root of) a `GEN` of type  $t$  and length  $l$ . Sets  $z[0]$ .

`GEN cgeti(long l)` allocates a `t_INT` of length  $l$  (including the 2 codewords). Sets  $z[0]$  only.

`GEN cgetr(long l)` allocates a `t_REAL` of length  $l$  (including the 2 codewords). Sets  $z[0]$  only.

`GEN cgetc(long prec)` allocates a `t_COMPLEX` whose real and imaginary parts are `t_REALs` of length  $prec$ .

`GEN cgetg_copy(GEN x, long *lx)` fast version of `cgetg`: allocate a `GEN` with the same type and length as  $x$ , setting  $*lx$  to `lg(x)` as a side-effect. (Only sets the first codeword.) This is a little faster than `cgetg` since we may reuse the bitmask in  $x[0]$  instead of recomputing it, and we do not need to check that the length does not overflow the possibilities of the implementation (since an object with that length already exists). Note that `cgetg` with arguments known at compile time, as in

```
cgetg(3, t_INTMOD)
```

will be even faster since the compiler will directly perform all computations and checks.

`GEN vectrunc_init(long l)` perform `cgetg(1,t_VEC)`, then set the length to 1 and return the result. This is used to implement vectors whose final length is easily bounded at creation time, that we intend to fill gradually using:

`void vectrunc_append(GEN x, GEN y)` assuming  $x$  was allocated using `vectrunc_init`, appends  $y$  as the last element of  $x$ , which grows in the process. The function is shallow: we append  $y$ , not a copy; it is equivalent to

```
long lx = lg(x); gel(x, lx) = y; setlg(x, lx+1);
```

Beware that the maximal size of  $x$  (the  $l$  argument to `vectrunc_init`) is unknown, hence unchecked, and stack corruption will occur if we append more than  $l - 1$  elements to  $x$ . Use the safer (but slower) `shallowconcat` when  $l$  is not easy to bound in advance.

An other possibility is simply to allocate using `cgetg(1, t)` then fill the components as they become available: this time the downside is that we do not obtain a correct `GEN` until the vector is complete. Almost no PARI function will be able to operate on it.

`void vectrunc_append_batch(GEN x, GEN y)` successively apply

```
vectrunc_append(x, gel(y, i))
```

for all elements of the vector  $y$ .

`GEN coltrunc_init(long l)` as `vectrunc_init` but perform `cgetg(1,t.COL)`.

`GEN vecsmalltrunc_init(long l)`

`void vecsmalltrunc_append(GEN x, long t)` analog to the above for a `t_VECSMALL` container.



### 5.2.2 Length conversions.

These routines convert a non-negative length to different units. Their behavior is undefined at negative integers.

`long ndec2nlong(long x)` converts a number of decimal digits to a number of words. Returns  $1 + \text{floor}(x \times \text{BITS\_IN\_LONG} \log_2 10)$ .

`long ndec2prec(long x)` converts a number of decimal digits to a number of codewords. This is equal to  $2 + \text{ndec2nlong}(x)$ .

`long ndec2nbits(long x)` converts a number of decimal digits to a number of bits.

`long prec2ndec(long x)` converts a number of codewords to a number of decimal digits.

`long nbits2nlong(long x)` converts a number of bits to a number of words. Returns the smallest word count containing  $x$  bits, i.e.  $\text{ceil}(x/\text{BITS\_IN\_LONG})$ .

`long nbits2ndec(long x)` converts a number of bits to a number of decimal digits.

`long nbits2lg(long x)` converts a number of bits to a length in code words. Currently an alias for `nbits2nlong`.

`long nbits2prec(long x)` converts a number of bits to a number of codewords. This is equal to  $2 + \text{nbits2nlong}(x)$ .

`long nbits2extraprec(long x)` converts a number of bits to the mantissa length of a `t_REAL` in codewords. This is currently an alias to `nbits2nlong`.

`long nchar2nlong(long x)` converts a number of bytes to number of words. Returns the smallest word count containing  $x$  bytes, i.e.  $\text{ceil}(x/\text{sizeof}(\text{long}))$ .

`long prec2nbits(long x)` converts a `t_REAL` length into a number of significant bits; returns  $(x - 2)\text{BITS\_IN\_LONG}$ .

`double prec2nbits_mul(long x, double y)` returns  $\text{prec2nbits}(x) \times y$ .

`long bit_accuracy(long x)` converts a length into a number of significant bits; currently an alias for `prec2nbits`.

`double bit_accuracy_mul(long x, double y)` returns  $\text{bit\_accuracy}(x) \times y$ .

`long realprec(GEN x)` length of a `t_REAL` in words; currently an alias for `lg`.

`long bit_prec(GEN x)` length of a `t_REAL` in bits.

`long precdbl(long prec)` given a length in words corresponding to a `t_REAL` precision, return the length corresponding to doubling the precision. Due to the presence of 2 code words, this is  $2(\text{prec} - 2) + 2$ .

### 5.2.3 Read type-dependent information.

`long typ(GEN x)` returns the type number of  $x$ . The header files included through `pari.h` define symbolic constants for the GEN types: `t_INT` etc. Never use their actual numerical values. E.g to determine whether  $x$  is a `t_INT`, simply check

```
if (typ(x) == t_INT) { }
```

The types are internally ordered and this simplifies the implementation of commutative binary operations (e.g addition, gcd). Avoid using the ordering directly, as it may change in the future; use type grouping functions instead (Section 5.2.6).

`const char* type_name(long t)` given a type number  $t$  this routine returns a string containing its symbolic name. E.g `type_name(t_INT)` returns `"t_INT"`. The return value is read-only.

`long lg(GEN x)` returns the length of  $x$  in `BITS_IN_LONG`-bit words.

`long lgefint(GEN x)` returns the effective length of the `t_INT`  $x$  in `BITS_IN_LONG`-bit words.

`long signe(GEN x)` returns the sign ( $-1$ ,  $0$  or  $1$ ) of  $x$ . Can be used for `t_INT`, `t_REAL`, `t_POL` and `t_SER` (for the last two types, only  $0$  or  $1$  are possible).

`long gsigne(GEN x)` returns the sign of a real number  $x$ , valid for `t_INT`, `t_REAL` as `signe`, but also for `t_FRAC` and `t_QUAD` of positive discriminants. Raise a type error if `typ(x)` is not among those.

`long expi(GEN x)` returns the binary exponent of the real number equal to the `t_INT`  $x$ . This is a special case of `gexpo`.

`long expo(GEN x)` returns the binary exponent of the `t_REAL`  $x$ .

`long mpxpo(GEN x)` returns the binary exponent of the `t_INT` or `t_REAL`  $x$ .

`long gexpo(GEN x)` same as `expo`, but also valid when  $x$  is not a `t_REAL` (returns the largest exponent found among the components of  $x$ ). When  $x$  is an exact  $0$ , this returns `-HIGHEXPOBIT`, which is lower than any valid exponent.

`long gexpo_safe(GEN x)` same as `gexpo`, but returns a value strictly less than `-HIGHEXPOBIT` when the exponent is not defined (e.g. for a `t_PADIC` or `t_INTMOD` component).

`long valp(GEN x)` returns the  $p$ -adic valuation (for a `t_PADIC`) or  $X$ -adic valuation (for a `t_SER`, taken with respect to the main variable) of  $x$ .

`long precp(GEN x)` returns the precision of the `t_PADIC`  $x$ .

`long varn(GEN x)` returns the variable number of the `t_POL` or `t_SER`  $x$  (between  $0$  and `MAXVARN`).

`long gvar(GEN x)` returns the main variable number when any variable at all occurs in the composite object  $x$  (the smallest variable number which occurs), and `NO_VARIABLE` otherwise.

`long gvar2(GEN x)` returns the variable number for the ring over which  $x$  is defined, e.g. if  $x \in \mathbb{Z}[a][b]$  return (the variable number for)  $a$ . Return `NO_VARIABLE` if  $x$  has no variable or is not defined over a polynomial ring.

`long degpol(GEN x)` is a simple macro returning `lg(x) - 3`. This is the degree of the `t_POL`  $x$  with respect to its main variable, *if* its leading coefficient is non-zero (a rational  $0$  is impossible, but an inexact  $0$  is allowed, as well as an exact modular  $0$ , e.g. `Mod(0,2)`). If  $x$  has no coefficients (rational  $0$  polynomial), its length is  $2$  and we return the expected  $-1$ .

`long lgpol(GEN x)` is equal to `degpol(x) + 1`. Used to loop over the coefficients of a `t_POL` in the following situation:

```
GEN xd = x + 2;
long i, l = lgpol(x);
for (i = 0; i < l; i++) foo( xd[i] ).
```

`long precision(GEN x)` If `x` is of type `t_REAL`, returns the precision of `x`, namely the length of `x` in `BITS_IN_LONG`-bit words if `x` is not zero, and a reasonable quantity obtained from the exponent of `x` if `x` is numerically equal to zero. If `x` is of type `t_COMPLEX`, returns the minimum of the precisions of the real and imaginary part. Otherwise, returns 0 (which stands for infinite precision).

`long lgcols(GEN x)` is equal to `lg(gel(x,1))`. This is the length of the columns of a `t_MAT` with at least one column.

`long nbrows(GEN x)` is equal to `lg(gel(x,1))-1`. This is the number of rows of a `t_MAT` with at least one column.

`long gprecision(GEN x)` as `precision` for scalars. Returns the lowest precision encountered among the components otherwise.

`long sizedigit(GEN x)` returns 0 if `x` is exactly 0. Otherwise, returns `gexpo(x)` multiplied by  $\log_{10}(2)$ . This gives a crude estimate for the maximal number of decimal digits of the components of `x`.

**5.2.4 Eval type-dependent information.** These routines convert type-dependent information to bitmask to fill the codewords of `GEN` objects (see Section 4.5). E.g for a `t_REAL` `z`:

```
z[1] = evalsigne(-1) | evalexpo(2)
```

Compatible components of a codeword for a given type can be OR-ed as above.

`ulong evaltyp(long x)` convert type `x` to bitmask (first codeword of all `GENs`)

`long evallg(long x)` convert length `x` to bitmask (first codeword of all `GENs`). Raise overflow error if `x` is so large that the corresponding length cannot be represented

`long _evallg(long x)` as `evallg` *without* the overflow check.

`ulong evalvarn(long x)` convert variable number `x` to bitmask (second codeword of `t_POL` and `t_SER`)

`long evalsigne(long x)` convert sign `x` (in  $-1, 0, 1$ ) to bitmask (second codeword of `t_INT`, `t_REAL`, `t_POL`, `t_SER`)

`long evalprecp(long x)` convert  $p$ -adic ( $X$ -adic) precision `x` to bitmask (second codeword of `t_PADIC`, `t_SER`). Raise overflow error if `x` is so large that the corresponding precision cannot be represented.

`long _evalprecp(long x)` same as `evalprecp` *without* the overflow check.

`long evalvalp(long x)` convert  $p$ -adic ( $X$ -adic) valuation `x` to bitmask (second codeword of `t_PADIC`, `t_SER`). Raise overflow error if `x` is so large that the corresponding valuation cannot be represented.

`long _evalvalp(long x)` same as `evalvalp` *without* the overflow check.

`long evalexpo(long x)` convert exponent  $x$  to bitmask (second codeword of `t_REAL`). Raise overflow error if  $x$  is so large that the corresponding exponent cannot be represented

`long _evalexpo(long x)` same as `evalexpo` *without* the overflow check.

`long evallgefint(long x)` convert effective length  $x$  to bitmask (second codeword `t_INT`). This should be less or equal than the length of the `t_INT`, hence there is no overflow check for the effective length.

**5.2.5 Set type-dependent information.** Use these functions and macros with extreme care since usually the corresponding information is set otherwise, and the components and further codeword fields (which are left unchanged) may not be compatible with the new information.

`void settyp(GEN x, long s)` sets the type number of  $x$  to  $s$ .

`void setlg(GEN x, long s)` sets the length of  $x$  to  $s$ . This is an efficient way of truncating vectors, matrices or polynomials.

`void setlgefint(GEN x, long s)` sets the effective length of the `t_INT`  $x$  to  $s$ . The number  $s$  must be less than or equal to the length of  $x$ .

`void setsigne(GEN x, long s)` sets the sign of  $x$  to  $s$ . If  $x$  is a `t_INT` or `t_REAL`,  $s$  must be equal to  $-1$ ,  $0$  or  $1$ , and if  $x$  is a `t_POL` or `t_SER`,  $s$  must be equal to  $0$  or  $1$ . No sanity check is made; in particular, setting the sign of a  $0$  `t_INT` to  $\pm 1$  creates an invalid object.

`void togglesign(GEN x)` sets the sign  $s$  of  $x$  to  $-s$ , in place.

`void togglesign_safe(GEN *x)` sets the  $s$  sign of  $*x$  to  $-s$ , in place, unless  $*x$  is one of the integer universal constants in which case replace  $*x$  by its negation (e.g. replace `gen_1` by `gen_m1`).

`void setabssign(GEN x)` sets the sign  $s$  of  $x$  to  $|s|$ , in place.

`void affectsign(GEN x, GEN y)` shortcut for `setsigne(y, signe(x))`. No sanity check is made; in particular, setting the sign of a  $0$  `t_INT` to  $\pm 1$  creates an invalid object.

`void affectsign_safe(GEN x, GEN *y)` sets the sign of  $*y$  to that of  $x$ , in place, unless  $*y$  is one of the integer universal constants in which case replace  $*y$  by its negation if needed (e.g. replace `gen_1` by `gen_m1` if  $x$  is negative). No other sanity check is made; in particular, setting the sign of a  $0$  `t_INT` to  $\pm 1$  creates an invalid object.

`void normalize_frac(GEN z)` assuming  $z$  is of the form `mkfrac(a,b)` with  $b \neq 0$ , make sure that  $b > 0$  by changing the sign of  $a$  in place if needed (use `togglesign`).

`void setexpo(GEN x, long s)` sets the binary exponent of the `t_REAL`  $x$  to  $s$ . The value  $s$  must be a 24-bit signed number.

`void setvalp(GEN x, long s)` sets the  $p$ -adic or  $X$ -adic valuation of  $x$  to  $s$ , if  $x$  is a `t_PADIC` or a `t_SER`, respectively.

`void setprec(GEN x, long s)` sets the  $p$ -adic precision of the `t_PADIC`  $x$  to  $s$ .

`void setvarn(GEN x, long s)` sets the variable number of the `t_POL` or `t_SER`  $x$  to  $s$  (where  $0 \leq s \leq \text{MAXVARN}$ ).

**5.2.6 Type groups.** In the following functions, `t` denotes the type of a `GEN`. They used to be implemented as macros, which could evaluate their argument twice; *no longer*: it is not inefficient to write

```
is_intreal_t(typ(x))

int is_recursive_t(long t) true iff t is a recursive type (the non-recursive types are t_INT,
t_REAL, t_STR, t_VECSMALL). Somewhat contrary to intuition, t_LIST is also non-recursive, ; see
the Developer's guide for details.

int is_intreal_t(long t) true iff t is t_INT or t_REAL.

int is_rational_t(long t) true iff t is t_INT or t_FRAC.

int is_real_t(long t) true iff t is t_INT or t_REAL or t_FRAC.

int is_vec_t(long t) true iff t is t_VEC or t_COL.

int is_matvec_t(long t) true iff t is t_MAT, t_VEC or t_COL.

int is_scalar_t(long t) true iff t is a scalar, i.e a t_INT, a t_REAL, a t_INTMOD, a t_FRAC, a
t_COMPLEX, a t_PADIC, a t_QUAD, or a t_POLMOD.

int is_extscalar_t(long t) true iff t is a scalar (see is_scalar_t) or t is t_POL.

int is_const_t(long t) true iff t is a scalar which is not t_POLMOD.

int is_noncalc_t(long t) true if generic operations (gadd, gmul) do not make sense for t: cor-
responds to types t_LIST, t_STR, t_VECSMALL, t_CLOSURE
```

**5.2.7 Accessors and components.** The first two functions return `GEN` components as copies on the stack:

`GEN compo(GEN x, long n)` creates a copy of the `n`-th true component (i.e. not counting the codewords) of the object `x`.

`GEN truecoeff(GEN x, long n)` creates a copy of the coefficient of degree `n` of `x` if `x` is a scalar, `t_POL` or `t_SER`, and otherwise of the `n`-th component of `x`.

On the contrary, the following routines return the address of a `GEN` component. No copy is made on the stack:

`GEN constant_coeff(GEN x)` returns the address of the constant coefficient of `t_POL` `x`. By convention, a 0 polynomial (whose `sign` is 0) has `gen_0` constant term.

`GEN leading_coeff(GEN x)` returns the address of the leading coefficient of `t_POL` `x`, i.e. the coefficient of largest index stored in the array representing `x`. This may be an inexact 0. By convention, return `gen_0` if the coefficient array is empty.

`GEN gel(GEN x, long i)` returns the address of the `x[i]` entry of `x`. (`el` stands for element.)

`GEN gcoeff(GEN x, long i, long j)` returns the address of the `x[i,j]` entry of `t_MAT` `x`, i.e. the coefficient at row `i` and column `j`.

`GEN gmael(GEN x, long i, long j)` returns the address of the `x[i][j]` entry of `x`. (`mael` stands for multidimensional array element.)

`GEN gmael2(GEN A, long x1, long x2)` is an alias for `gmael`. Similar macros `gmael3`, `gmael4`, `gmael5` are available.

## 5.3 Global numerical constants.

These are defined in the various public PARI headers.

### 5.3.1 Constants related to word size.

long BITS\_IN\_LONG =  $2^{\text{TWOPOTBITS\_IN\_LONG}}$ : number of bits in a long (32 or 64).

long BITS\_IN\_HALFLONG: BITS\_IN\_LONG divided by 2.

long LONG\_MAX: the largest positive long.

ulong ULONG\_MAX: the largest ulong.

long DEFAULTPREC: the length (lg) of a t\_REAL with 64 bits of accuracy

long MEDDEFAULTPREC: the length (lg) of a t\_REAL with 128 bits of accuracy

long BIGDEFAULTPREC: the length (lg) of a t\_REAL with 192 bits of accuracy

ulong HIGHBIT: the largest power of 2 fitting in an ulong.

ulong LOWMASK: bitmask yielding the least significant bits.

ulong HIGHMASK: bitmask yielding the most significant bits.

The last two are used to implement the following convenience macros, returning half the bits of their operand:

ulong LOWWORD(ulong a) returns least significant bits.

ulong HIGHWORD(ulong a) returns most significant bits.

Finally

long divsBIL(long n) returns the Euclidean quotient of  $n$  by BITS\_IN\_LONG (with non-negative remainder).

long remsBIL(n) returns the (non-negative) Euclidean remainder of  $n$  by BITS\_IN\_LONG

long dvmdsBIL(long n, long \*r)

ulong dvmduBIL(ulong n, ulong \*r) sets  $r$  to remsBIL( $n$ ) and returns divsBIL( $n$ ).

### 5.3.2 Masks used to implement the GEN type.

These constants are used by higher level macros, like typ or lg:

EXPOnumBITS, LGnumBITS, SIGNnumBITS, TYPnumBITS, VALPnumBITS, VARNumBITS: number of bits used to encode expo, lg, signe, typ, valp, varn.

PRECPSHIFT, SIGNSHIFT, TYPSHIFT, VARNSHIFT: shifts used to recover or encode precp, varn, typ, signe

CLONEBIT, EXPOBITS, LGBITS, PRECPBITS, SIGNBITS, TYPBITS, VALPBITS, VARNBITS: bitmasks used to extract isclone, expo, lg, precp, signe, typ, valp, varn from GEN codewords.

MAXVARN: the largest possible variable number.

NO\_VARIABLE: sentinel returned by gvar( $x$ ) when  $x$  does not contain any polynomial; has a lower priority than any valid variable number.

HIGHEXPOBIT: a power of 2, one more than the largest possible exponent for a t\_REAL.

HIGHVALPBIT: a power of 2, one more than the largest possible valuation for a t\_PADIC or a t\_SER.

### 5.3.3 $\log 2$ , $\pi$ .

These are double approximations to useful constants:

M\_PI:  $\pi$ .

M\_LN2:  $\log 2$ .

LOG10\_2:  $\log 2 / \log 10$ .

LOG2\_10:  $\log 10 / \log 2$ .

## 5.4 Iterating over small primes, low-level interface.

One of the methods used by the high-level prime iterator (see Section 4.8.2), is a precomputed table. Its direct use is deprecated, but documented here.

After `pari_init(size, maxprime)`, a “prime table” is initialized with the successive *differences* of primes up to (possibly just a little beyond) `maxprime`. The prime table occupies roughly `maxprime / log(maxprime)` bytes in memory, so be sensible when choosing `maxprime`; it is 500000 by default under `gp` and there is no real benefit in choosing a much larger value: the high-level iterator provide *fast* access to primes up to the *square* of `maxprime`. In any case, the implementation requires that `maxprime < 2BITS_IN_LONG - 2048`, whatever memory is available.

PARI currently guarantees that the first 6547 primes, up to and including 65557, are present in the table, even if you set `maxprime` to zero. in the `pari_init` call.

Some convenience functions:

`ulong maxprime()` the largest prime computable using our prime table.

`void maxprime_check(ulong B)` raise an error if `maxprime()` is  $< B$ .

After the following initializations (the names *p* and *ptr* are arbitrary of course)

```
byteptr ptr = diffptr;
ulong p = 0;
```

calling the macro `NEXT_PRIME_VIADIFF_CHECK(p, ptr)` repeatedly will assign the successive prime numbers to *p*. Overrunning the prime table boundary will raise the error `e_MAXPRIME`, which just prints the error message:

```
*** not enough precomputed primes, need primelimit ~c
```

(for some numerical value *c*), then the macro aborts the computation. The alternative macro `NEXT_PRIME_VIADIFF` operates in the same way, but will omit that check, and is slightly faster. It should be used in the following way:

```
byteptr ptr = diffptr;
ulong p = 0;

if (maxprime() < goal) pari_err_MAXPRIME(goal); /* not enough primes */
while (p <= goal) /* run through all primes up to goal */
{
    NEXT_PRIME_VIADIFF(p, ptr);
    ...
}
```

Here, we use the general error handling function `pari_err` (see Section 4.7.3), with the codeword `e_MAXPRIME`, raising the “not enough primes” error. This could be rewritten as

```
maxprime_check(goal);
while (p <= goal) /* run through all primes up to goal */
{
    NEXT_PRIME_VIADIFF(p, ptr);
    ...
}
```

`byteptr initprimes(ulong maxprime, long *L, ulong *lastp)` computes a (malloc’ed) “prime table”, in fact a table of all prime differences for  $p < \text{maxprime}$  (and possibly a little beyond). Set  $L$  to the table length (argument to `malloc`), and  $lastp$  to the last prime in the table.

`void initprimetable(ulong maxprime)` computes a prime table (of all prime differences for  $p < \text{maxprime}$ ) and assign it to the global variable `diffptr`. Don’t change `diffptr` directly, call this function instead. This calls `initprimes` and updates internal data recording the table size.

`ulong init_primepointer_geq(ulong a, byteptr *pd)` returns the smallest prime  $p \geq a$ , and sets  $*pd$  to the proper offset of `diffptr` so that `NEXT_PRIME_VIADIFF(p, *pd)` correctly returns `unextprime(p + 1)`.

`ulong init_primepointer_gt(ulong a, byteptr *pd)` returns the smallest prime  $p > a$ .

`ulong init_primepointer_leq(ulong a, byteptr *pd)` returns the largest prime  $p \leq a$ .

`ulong init_primepointer_lt(ulong a, byteptr *pd)` returns the largest prime  $p < a$ .

## 5.5 Handling the PARI stack.

### 5.5.1 Allocating memory on the stack.

`GEN cgetg(long n, long t)` allocates memory on the stack for an object of length  $n$  and type  $t$ , and initializes its first codeword.

`GEN cgeti(long n)` allocates memory on the stack for a `t_INT` of length  $n$ , and initializes its first codeword. Identical to `cgetg(n, t_INT)`.

`GEN cgetr(long n)` allocates memory on the stack for a `t_REAL` of length  $n$ , and initializes its first codeword. Identical to `cgetg(n, t_REAL)`.

`GEN cgetc(long n)` allocates memory on the stack for a `t_COMPLEX`, whose real and imaginary parts are `t_REALs` of length  $n$ .

`GEN cgetp(GEN x)` creates space sufficient to hold the `t_PADIC`  $x$ , and sets the prime  $p$  and the  $p$ -adic precision to those of  $x$ , but does not copy (the  $p$ -adic unit or zero representative and the modulus of)  $x$ .

`GEN new_chunk(size_t n)` allocates a `GEN` with  $n$  components, *without* filling the required code words. This is the low-level constructor underlying `cgetg`, which calls `new_chunk` then sets the first code word. It works by simply returning the address  $((\text{GEN})\text{avma}) - n$ , after checking that it is larger than  $(\text{GEN})\text{bot}$ .

`void new_chunk_resize(size_t x)` this function is called by `new_chunk` when the PARI stack overflows. There is no need to call it manually. It will either extend the stack or report an `e_STACK` error.



`char* stack_malloc(size_t n)` allocates memory on the stack for  $n$  chars (*not*  $n$  GENs). This is faster than using `malloc`, and easier to use in most situations when temporary storage is needed. In particular there is no need to `free` individually all variables thus allocated: a simple `avma = oldavma` might be enough. On the other hand, beware that this is not permanent independent storage, but part of the stack. The memory is aligned on `sizeof(long)` bytes boundaries.

`char* stack_malloc_align(size_t n, long k)` as `stack_malloc`, but the memory is aligned on  $k$  bytes boundaries. The number  $k$  must be a multiple of the `sizeof(long)`.

`char* stack_calloc(size_t n)` as `stack_malloc`, setting the memory to zero.

Objects allocated through these last three functions cannot be `gerepile`'d, since they are not yet valid GENs: their codewords must be filled first.

`GEN cgetalloc(long t, size_t l)`, same as `cgetg(t, l)`, except that the result is allocated using `pari_malloc` instead of the PARI stack. The resulting GEN is now impervious to garbage collecting routines, but should be freed using `pari_free`.

### 5.5.2 Stack-independent binary objects.

`GENbin* copy_bin(GEN x)` copies  $x$  into a malloc'ed structure suitable for stack-independent binary transmission or storage. The object obtained is architecture independent provided, `sizeof(long)` remains the same on all PARI instances involved, as well as the multiprecision kernel (either native or GMP).

`GENbin* copy_bin_canon(GEN x)` as `copy_bin`, ensuring furthermore that the binary object is independent of the multiprecision kernel. Slower than `copy_bin`.

`GEN bin_copy(GENbin *p)` assuming  $p$  was created by `copy_bin(x)` (not necessarily by the same PARI instance: transmission or external storage may be involved), restores  $x$  on the PARI stack.

The routine `bin_copy` transparently encapsulate the following functions:

`GEN GENbinbase(GENbin *p)` the GEN data actually stored in  $p$ . All addresses are stored as offsets with respect to a common reference point, so the resulting GEN is unusable unless it is a non-recursive type; private low-level routines must be called first to restore absolute addresses.

`void shiftaddress(GEN x, long dec)` converts relative addresses to absolute ones.

`void shiftaddress_canon(GEN x, long dec)` converts relative addresses to absolute ones, and converts leaves from a canonical form to the one specific to the multiprecision kernel in use. The `GENbin` type stores whether leaves are stored in canonical form, so `bin_copy` can call the right variant.

Objects containing closures are harder to e.g. copy and save to disk, since closures contain pointers to libpari functions that will not be valid in another gp instance: there is little chance for them to be loaded at the exact same address in memory. Such objects must be saved along with a linking table.

`GEN copybin_unlink(GEN C)` returns a linking table allowing to safely store and transmit `t_CLOSURE` objects in  $C$ . If  $C = \text{NULL}$  return a linking table corresponding to the content of all gp variables.  $C$  may then be dumped to disk in binary form, for instance.

`void bincopy_relink(GEN C, GEN V)` given a binary object  $C$ , as dumped by `writebin` and read back into a session, and a linking table  $V$ , restore all closures contained in  $C$  (function pointers are translated to their current value).

**5.5.3 Garbage collection.** See Section 4.3 for a detailed explanation and many examples.

`void cgiv(GEN x)` frees object `x`, assuming it is the last created on the stack.

`GEN gerepile(pari_sp p, pari_sp q, GEN x)` general garbage collector for the stack.

`void gerepileall(pari_sp av, int n, ...)` cleans up the stack from `av` on (i.e from `avma` to `av`), preserving the `n` objects which follow in the argument list (of type `GEN*`). For instance, `gerepileall(av, 2, &x, &y)` preserves `x` and `y`.

`void gerepileallsp(pari_sp av, pari_sp ltop, int n, ...)` cleans up the stack between `av` and `ltop`, updating the `n` elements which follow `n` in the argument list (of type `GEN*`). Check that the elements of `g` have no component between `av` and `ltop`, and assumes that no garbage is present between `avma` and `ltop`. Analogous to (but faster than) `gerepileall` otherwise.

`GEN gerepilecopy(pari_sp av, GEN x)` cleans up the stack from `av` on, preserving the object `x`. Special case of `gerepileall` (case `n = 1`), except that the routine returns the preserved `GEN` instead of updating its address through a pointer.

`void gerepilemany(pari_sp av, GEN* g[], int n)` alternative interface to `gerepileall`. The preserved `GENs` are the elements of the array `g` of length `n`: `g[0]`, `g[1]`, ..., `g[n-1]`. Obsolete: no more efficient than `gerepileall`, error-prone, and clumsy (need to declare an extra `GEN *g`).

`void gerepilemanysp(pari_sp av, pari_sp ltop, GEN* g[], int n)` alternative interface to `gerepileallsp`. Obsolete.

`void gerepilecoeffs(pari_sp av, GEN x, int n)` cleans up the stack from `av` on, preserving `x[0]`, ..., `x[n-1]` (which are `GENs`).

`void gerepilecoeffssp(pari_sp av, pari_sp ltop, GEN x, int n)` cleans up the stack from `av` to `ltop`, preserving `x[0]`, ..., `x[n-1]` (which are `GENs`). Same assumptions as in `gerepilemanysp`, of which this is a variant. For instance

```
z = cgetg(3, t_COMPLEX);
av = avma; garbage(); ltop = avma;
z[1] = fun1();
z[2] = fun2();
gerepilecoeffssp(av, ltop, z + 1, 2);
return z;
```

cleans up the garbage between `av` and `ltop`, and connects `z` and its two components. This is marginally more efficient than the standard

```
av = avma; garbage(); ltop = avma;
z = cgetg(3, t_COMPLEX);
z[1] = fun1();
z[2] = fun2(); return gerepile(av, ltop, z);
```

`GEN gerepileupto(pari_sp av, GEN q)` analogous to (but faster than) `gerepilecopy`. Assumes that `q` is connected and that its root was created before any component. If `q` is not on the stack, this is equivalent to `avma = av`; in particular, sentinels which are not even proper `GENs` such as `q = NULL` are allowed.

`GEN gerepileuptoint(pari_sp av, GEN q)` analogous to (but faster than) `gerepileupto`. Assumes further that `q` is a `t_INT`. The length and effective length of the resulting `t_INT` are equal.

GEN `gerepileuptoleaf(pari_sp av, GEN q)` analogous to (but faster than) `gerepileupto`. Assumes further that `q` is a leaf, i.e a non-recursive type (`is_recursive_t(typ(q))` is non-zero). Contrary to `gerepileuptoint` and `gerepileupto`, `gerepileuptoleaf` leaves length and effective length of a `t_INT` unchanged.

#### 5.5.4 Garbage collection: advanced use.

`void stackdummy(pari_sp av, pari_sp ltop)` inhibits the memory area between `av` *included* and `ltop` *excluded* with respect to `gerepile`, in order to avoid a call to `gerepile(av, ltop, ...)`. The stack space is not reclaimed though.

More precisely, this routine assumes that `av` is recorded earlier than `ltop`, then marks the specified stack segment as a non-recursive type of the correct length. Thus `gerepile` will not inspect the zone, at most copy it. To be used in the following situation:

```
av0 = avma; z = cgetg(t_VEC, 3);
gel(z,1) = HUGE(); av = avma; garbage(); ltop = avma;
gel(z,2) = HUGE(); stackdummy(av, ltop);
```

Compared to the orthodox

```
gel(z,2) = gerepile(av, ltop, gel(z,2));
```

or even more wasteful

```
z = gerepilecopy(av0, z);
```

we temporarily lose `(av - ltop)` words but save a costly `gerepile`. In principle, a garbage collection higher up the call chain should reclaim this later anyway.

Without the `stackdummy`, if the `[av, ltop]` zone is arbitrary (not even valid GENs as could happen after direct truncation via `setlg`), we would leave dangerous data in the middle of `z`, which would be a problem for a later

```
gerepile(..., ... , z);
```

And even if it were made of valid GENs, inhibiting the area makes sure `gerepile` will not inspect their components, saving time.

Another natural use in low-level routines is to “shorten” an existing GEN `z` to its first `n - 1` components:

```
setlg(z, n);
stackdummy((pari_sp)(z + lg(z)), (pari_sp)(z + n));
```

or to its last `n` components:

```
long L = lg(z) - n, tz = typ(z);
stackdummy((pari_sp)(z + L), (pari_sp)z);
z += L; z[0] = evaltyp(tz) | evallg(L);
```

The first scenario (safe shortening an existing GEN) is in fact so common, that we provide a function for this:

`void fixlg(GEN z, long ly)` a safe variant of `setlg(z, ly)`. If `ly` is larger than `lg(z)` do nothing. Otherwise, shorten `z` in place, using `stackdummy` to avoid later `gerepile` problems.

`GEN gcopy_avma(GEN x, pari_sp *AVMA)` return a copy of  $x$  as from `gcopy`, except that we pretend that initially `avma` is `*AVMA`, and that `*AVMA` is updated accordingly (so that the total size of  $x$  is the difference between the two successive values of `*AVMA`). It is not necessary for `*AVMA` to initially point on the stack: `gclone` is implemented using this mechanism.

`GEN icopy_avma(GEN x, pari_sp av)` analogous to `gcopy_avma` but simpler: assume  $x$  is a `t_INT` and return a copy allocated as if initially we had `avma` equal to `av`. There is no need to pass a pointer and update the value of the second argument: the new (fictitious) `avma` is just the return value (typecast to `pari_sp`).

### 5.5.5 Debugging the PARI stack.

`int chk_gerepileupto(GEN x)` returns 1 if  $x$  is suitable for `gerepileupto`, and 0 otherwise. In the latter case, print a warning explaining the problem.

`void dbg_gerepile(pari_sp ltop)` outputs the list of all objects on the stack between `avma` and `ltop`, i.e. the ones that would be inspected in a call to `gerepile(...,ltop,...)`.

`void dbg_gerepileupto(GEN q)` outputs the list of all objects on the stack that would be inspected in a call to `gerepileupto(...,q)`.

### 5.5.6 Copies.

`GEN gcopy(GEN x)` creates a new copy of  $x$  on the stack.

`GEN gcopy_lg(GEN x, long l)` creates a new copy of  $x$  on the stack, pretending that `lg(x)` is `l`, which must be less than or equal to `lg(x)`. If equal, the function is equivalent to `gcopy(x)`.

`int isonstack(GEN x)` true iff  $x$  belongs to the stack.

`void copyifstack(GEN x, GEN y)` sets  $y = gcopy(x)$  if  $x$  belongs to the stack, and  $y = x$  otherwise. This macro evaluates its arguments once, contrary to

```
y = isonstack(x)? gcopy(x): x;
```

`void icopyifstack(GEN x, GEN y)` as `copyifstack` assuming  $x$  is a `t_INT`.

### 5.5.7 Simplify.

`GEN simplify(GEN x)` you should not need that function in library mode. One rather uses:

`GEN simplify_shallow(GEN x)` shallow, faster, version of `simplify`.

## 5.6 The PARI heap.

### 5.6.1 Introduction.

It is implemented as a doubly-linked list of `malloc`'ed blocks of memory, equipped with reference counts. Each block has type `GEN` but need not be a valid `GEN`: it is a chunk of data preceded by a hidden header (meaning that we allocate  $x$  and return  $x + \text{headersize}$ ). A *clone*, created by `gclone`, is a block which is a valid `GEN` and whose *clone bit* is set.

### 5.6.2 Public interface.

`GEN newblock(size_t n)` allocates a block of  $n$  words (not bytes).

`void killblock(GEN x)` deletes the block  $x$  created by `newblock`. Fatal error if  $x$  not a block.

`GEN gclone(GEN x)` creates a new permanent copy of  $x$  on the heap (allocated using `newblock`). The *clone bit* of the result is set.

`GEN gcloneref(GEN x)` if  $x$  is not a clone, clone it and return the result; otherwise, increase the clone reference count and return  $x$ .

`void gunclone(GEN x)` deletes a clone. Deletion at first only decreases the reference count by 1. If the count remains positive, no further action is taken; if the count becomes zero, then the clone is actually deleted. In the current implementation, this is an alias for `killblock`, but it is cleaner to kill clones (valid `GEN`s) using this function, and other blocks using `killblock`.

`void gunclone_deep(GEN x)` is only useful in the context of the GP interpreter which may replace arbitrary components of container types (`t_VEC`, `t_COL`, `t_MAT`, `t_LIST`) by clones. If  $x$  is such a container, the function recursively deletes all clones among the components of  $x$ , then unclones  $x$ . Useless in library mode: simply use `gunclone`.

`void traverseheap(void(*f)(GEN, void*), void *data)` this applies  $f(x, \text{data})$  to each object  $x$  on the PARI heap, most recent first. Mostly for debugging purposes.

`GEN getheap()` a simple wrapper around `traverseheap`. Returns a two-component row vector giving the number of objects on the heap and the amount of memory they occupy in long words.

`GEN cgetg_block(long x, long y)` as `cgetg(x,y)`, creating the return value as a **block**, not on the PARI stack.

`GEN cgetr_block(long prec)` as `cgetr(prec)`, creating the return value as a **block**, not on the PARI stack.

**5.6.3 Implementation note.** The hidden block header is manipulated using the following private functions:

`void* bl_base(GEN x)` returns the pointer that was actually allocated by `malloc` (can be freed).

`long bl_refc(GEN x)` the reference count of  $x$ : the number of pointers to this block. Decrement in `killblock`, incremented by the private function `void gclone_refc(GEN x)`; block is freed when the reference count reaches 0.

`long bl_num(GEN x)` the index of this block in the list of all blocks allocated so far (including freed blocks). Uniquely identifies a block until  $2^{\text{BITS\_IN\_LONG}}$  blocks have been allocated and this wraps around.

GEN `bl_next`(GEN `x`) the block *after* `x` in the linked list of blocks (NULL if `x` is the last block allocated not yet killed).

GEN `bl_prev`(GEN `x`) the block allocated *before* `x` (never NULL).

We documented the last four routines as functions for clarity (and type checking) but they are actually macros yielding valid lvalues. It is allowed to write `bl_refc(x)++` for instance.

## 5.7 Handling user and temp variables.

Low-level implementation of user / temporary variables is liable to change. We describe it nevertheless for completeness. Currently variables are implemented by a single array of values divided in 3 zones: `0-nvar` (user variables), `max_avail-MAXVARN` (temporary variables), and `nvar+1-max_avail-1` (pool of free variable numbers).

### 5.7.1 Low-level.

void `pari_var_init()`: a small part of `pari_init`. Resets variable counters `nvar` and `max_avail`, notwithstanding existing variables! In effect, this even deletes `x`. Don't use it.

void `pari_var_close`(void) attached destructor, called by `pari_close`.

long `pari_var_next()`: returns `nvar`, the number of the next user variable we can create.

long `pari_var_next_temp()` returns `max_avail`, the number of the next temp variable we can create.

long `pari_var_create`(entree \*`ep`) low-level initialization of an EpVAR. Return the attached (new) variable number.

GEN `vars_sort_inplace`(GEN `z`) given a `t_VECSMALL` `z` of variable numbers, sort `z` in place according to variable priorities (highest priority comes first).

GEN `vars_to_RgXV`(GEN `h`) given a `t_VECSMALL` `z` of variable numbers, return the `t_VEC` of `pol_x(z[i])`.

### 5.7.2 User variables.

long `fetch_user_var`(char \*`s`) returns a user variable whose name is `s`, creating it is needed (and using an existing variable otherwise). Returns its variable number.

GEN `fetch_var_value`(long `v`) returns a shallow copy of the current value of the variable numbered `v`. Return NULL for a temporary variable.

entree\* `is_entry`(const char \*`s`) returns the `entree*` attached to an identifier `s` (variable or function), from the interpreter hashtables. Return NULL is the identifier is unknown.

### 5.7.3 Temporary variables.

long `fetch_var`(void) returns the number of a new temporary variable (decreasing `max_avail`).

long `delete_var`(void) delete latest temp variable created and return the number of previous one.

void `name_var`(long `n`, char \*`s`) rename temporary variable number `n` to `s`; mostly useful for nicer printout. Error when trying to rename a user variable.

## 5.8 Adding functions to PARI.

**5.8.1 Nota Bene.** As mentioned in the `COPYING` file, modified versions of the PARI package can be distributed under the conditions of the GNU General Public License. If you do modify PARI, however, it is certainly for a good reason, and we would like to know about it, so that everyone can benefit from your changes. There is then a good chance that your improvements are incorporated into the next release.

We classify changes to PARI into four rough classes, where changes of the first three types are almost certain to be accepted. The first type includes all improvements to the documentation, in a broad sense. This includes correcting typos or inaccuracies of course, but also items which are not really covered in this document, e.g. if you happen to write a tutorial, or pieces of code exemplifying fine points unduly omitted in the present manual.

The second type is to expand or modify the configuration routines and skeleton files (the `Configure` script and anything in the `config/` subdirectory) so that compilation is possible (or easier, or more efficient) on an operating system previously not catered for. This includes discovering and removing idiosyncrasies in the code that would hinder its portability.

The third type is to modify existing (mathematical) code, either to correct bugs, to add new functionality to existing functions, or to improve their efficiency.

Finally the last type is to add new functions to PARI. We explain here how to do this, so that in particular the new function can be called from `gp`.

**5.8.2 Coding guidelines.** Code your function in a file of its own, using as a guide other functions in the PARI sources. One important thing to remember is to clean the stack before exiting your main function, since otherwise successive calls to the function clutters the stack with unnecessary garbage, and stack overflow occurs sooner. Also, if it returns a `GEN` and you want it to be accessible to `gp`, you have to make sure this `GEN` is suitable for `gerepileupto` (see Section 4.3).

If error messages or warnings are to be generated in your function, use `pari_err` and `pari_warn` respectively. Recall that `pari_err` does not return but ends with a `longjmp` statement. As well, instead of explicit `printf` / `fprintf` statements, use the following encapsulated variants:

`void pari_putc(char c):` write character `c` to the output stream.

`void pari_puts(char *s):` write `s` to the output stream.

`void pari_printf(const char *fmt, ...):` write following arguments to the output stream, according to the conversion specifications in format `fmt` (see `printf`).

`void err_printf(const char *fmt, ...):` as `pari_printf`, writing to PARI's current error stream.

`void err_flush(void)` flush error stream.

Declare all public functions in an appropriate header file, if you want to access them from C. The other functions should be declared `static` in your file.

Your function is now ready to be used in library mode after compilation and creation of the library. If possible, compile it as a shared library (see the `Makefile` coming with the `extgcd` example in the distribution). It is however still inaccessible from `gp`.

**5.8.3 GP prototypes, parser codes.** A *GP prototype* is a character string describing all the GP parser needs to know about the function prototype. It contains a sequence of the following atoms:

- Return type: **GEN** by default (must be valid for **gerepileupto**), otherwise the following can appear as the *first* char of the code string:

```

i      return int
l      return long
u      return ulong
v      return void
m      return a GEN which is not gerepile-safe.

```

The **m** code is used for member functions, to avoid unnecessary copies. A copy opcode is generated by the compiler if the result needs to be kept safe for later use.

- Mandatory arguments, appearing in the same order as the input arguments they describe:

```

G      GEN
&      *GEN
L      long (we implicitly typecast int to long)
U      ulong
V      loop variable
n      variable, expects a variable number (a long, not an *entree)
W      a GEN which is a lvalue to be modified in place (for t_LIST)
r      raw input (treated as a string without quotes). Quoted args are copied as strings
        Stops at first unquoted ')' or ', '. Special chars can be quoted using '\ '
        Example: aa"b\n)"c yields the string "aab\n)c"
s      expanded string. Example: Pi"x"2 yields "3.142x2"
        Unquoted components can be of any PARI type, converted to string following
        current output format
I      closure whose value is ignored, as in for loops,
        to be processed by void closure_evalvoid(GEN C)
E      closure whose value is used, as in sum loops,
        to be processed by void closure_evalgen(GEN C)
J      implicit function of arity 1, as in parsum loops,
        to be processed by void closure_callgen1(GEN C)

```

A *closure* is a GP function in compiled (bytecode) form. It can be efficiently evaluated using the **closure\_evalxxx** functions.

- Automatic arguments:

```

f      Fake *long. C function requires a pointer but we do not use the resulting long
b      current real precision in bits
p      current real precision in words
P      series precision (default seriesprecision, global variable precdl for the library)
C      lexical context (internal, for eval, see localvars_read_str)

```

- Syntax requirements, used by functions like **for**, **sum**, etc.:

```

=      separator = required at this point (between two arguments)

```

- Optional arguments and default values:

```

E*     any number of expressions, possibly 0 (see E)
s*     any number of strings, possibly 0 (see s)

```



`Dxxx` argument can be omitted and has a default value

The `E*` code reads all remaining arguments in closure context and passes them as a single `t_VEC`. The `s*` code reads all remaining arguments in *string context* and passes the list of strings as a single `t_VEC`. The automatic concatenation rules in string context are implemented so that adjacent strings are read as different arguments, as if they had been comma-separated. For instance, if the remaining argument sequence is: `"xx" 1, "yy"`, the `s*` atom sends `[a, b, c]`, where *a*, *b*, *c* are GENs of type `t_STR` (content `"xx"`), `t_INT` (equal to 1) and `t_STR` (content `"yy"`).

The format to indicate a default value (atom starts with a D) is `"Dvalue,type,"`, where *type* is the code for any mandatory atom (previous group), *value* is any valid GP expression which is converted according to *type*, and the ending comma is mandatory. For instance `D0,L`, stands for "this optional argument is converted to a `long`, and is 0 by default". So if the user-given argument reads `1 + 3` at this point, `4L` is sent to the function; and `0L` if the argument is omitted. The following special notations are available:

- `DG` optional GEN, send NULL if argument omitted.
- `D&` optional \*GEN, send NULL if argument omitted.  
The argument must be prefixed by `&`.
- `DI, DE` optional closure, send NULL if argument omitted.
- `DP` optional `long`, send `precdl` if argument omitted.
- `DV` optional \*entree, send NULL if argument omitted.
- `Dn` optional variable number, -1 if omitted.
- `Dr` optional raw string, send NULL if argument omitted.
- `Ds` optional `char *`, send NULL if argument omitted.

**Hardcoded limit.** C functions using more than 20 arguments are not supported. Use vectors if you really need that many parameters.

When the function is called under `gp`, the prototype is scanned and each time an atom corresponding to a mandatory argument is met, a user-given argument is read (`gp` outputs an error message if the argument was missing). Each time an optional atom is met, a default value is inserted if the user omits the argument. The "automatic" atoms fill in the argument list transparently, supplying the current value of the corresponding variable (or a dummy pointer).

For instance, here is how you would code the following prototypes, which do not involve default values:

```
GEN f(GEN x, GEN y, long prec)  ----> "GGp"
void f(GEN x, GEN y, long prec)  ----> "vGGp"
void f(GEN x, long y, long prec) ----> "vGLp"
long f(GEN x)                    ----> "lG"
int f(long x)                    ----> "iL"
```

If you want more examples, `gp` gives you easy access to the parser codes attached to all GP functions: just type `\h function`. You can then compare with the C prototypes as they stand in `paridecl.h`.

**Remark.** If you need to implement complicated control statements (probably for some improved summation functions), you need to know how the parser implements closures and lexicals and how the evaluator lets you deal with them, in particular the `push_lex` and `pop_lex` functions. Check their descriptions and adapt the source code in `language/sumiter.c` and `language/intnum.c`.

#### 5.8.4 Integration with `gp` as a shared module.

In this section we assume that your Operating System is supported by `install`. You have written a function in C following the guidelines in Section 5.8.2; in case the function returns a `GEN`, it must satisfy `gerepileupto` assumptions (see Section 4.3).

You then succeeded in building it as part of a shared library and want to finally tell `gp` about your function. First, find a name for it. It does not have to match the one used in library mode, but consistency is nice. It has to be a valid GP identifier, i.e. use only alphabetic characters, digits and the underscore character (`_`), the first character being alphabetic.

Then figure out the correct parser code corresponding to the function prototype (as explained in Section 5.8.3) and write a GP script like the following:

```
install(libname, code, gpname, library)
addhelp(gpname, "some help text")
```

The `addhelp` part is not mandatory, but very useful if you want others to use your module. `libname` is how the function is named in the library, usually the same name as one visible from C.

Read that file from your `gp` session, for instance from your preferences file (or `gprc`), and that's it. You can now use the new function `gpname` under `gp`, and we would very much like to hear about it!

**Example.** A complete description could look like this:

```
{
  install(bnfini0, "GD0,L,DGp", ClassGroupInit, "libpari.so");
  addhelp(ClassGroupInit, "ClassGroupInit(P,{flag=0},{data=[]}):
    compute the necessary data for ...");
}
```

which means we have a function `ClassGroupInit` under `gp`, which calls the library function `bnfini0`. The function has one mandatory argument, and possibly two more (two `'D'` in the code), plus the current real precision. More precisely, the first argument is a `GEN`, the second one is converted to a `long` using `itos` (0 is passed if it is omitted), and the third one is also a `GEN`, but we pass `NULL` if no argument was supplied by the user. This matches the C prototype (from `paridecl.h`):

```
GEN bnfini0(GEN P, long flag, GEN data, long prec)
```

This function is in fact coded in `basemath/buch2.c`, and is in this case completely identical to the GP function `bnfinit` but `gp` does not need to know about this, only that it can be found somewhere in the shared library `libpari.so`.

**Important note.** You see in this example that it is the function's responsibility to correctly interpret its operands: `data = NULL` is interpreted *by the function* as an empty vector. Note that since `NULL` is never a valid `GEN` pointer, this trick always enables you to distinguish between a default value and actual input: the user could explicitly supply an empty vector!

### 5.8.5 Library interface for `install`.

There is a corresponding library interface for this `install` functionality, letting you expand the GP parser/evaluator available in the library with new functions from your C source code. Functions such as `gp_read_str` may then evaluate a GP expression sequence involving calls to these new function!

```
entree * install(void *f, const char *gpname, const char *code)
```

where `f` is the (address of the) function (cast to `void*`), `gpname` is the name by which you want to access your function from within your GP expressions, and `code` is as above.

### 5.8.6 Integration by patching `gp`.

If `install` is not available, and installing Linux or a BSD operating system is not an option (why?), you have to hardcode your function in the `gp` binary. Here is what needs to be done:

- Fetch the complete sources of the PARI distribution.
- Drop the function source code module in an appropriate directory (a priori `src/modules`), and declare all public functions in `src/headers/paridecl.h`.
- Choose a help section and add a file `src/functions/section/gpname` containing the following, keeping the notation above:

```
Function:  gpname
Section:   section
C-Name:    libname
Prototype: code
Help:      some help text
```

(If the help text does not fit on a single line, continuation lines must start by a whitespace character.) Two GP2C-related fields (`Description` and `Wrapper`) are also available to improve the code GP2C generates when compiling scripts involving your function. See the GP2C documentation for details.

- Launch `Configure`, which should pick up your C files and build an appropriate `Makefile`. At this point you can recompile `gp`, which will first rebuild the functions database.

**Example.** We reuse the `ClassGroupInit` / `bnfinit0` from the preceding section. Since the C source code is already part of PARI, we only need to add a file

```
functions/number_fields/ClassGroupInit
```

containing the following:

```
Function: ClassGroupInit
Section: number_fields
C-Name: bnfinit0
Prototype: GD0,L,DGp
Help: ClassGroupInit(P,{flag=0},{tech=[]}): this routine does ...
```

and recompile `gp`.

## 5.9 Globals related to PARI configuration.

### 5.9.1 PARI version numbers.

`paricfg_version_code` encodes in a single `long`, the Major and minor version numbers as well as the patchlevel.

`long PARI_VERSION(long M, long m, long p)` produces the version code attached to release  $M.m.p$ . Each code identifies a unique PARI release, and corresponds to the natural total order on the set of releases (bigger code number means more recent release).

`PARI_VERSION_SHIFT` is the number of bits used to store each of the integers  $M, m, p$  in the version code.

`paricfg_vcsversion` is a version string related to the revision control system used to handle your sources, if any. For instance `git-commit hash` if compiled from a git repository.

The two character strings `paricfg_version` and `paricfg_buildinfo`, correspond to the first two lines printed by `gp` just before the Copyright message. The character string `paricfg_com-piledat` is the date of compilation which appears on the next line. The character string `par-icfg_mt_engine` is the name of the threading engine on the next line.

`GEN pari_version()` returns the version number as a PARI object, a `t_VEC` with three `t_INT` and one `t_STR` components.

### 5.9.2 Miscellaneous.

`paricfg_datadir`: character string. The location of PARI's `datadir`.

`paricfg_gphelp`: character string. The name of an external help command for ?? (such as the `gphelp` script)

## Chapter 6:

### Arithmetic kernel: Level 0 and 1

#### 6.1 Level 0 kernel (operations on ulongs).

**6.1.1 Micro-kernel.** The Level 0 kernel simulates basic operations of the 68020 processor on which PARI was originally implemented. They need “global” `ulong` variables `overflow` (which will contain only 0 or 1) and `hiremainder` to function properly. A routine using one of these lowest-level functions where the description mentions either `hiremainder` or `overflow` must declare the corresponding

```
LOCAL_HIREMAINDER; /* provides 'hiremainder' */
LOCAL_OVERFLOW;    /* provides 'overflow' */
```

in a declaration block. Variables `hiremainder` and `overflow` then become available in the enclosing block. For instance a loop over the powers of an `ulong p` protected from overflows could read

```
while (pk < lim)
{
    LOCAL_HIREMAINDER;
    ...
    pk = mulll(pk, p); if (hiremainder) break;
}
```

For most architectures, the functions mentioned below are really chunks of inlined assembler code, and the above ‘global’ variables are actually local register values.

`ulong addll(ulong x, ulong y)` adds `x` and `y`, returns the lower `BITS_IN_LONG` bits and puts the carry bit into `overflow`.

`ulong addllx(ulong x, ulong y)` adds `overflow` to the sum of the `x` and `y`, returns the lower `BITS_IN_LONG` bits and puts the carry bit into `overflow`.

`ulong subll(ulong x, ulong y)` subtracts `x` and `y`, returns the lower `BITS_IN_LONG` bits and put the carry (borrow) bit into `overflow`.

`ulong subllx(ulong x, ulong y)` subtracts `overflow` from the difference of `x` and `y`, returns the lower `BITS_IN_LONG` bits and puts the carry (borrow) bit into `overflow`.

`int bffo(ulong x)` returns the number of leading zero bits in `x`. That is, the number of bit positions by which it would have to be shifted left until its leftmost bit first becomes equal to 1, which can be between 0 and `BITS_IN_LONG - 1` for nonzero `x`. When `x` is 0, the result is undefined.

`ulong mulll(ulong x, ulong y)` multiplies `x` by `y`, returns the lower `BITS_IN_LONG` bits and stores the high-order `BITS_IN_LONG` bits into `hiremainder`.

`ulong addmul(ulong x, ulong y)` adds `hiremainder` to the product of `x` and `y`, returns the lower `BITS_IN_LONG` bits and stores the high-order `BITS_IN_LONG` bits into `hiremainder`.

`ulong divll(ulong x, ulong y)` returns the quotient of  $(\text{hiremainder} * 2^{\text{BITS\_IN\_LONG}}) + x$  by  $y$  and stores the remainder into `hiremainder`. An error occurs if the quotient cannot be represented by an `ulong`, i.e. if initially  $\text{hiremainder} \geq y$ .

`long hammingl(ulong x)` returns the Hamming weight of  $x$ , i.e. the number of non-zero bits in its binary expansion.

**Obsolete routines.** Those functions are awkward and no longer used; they are only provided for backward compatibility:

`ulong shiffl(ulong x, ulong y)` returns  $x$  shifted left by  $y$  bits, i.e.  $x \ll y$ , where we assume that  $0 \leq y \leq \text{BITS\_IN\_LONG}$ . The global variable `hiremainder` receives the bits that were shifted out, i.e.  $x \gg (\text{BITS\_IN\_LONG} - y)$ .

`ulong shifflr(ulong x, ulong y)` returns  $x$  shifted right by  $y$  bits, i.e.  $x \gg y$ , where we assume that  $0 \leq y \leq \text{BITS\_IN\_LONG}$ . The global variable `hiremainder` receives the bits that were shifted out, i.e.  $x \ll (\text{BITS\_IN\_LONG} - y)$ .

**6.1.2 Modular kernel.** The following routines are not part of the level 0 kernel per se, but implement modular operations on words in terms of the above. They are written so that no overflow may occur. Let  $m \geq 1$  be the modulus; all operands representing classes modulo  $m$  are assumed to belong to  $[0, m - 1]$ . The result may be wrong for a number of reasons otherwise: it may not be reduced, overflow can occur, etc.

`int odd(ulong x)` returns 1 if  $x$  is odd, and 0 otherwise.

`int both_odd(ulong x, ulong y)` returns 1 if  $x$  and  $y$  are both odd, and 0 otherwise.

`ulong invmod2BIL(ulong x)` returns the smallest positive representative of  $x^{-1} \bmod 2^{\text{BITS\_IN\_LONG}}$ , assuming  $x$  is odd.

`ulong Fl_add(ulong x, ulong y, ulong m)` returns the smallest non-negative representative of  $x + y$  modulo  $m$ .

`ulong Fl_neg(ulong x, ulong m)` returns the smallest non-negative representative of  $-x$  modulo  $m$ .

`ulong Fl_sub(ulong x, ulong y, ulong m)` returns the smallest non-negative representative of  $x - y$  modulo  $m$ .

`long Fl_center(ulong x, ulong m, ulong mo2)` returns the representative in  $] -m/2, m/2]$  of  $x$  modulo  $m$ . Assume  $0 \leq x < m$  and  $\text{mo2} = m \gg 1$ .

`ulong Fl_mul(ulong x, ulong y, ulong m)` returns the smallest non-negative representative of  $xy$  modulo  $m$ .

`ulong Fl_double(ulong x, ulong m)` returns  $2x$  modulo  $m$ .

`ulong Fl_triple(ulong x, ulong m)` returns  $3x$  modulo  $m$ .

`ulong Fl_half(ulong x, ulong m)` returns  $z$  such that  $2z = x$  modulo  $m$  assuming such  $z$  exists.

`ulong Fl_sqr(ulong x, ulong m)` returns the smallest non-negative representative of  $x^2$  modulo  $m$ .

`ulong Fl_inv(ulong x, ulong m)` returns the smallest positive representative of  $x^{-1}$  modulo  $m$ . If  $x$  is not invertible mod  $m$ , raise an exception.

`ulong Fl_invsafe(ulong x, ulong m)` returns the smallest positive representative of  $x^{-1}$  modulo  $m$ . If  $x$  is not invertible mod  $m$ , return 0 (which is ambiguous if  $m = 1$ ).

`ulong Fl_invgen(ulong x, ulong m, ulong *pg)` set `*pg` to  $g = \gcd(x, m)$  and return  $u$  in  $(\mathbf{Z}/m\mathbf{Z})^*$  such that  $xu = g$  modulo  $m$ . We have  $g = 1$  if and only if  $x$  is invertible, and in this case  $u$  is its inverse.

`ulong Fl_div(ulong x, ulong y, ulong m)` returns the smallest non-negative representative of  $xy^{-1}$  modulo  $m$ . If  $y$  is not invertible mod  $m$ , raise an exception.

`ulong Fl_powu(ulong x, ulong n, ulong m)` returns the smallest non-negative representative of  $x^n$  modulo  $m$ .

`GEN Fl_powers(ulong x, long n, ulong p)` returns  $[x^0, \dots, x^n]$  modulo  $m$ , as a `t_VECSMALL`.

`ulong Fl_sqrt(ulong x, ulong p)` returns the square root of  $x$  modulo  $p$  (smallest non-negative representative). Assumes  $p$  to be prime, and  $x$  to be a square modulo  $p$ .

`ulong Fl_sqrtl(ulong x, ulong l, ulong p)` returns a  $l$ -th root of  $x$  modulo  $p$ . Assumes  $p$  to be prime and  $p \equiv 1 \pmod{l}$ , and  $x$  to be a  $l$ -th power modulo  $p$ .

`ulong Fl_sqrtn(ulong a, ulong n, ulong p, ulong *zn)` returns `ULONG_MAX` if  $a$  is not an  $n$ -th power residue mod  $p$ . Otherwise, returns an  $n$ -th root of  $a$ ; if `zn` is non-NULL set it to a primitive  $m$ -th root of 1,  $m = \gcd(p-1, n)$  allowing to compute all  $m$  solutions in  $\mathbf{F}_p$  of the equation  $x^n = a$ .

`ulong Fl_log(ulong a, ulong g, ulong ord, ulong p)` Let  $g$  such that  $g^{\text{ord}} \equiv 1 \pmod{p}$ . Return an integer  $e$  such that  $a^e \equiv g \pmod{p}$ . If  $e$  does not exist, the result is undefined.

`ulong Fl_order(ulong a, ulong o, ulong p)` returns the order of the  $\mathbf{F}_p$   $a$ . It is assumed that  $o$  is a multiple of the order of  $a$ , 0 being allowed (no non-trivial information).

`ulong random_Fl(ulong p)` returns a pseudo-random integer uniformly distributed in  $0, 1, \dots, p-1$ .

`ulong pgener_Fl(ulong p)` returns the smallest primitive root modulo  $p$ , assuming  $p$  is prime.

`ulong pgener_Zl(ulong p)` returns the smallest primitive root modulo  $p^k$ ,  $k > 1$ , assuming  $p$  is an odd prime.

`ulong pgener_Fl_local(ulong p, GEN L)`, see `gener_Fp_local`,  $L$  is an `Flv`.

### 6.1.3 Modular kernel with “precomputed inverse”.

This is based on an algorithm by T. Grandlund and N. Möller in “Improved division by invariant integers” <http://gmplib.org/~tege/division-paper.pdf>.

In the following, we set  $B = \text{BITS\_IN\_LONG}$ .

`ulong get_Fl_red(ulong p)` returns a pseudo inverse  $pi$  for  $p$

`ulong divll_pre(ulong x, ulong p, ulong yi)` as `divll`, where  $yi$  is the pseudo inverse of  $y$ .

`ulong remll_pre(ulong u1, ulong u0, ulong p, ulong pi)` returns the Euclidean remainder of  $u_1 2^B + u_0$  modulo  $p$ , assuming  $pi$  is the pseudo inverse of  $p$ . This function is faster if  $u_1 < p$ .

`ulong remlll_pre(ulong u2, ulong u1, ulong u0, ulong p, ulong pi)` returns the Euclidean remainder of  $u_2 2^{2B} + u_1 2^B + u_0$  modulo  $p$ , assuming  $pi$  is the pseudo inverse of  $p$ .

`ulong Fl_sqr_pre(ulong x, ulong p, ulong pi)` returns  $x^2$  modulo  $p$ , assuming  $pi$  is the pseudo inverse of  $p$ .

`ulong Fl_mul_pre(ulong x, ulong y, ulong p, ulong pi)` returns  $xy$  modulo  $p$ , assuming  $pi$  is the pseudo inverse of  $p$ .

`ulong Fl_addmul_pre(ulong a, ulong b, ulong c, ulong p, ulong pi)` returns  $a+bc$  modulo  $p$ , assuming  $pi$  is the pseudo inverse of  $p$ .

`ulong Fl_addmulmul_pre(ulong a, ulong b, ulong c, ulong d, ulong p, ulong pi)` returns  $ab+cd$  modulo  $p$ , assuming  $pi$  is the pseudo inverse of  $p$ .

`ulong Fl_powu_pre(ulong x, ulong n, ulong p, ulong pi)` returns  $x^n$  modulo  $p$ , assuming  $pi$  is the pseudo inverse of  $p$ .

`GEN Fl_powers_pre(ulong x, long n, ulong p, ulong pi)` returns the vector  $(t\_VECSMALL)(x^0, \dots, x^n)$ , assuming  $pi$  is the pseudo inverse of  $p$ .

`ulong Fl_log_pre(ulong a, ulong g, ulong ord, ulong p, ulong pi)` as `Fl_log`, assuming  $pi$  is the pseudo inverse of  $p$ .

`ulong Fl_sqrt_pre(ulong x, ulong p, ulong pi)` returns a square root of  $x$  modulo  $p$ , assuming  $pi$  is the pseudo inverse of  $p$ . See `Fl_sqrt`.

`ulong Fl_sqrtl_pre(ulong x, ulong l, ulong p, ulong pi)` returns a  $l$ -th root of  $x$  modulo  $p$ , assuming  $pi$  is the pseudo inverse of  $p$ ,  $p$  prime and  $p \equiv 1 \pmod{l}$ , and  $x$  to be a  $l$ -th power modulo  $p$ .

`ulong Fl_sqrtn_pre(ulong x, ulong n, ulong p, ulong *zn)` See `Fl_sqrtn`, assuming  $pi$  is the pseudo inverse of  $p$ .

`ulong Fl_2gener_pre(ulong p, ulong pi)` return a generator of the 2-Sylow subgroup of  $\mathbf{F}_p^*$ . To use with `Fl_sqrt_pre-i`.

`ulong Fl_sqrt_pre-i(ulong x, ulong s2, ulong p, ulong pi)` as `Fl_sqrt_pre` where  $s2$  is the element returned by `Fl_2gener_pre`.

#### 6.1.4 Switching between `Fl_xxx` and standard operators.

Even though the `Fl_xxx` routines are efficient, they are slower than ordinary `long` operations, using the standard `+`, `%`, etc. operators. The following macro is used to choose in a portable way the most efficient functions for given operands:

`int SMALL_ULONG(ulong p)` true if  $2p^2 < 2^{\text{BITS\_IN\_LONG}}$ . In that case, it is possible to use ordinary operators efficiently. If  $p < 2^{\text{BITS\_IN\_LONG}}$ , one may still use the `Fl_xxx` routines. Otherwise, one must use generic routines. For instance, the scalar product of the GENs  $x$  and  $y$  mod  $p$  could be computed as follows.

```
long i, l = lg(x);
if (lgefint(p) > 3)
{ /* arbitrary */
  GEN s = gen_0;
  for (i = 1; i < l; i++) s = addii(s, mulii(gel(x,i), gel(y,i)));
  return modii(s, p).
}
else
{
  ulong s = 0, pp = itou(p);
```



```

x = ZV_to_Flv(x, pp);
y = ZV_to_Flv(y, pp);
if (SMALL_ULONG(pp))
{ /* very small */
  for (i = 1; i < l; i++)
  {
    s += x[i] * y[i];
    if (s & HIGHBIT) s %= pp;
  }
  s %= pp;
}
else
{ /* small */
  for (i = 1; i < l; i++)
    s = Fl_add(s, Fl_mul(x[i], y[i], pp), pp);
}
return utoi(s);
}

```

In effect, we have three versions of the same code: very small, small, and arbitrary inputs. The very small and arbitrary variants use lazy reduction and reduce only when it becomes necessary: when overflow might occur (very small), and at the very end (very small, arbitrary).

## 6.2 Level 1 kernel (operations on longs, integers and reals).

**Note.** Some functions consist of an elementary operation, immediately followed by an assignment statement. They will be introduced as in the following example:

`GEN gadd[z](GEN x, GEN y[, GEN z])` followed by the explicit description of the function

`GEN gadd(GEN x, GEN y)`

which creates its result on the stack, returning a `GEN` pointer to it, and the parts in brackets indicate that there exists also a function

`void gaddz(GEN x, GEN y, GEN z)`

which assigns its result to the pre-existing object `z`, leaving the stack unchanged. These assignment variants are kept for backward compatibility but are inefficient: don't use them.

### 6.2.1 Creation.

`GEN cgeti(long n)` allocates memory on the PARI stack for a `t_INT` of length `n`, and initializes its first codeword. Identical to `cgetg(n,t_INT)`.

`GEN cgetipos(long n)` allocates memory on the PARI stack for a `t_INT` of length `n`, and initializes its two codewords. The sign of `n` is set to 1.

`GEN cgetineg(long n)` allocates memory on the PARI stack for a negative `t_INT` of length `n`, and initializes its two codewords. The sign of `n` is set to  $-1$ .

`GEN cgetr(long n)` allocates memory on the PARI stack for a `t_REAL` of length `n`, and initializes its first codeword. Identical to `cgetg(n,t_REAL)`.

GEN `cgetc(long n)` allocates memory on the PARI stack for a `t_COMPLEX`, whose real and imaginary parts are `t_REAL`s of length `n`.

GEN `real_1(long prec)` create a `t_REAL` equal to 1 to `prec` words of accuracy.

GEN `real_1_bit(long bitprec)` create a `t_REAL` equal to 1 to `bitprec` bits of accuracy.

GEN `real_m1(long prec)` create a `t_REAL` equal to  $-1$  to `prec` words of accuracy.

GEN `real_0_bit(long bit)` create a `t_REAL` equal to 0 with exponent  $-\text{bit}$ .

GEN `real_0(long prec)` is a shorthand for

```
real_0_bit( -prec2nbits(prec) )
```

GEN `int2n(long n)` creates a `t_INT` equal to  $1 \ll n$  (i.e  $2^n$  if  $n \geq 0$ , and 0 otherwise).

GEN `int2u(ulong n)` creates a `t_INT` equal to  $2^n$ .

GEN `int2um1(long n)` creates a `t_INT` equal to  $2^n - 1$ .

GEN `real2n(long n, long prec)` create a `t_REAL` equal to  $2^n$  to `prec` words of accuracy.

GEN `real_m2n(long n, long prec)` create a `t_REAL` equal to  $-2^n$  to `prec` words of accuracy.

GEN `strtoi(char *s)` convert the character string `s` to a non-negative `t_INT`. Decimal numbers, hexadecimal numbers prefixed by `0x` and binary numbers prefixed by `0b` are allowed. The string `s` consists exclusively of digits: no leading sign, no whitespace. Leading zeroes are discarded.

GEN `strtord(char *s, long prec)` convert the character string `s` to a non-negative `t_REAL` of precision `prec`. The string `s` consists exclusively of digits and optional decimal point and exponent (`e` or `E`): no leading sign, no whitespace. Leading zeroes are discarded.

**6.2.2 Assignment.** In this section, the `z` argument in the `z`-functions must be of type `t_INT` or `t_REAL`.

`void mpaff(GEN x, GEN z)` assigns `x` into `z` (where `x` and `z` are `t_INT` or `t_REAL`). Assumes that  $\lg(z) > 2$ .

`void affii(GEN x, GEN z)` assigns the `t_INT` `x` into the `t_INT` `z`.

`void affir(GEN x, GEN z)` assigns the `t_INT` `x` into the `t_REAL` `z`. Assumes that  $\lg(z) > 2$ .

`void affiz(GEN x, GEN z)` assigns `t_INT` `x` into `t_INT` or `t_REAL` `z`. Assumes that  $\lg(z) > 2$ .

`void affsi(long s, GEN z)` assigns the `long` `s` into the `t_INT` `z`. Assumes that  $\lg(z) > 2$ .

`void affsr(long s, GEN z)` assigns the `long` `s` into the `t_REAL` `z`. Assumes that  $\lg(z) > 2$ .

`void affsz(long s, GEN z)` assigns the `long` `s` into the `t_INT` or `t_REAL` `z`. Assumes that  $\lg(z) > 2$ .

`void affui(ulong u, GEN z)` assigns the `ulong` `u` into the `t_INT` `z`. Assumes that  $\lg(z) > 2$ .

`void affur(ulong u, GEN z)` assigns the `ulong` `u` into the `t_REAL` `z`. Assumes that  $\lg(z) > 2$ .

`void affrr(GEN x, GEN z)` assigns the `t_REAL` `x` into the `t_REAL` `z`.

`void affgr(GEN x, GEN z)` assigns the scalar `x` into the `t_REAL` `z`, if possible.

The function `affrs` and `affri` do not exist. So don't use them.

`void affrr_fixlg(GEN y, GEN z)` a variant of `affrr`. First shorten  $z$  so that it is no longer than  $y$ , then assigns  $y$  to  $z$ . This is used in the following scenario: room is reserved for the result but, due to cancellation, fewer words of accuracy are available than had been anticipated; instead of appending meaningless 0s to the mantissa, we store what was actually computed.

Note that shortening  $z$  is not quite straightforward, since `setlg(z, ly)` would leave garbage on the stack, which `gerepile` might later inspect. It is done using

`void fixlg(GEN z, long ly)` see `stackdummy` and the examples that follow.

### 6.2.3 Copy.

`GEN icopy(GEN x)` copy relevant words of the `t_INT`  $x$  on the stack: the length and effective length of the copy are equal.

`GEN rcopy(GEN x)` copy the `t_REAL`  $x$  on the stack.

`GEN leafcopy(GEN x)` copy the leaf  $x$  on the stack (works in particular for `t_INTs` and `t_REALs`). Contrary to `icopy`, `leafcopy` preserves the original length of a `t_INT`. The obsolete form `GEN mpcopy(GEN x)` is still provided for backward compatibility.

This function also works on recursive types, copying them as if they were leaves, i.e. making a shallow copy in that case: the components of the copy point to the same data as the component of the source; see also `shallowcopy`.

`GEN leafcopy_avma(GEN x, pari_sp av)` analogous to `gcopy_avma` but simpler: assume  $x$  is a leaf and return a copy allocated as if initially we had `avma` equal to `av`. There is no need to pass a pointer and update the value of the second argument: the new (fictitious) `avma` is just the return value (typecast to `pari_sp`).

`GEN icopyspec(GEN x, long nx)` copy the  $nx$  words  $x[2], \dots, x[nx+1]$  to make up a new `t_INT`. Set the sign to 1.

### 6.2.4 Conversions.

`GEN itor(GEN x, long prec)` converts the `t_INT`  $x$  to a `t_REAL` of length `prec` and return the latter. Assumes that `prec > 2`.

`long itos(GEN x)` converts the `t_INT`  $x$  to a `long` if possible, otherwise raise an exception. We consider the conversion to be possible if and only if  $|x| \leq \text{LONG\_MAX}$ , i.e.  $|x| < 2^{63}$  on a 64-bit architecture. Since the range is symmetric, the output of `itos` can safely be negated.

`long itos_or_0(GEN x)` converts the `t_INT`  $x$  to a `long` if possible, otherwise return 0.

`int is_bigint(GEN n)` true if `itos(n)` would give an error.

`ulong itou(GEN x)` converts the `t_INT`  $|x|$  to an `ulong` if possible, otherwise raise an exception. The conversion is possible if and only if `lgefint(x) ≤ 3`.

`long itou_or_0(GEN x)` converts the `t_INT`  $|x|$  to an `ulong` if possible, otherwise return 0.

`GEN stoi(long s)` creates the `t_INT` corresponding to the `long s`.

`GEN stor(long s, long prec)` converts the `long s` into a `t_REAL` of length `prec` and return the latter. Assumes that `prec > 2`.

`GEN utoi(ulong s)` converts the `ulong s` into a `t_INT` and return the latter.

GEN `utoipos(ulong s)` converts the *non-zero* `ulong s` into a `t_INT` and return the latter.

GEN `utoineg(ulong s)` converts the *non-zero* `ulong s` into the `t_INT`  $-s$  and return the latter.

GEN `utor(ulong s, long prec)` converts the `ulong s` into a `t_REAL` of length `prec` and return the latter. Assumes that `prec > 2`.

GEN `rtor(GEN x, long prec)` converts the `t_REAL x` to a `t_REAL` of length `prec` and return the latter. If `prec < lg(x)`, round properly. If `prec > lg(x)`, pad with zeroes. Assumes that `prec > 2`.

The following function is also available as a special case of `mkintn`:

GEN `uu32toi(ulong a, ulong b)` returns the GEN equal to  $2^{32}a + b$ , *assuming* that  $a, b < 2^{32}$ . This does not depend on `sizeof(long)`: the behavior is as above on both 32 and 64-bit machines.

GEN `uu32toineg(ulong a, ulong b)` returns the GEN equal to  $-(2^{32}a + b)$ , *assuming* that  $a, b < 2^{32}$  and that one of  $a$  or  $b$  is positive. This does not depend on `sizeof(long)`: the behavior is as above on both 32 and 64-bit machines.

GEN `uutoi(ulong a, ulong b)` returns the GEN equal to  $2^{\text{BITS\_IN\_LONG}}a + b$ .

GEN `uutoineg(ulong a, ulong b)` returns the GEN equal to  $-(2^{\text{BITS\_IN\_LONG}}a + b)$ .

**6.2.5 Integer parts.** The following four functions implement the conversion from `t_REAL` to `t_INT` using standard rounding modes. Contrary to usual semantics (complement the mantissa with an infinite number of 0), they will raise an error *precision loss in truncation* if the `t_REAL` represents a range containing more than one integer.

GEN `ceilr(GEN x)` smallest integer larger or equal to the `t_REAL x` (i.e. the `ceil` function).

GEN `floorr(GEN x)` largest integer smaller or equal to the `t_REAL x` (i.e. the `floor` function).

GEN `roundr(GEN x)` rounds the `t_REAL x` to the nearest integer (towards  $+\infty$  in case of tie).

GEN `truncr(GEN x)` truncates the `t_REAL x` (not the same as `floorr` if  $x$  is negative).

The following four function are analogous, but can also treat the trivial case when the argument is a `t_INT`:

GEN `mpceil(GEN x)` as `ceilr` except that  $x$  may be a `t_INT`.

GEN `mpfloor(GEN x)` as `floorr` except that  $x$  may be a `t_INT`.

GEN `mpround(GEN x)` as `roundr` except that  $x$  may be a `t_INT`.

GEN `mptrunc(GEN x)` as `truncr` except that  $x$  may be a `t_INT`.

GEN `diviiround(GEN x, GEN y)` if  $x$  and  $y$  are `t_INT`s, returns the quotient  $x/y$  of  $x$  and  $y$ , rounded to the nearest integer. If  $x/y$  falls exactly halfway between two consecutive integers, then it is rounded towards  $+\infty$  (as for `roundr`).

GEN `ceil_safe(GEN x)`,  $x$  being a real number (not necessarily a `t_REAL`) returns the smallest integer which is larger than any possible incarnation of  $x$ . (Recall that a `t_REAL` represents an interval of possible values.) Note that `gceil` raises an exception if the input accuracy is too low compared to its magnitude.

GEN `floor_safe(GEN x)`,  $x$  being a real number (not necessarily a `t_REAL`) returns the largest integer which is smaller than any possible incarnation of  $x$ . (Recall that a `t_REAL` represents an

interval of possible values.) Note that `gfloor` raises an exception if the input accuracy is too low compared to its magnitude.

`GEN trunc_safe(GEN x)`,  $x$  being a real number (not necessarily a `t_REAL`) returns the integer with the largest absolute value, which is closer to 0 than any possible incarnation of  $x$ . (Recall that a `t_REAL` represents an interval of possible values.)

`GEN roundr_safe(GEN x)` rounds the `t_REAL`  $x$  to the nearest integer (towards  $+\infty$ ). Complement the mantissa with an infinite number of 0 before rounding, hence never raise an exception.

### 6.2.6 2-adic valuations and shifts.

`long vals(long s)` 2-adic valuation of the `long`  $s$ . Returns  $-1$  if  $s$  is equal to 0.

`long vali(GEN x)` 2-adic valuation of the `t_INT`  $x$ . Returns  $-1$  if  $x$  is equal to 0.

`GEN mpshift(GEN x, long n)` shifts the `t_INT` or `t_REAL`  $x$  by  $n$ . If  $n$  is positive, this is a left shift, i.e. multiplication by  $2^n$ . If  $n$  is negative, it is a right shift by  $-n$ , which amounts to the truncation of the quotient of  $x$  by  $2^{-n}$ .

`GEN shifti(GEN x, long n)` shifts the `t_INT`  $x$  by  $n$ .

`GEN shiftr(GEN x, long n)` shifts the `t_REAL`  $x$  by  $n$ .

`void shiftr_inplace(GEN x, long n)` shifts the `t_REAL`  $x$  by  $n$ , in place.

`GEN trunc2nr(GEN x, long n)` given a `t_REAL`  $x$ , returns `truncr(shiftr(x,n))`, but faster, without leaving garbage on the stack and never raising a *precision loss in truncation* error. Called by `gtrunc2n`.

`GEN trunc2nr_lg(GEN x, long lx, long n)` given a `t_REAL`  $x$ , returns `trunc2nr(x,n)`, pretending that the length of  $x$  is  $lx$ , which must be  $\leq \lg(x)$ .

`GEN mantissa2nr(GEN x, long n)` given a `t_REAL`  $x$ , returns the mantissa of  $x2^n$  (disregards the exponent of  $x$ ). Equivalent to

`trunc2nr(x, n-expo(x)+bit_prec(x)-1)`

`GEN mantissa_real(GEN z, long *e)` returns the mantissa  $m$  of  $z$ , and sets `*e` to the exponent `bit_accuracy(lg(z)) - 1 - expo(z)`, so that  $z = m/2^e$ .

**Low-level.** In the following two functions,  $s$ (ource) and  $t$ (arget) need not be valid GENs (in practice, they usually point to some part of a `t_REAL` mantissa): they are considered as arrays of words representing some mantissa, and we shift globally  $s$  by  $n > 0$  bits, storing the result in  $t$ . We assume that  $m \leq M$  and only access  $s[m], s[m+1], \dots, s[M]$  (read) and likewise for  $t$  (write); we may have  $s = t$  but more general overlaps are not allowed. The word  $f$  is concatenated to  $s$  to supply extra bits.

`void shift_left(GEN t, GEN s, long m, long M, ulong f, ulong n)` shifts the mantissa

$s[m], s[m+1], \dots, s[M], f$

left by  $n$  bits.

`void shift_right(GEN t, GEN s, long m, long M, ulong f, ulong n)` shifts the mantissa

$f, s[m], s[m+1], \dots, s[M]$

right by  $n$  bits.

### 6.2.7 From t\_INT to bits or digits in base $2^k$ and back.

GEN `binary_zv`(GEN `x`) given a `t_INT` `x`, return a `t_VECSMALL` of bits, from most significant to least significant.

GEN `binary_2k`(GEN `x`, long `k`) given a `t_INT` `x`, and  $k > 0$ , return a `t_VEC` of digits of `x` in base  $2^k$ , as `t_INT`s, from most significant to least significant.

GEN `binary_2k_nv`(GEN `x`, long `k`) given a `t_INT` `x`, and  $0 < k < \text{BITS\_IN\_LONG}$ , return a `t_VECSMALL` of digits of `x` in base  $2^k$ , as `ulong`s, from most significant to least significant.

GEN `bits_to_int`(GEN `x`, long `l`) given a vector `x` of `l` bits (as a `t_VECSMALL` or even a pointer to a part of a larger vector, so not a proper GEN), return the integer  $\sum_{i=1}^l x[i]2^{l-i}$ , as a `t_INT`.

`ulong bits_to_u`(GEN `v`, long `l`) same as `bits_to_int`, where  $l < \text{BITS\_IN\_LONG}$ , so we can return an `ulong`.

GEN `fromdigitsu`(GEN `x`, GEN `B`) given a `t_VECSMALL` `x` of length `l` and a `t_INT` `B`, return the integer  $\sum_{i=1}^l x[i]B^{l-i}$ , as a `t_INT`, where the `x[i]` are seen as unsigned integers.

GEN `fromdigits_2k`(GEN `x`, long `k`) converse of `binary_2k`; given a `t_VEC` `x` of length `l` and a positive long `k`, where each `x[i]` is a `t_INT` with  $0 \leq x[i] < 2^k$ , return the integer  $\sum_{i=1}^l x[i]2^{k(l-i)}$ , as a `t_INT`.

GEN `nv_fromdigits_2k`(GEN `x`, long `k`) as `fromdigits_2k`, but with `x` being a `t_VECSMALL` and each `x[i]` being a `ulong` with  $0 \leq x[i] < 2^{\min\{k, \text{BITS\_IN\_LONG}\}}$ . Here `k` may be any positive long, and the `x[i]` are regarded as `k`-bit integers by truncating or extending with zeroes.

**6.2.8 Integer valuation.** For integers `x` and `p`, such that  $x \neq 0$  and  $|p| > 1$ , we define  $v_p(x)$  to be the largest integer exponent `e` such that  $p^e$  divides `x`. If `p` is prime, this is the ordinary valuation of `x` at `p`.

long `Z_pvalrem`(GEN `x`, GEN `p`, GEN `*r`) applied to `t_INT`s  $x \neq 0$  and `p`,  $|p| > 1$ , returns  $e := v_p(x)$ . The quotient  $x/p^e$  is returned in `*r`. If `p` is a prime, `*r` is the prime-to-`p` part of `x`.

long `Z_pval`(GEN `x`, GEN `p`) as `Z_pvalrem` but only returns  $v_p(x)$ .

long `Z_lvalrem`(GEN `x`, `ulong p`, GEN `*r`) as `Z_pvalrem`, except that `p` is an `ulong` ( $p > 1$ ).

long `Z_lvalrem_stop`(GEN `*x`, `ulong p`, int `*stop`) assume  $x > 0$ ; returns  $e := v_p(x)$  and replaces `x` by  $x/p^e$ . Set `stop` to 1 if the new value of `x` is  $< p^2$  (and 0 otherwise). To be used when trial dividing `x` by successive primes: the `stop` condition is cheaply tested while testing whether `p` divides `x` (is the quotient less than `p`?), and allows to decide that `n` is prime if no prime  $< p$  divides `n`. Not memory-clean.

long `Z_lval`(GEN `x`, `ulong p`) as `Z_pval`, except that `p` is an `ulong` ( $p > 1$ ).

long `u_lvalrem`(`ulong x`, `ulong p`, `ulong *r`) as `Z_pvalrem`, except the inputs/outputs are now `ulong`s.

long `u_lvalrem_stop`(`ulong *n`, `ulong p`, int `*stop`) as `Z_pvalrem_stop`.

long `u_pvalrem`(`ulong x`, GEN `p`, `ulong *r`) as `Z_pvalrem`, except `x` and `r` are now `ulong`s.

long `u_lval`(`ulong x`, `ulong p`) as `Z_pval`, except the inputs are now `ulong`s.

long `u_pval`(`ulong x`, GEN `p`) as `Z_pval`, except `x` is now an `ulong`.

`long z_lval(long x, ulong p)` as `u_lval`, for signed `x`.  
`long z_lvalrem(long x, ulong p)` as `u_lvalrem`, for signed `x`.  
`long z_pval(long x, GEN p)` as `Z_pval`, except `x` is now a `long`.  
`long z_pvalrem(long x, GEN p)` as `Z_pvalrem`, except `x` is now a `long`.  
`long Q_pval(GEN x, GEN p)` valuation at the `t_INT` `p` of the `t_INT` or `t_FRAC` `x`.  
`long factorial_lval(ulong n, ulong p)` returns  $v_p(n!)$ , assuming  $p$  is prime.

The following convenience functions generalize `Z_pval` and its variants to “containers” (`ZV` and `ZX`):

`long ZV_pvalrem(GEN x, GEN p, GEN *r)` `x` being a `ZV` (a vector of `t_INT`s), return the min  $v$  of the valuations of its components and set `*r` to  $x/p^v$ . Infinite loop if `x` is the zero vector. This function is not stack clean.

`long ZV_pval(GEN x, GEN p)` as `ZV_pvalrem` but only returns the “valuation”.

`int ZV_Z_dvd(GEN x, GEN p)` returns 1 if  $p$  divides all components of `x` and 0 otherwise. Faster than testing `ZV_pval(x,p) >= 1`.

`long ZV_lvalrem(GEN x, ulong p, GEN *px)` as `ZV_pvalrem`, except that `p` is an `ulong` ( $p > 1$ ). This function is not stack-clean.

`long ZV_lval(GEN x, ulong p)` as `ZV_pval`, except that `p` is an `ulong` ( $p > 1$ ).

`long ZX_pvalrem(GEN x, GEN p, GEN *r)` as `ZV_pvalrem`, for a `ZX` `x` (a `t_POL` with `t_INT` coefficients). This function is not stack-clean.

`long ZX_pval(GEN x, GEN p)` as `ZV_pval` for a `ZX` `x`.

`long ZX_lvalrem(GEN x, ulong p, GEN *px)` as `ZV_lvalrem`, a `ZX` `x`. This function is not stack-clean.

`long ZX_lval(GEN x, ulong p)` as `ZX_pval`, except that `p` is an `ulong` ( $p > 1$ ).

**6.2.9 Generic unary operators.** Let “*op*” be a unary operation among

- **neg**: negation ( $-x$ ).
- **abs**: absolute value ( $|x|$ ).
- **sqr**: square ( $x^2$ ).

The names and prototypes of the low-level functions corresponding to *op* are as follows. The result is of the same type as `x`.

`GEN opi(GEN x)` creates the result of *op* applied to the `t_INT` `x`.

`GEN opr(GEN x)` creates the result of *op* applied to the `t_REAL` `x`.

`GEN mpop(GEN x)` creates the result of *op* applied to the `t_INT` or `t_REAL` `x`.

Complete list of available functions:

`GEN absi(GEN x), GEN absr(GEN x), GEN mpabs(GEN x)`

`GEN negi(GEN x), GEN negr(GEN x), GEN mpneg(GEN x)`

GEN sqri(GEN x), GEN sqrr(GEN x), GEN mpsqr(GEN x)

GEN absi\_shallow(GEN x)  $x$  being a `t_INT`, returns a shallow copy of  $|x|$ , in particular returns  $x$  itself when  $x \geq 0$ , and `negi(x)` otherwise.

GEN mpabs\_shallow(GEN x)  $x$  being a `t_INT` or a `t_REAL`, returns a shallow copy of  $|x|$ , in particular returns  $x$  itself when  $x \geq 0$ , and `mpneg(x)` otherwise.

Some miscellaneous routines:

GEN sqrs(long x) returns  $x^2$ .

GEN squu(ulong x) returns  $x^2$ .

#### 6.2.10 Comparison operators.

long minss(long x, long y)

ulong minuu(ulong x, ulong y)

double mindd(double x, double y) returns the min of  $x$  and  $y$ .

long maxss(long x, long y)

ulong maxuu(ulong x, ulong y)

double maxdd(double x, double y) returns the max of  $x$  and  $y$ .

int mpcmp(GEN x, GEN y) compares the `t_INT` or `t_REAL`  $x$  to the `t_INT` or `t_REAL`  $y$ . The result is the sign of  $x - y$ .

int cmpii(GEN x, GEN y) compares the `t_INT`  $x$  to the `t_INT`  $y$ .

int cmpir(GEN x, GEN y) compares the `t_INT`  $x$  to the `t_REAL`  $y$ .

int cmpis(GEN x, long s) compares the `t_INT`  $x$  to the `long`  $s$ .

int cmpiu(GEN x, ulong s) compares the `t_INT`  $x$  to the `ulong`  $s$ .

int cmpsi(long s, GEN x) compares the `long`  $s$  to the `t_INT`  $x$ .

int cmpui(ulong s, GEN x) compares the `ulong`  $s$  to the `t_INT`  $x$ .

int cmpsr(long s, GEN x) compares the `long`  $s$  to the `t_REAL`  $x$ .

int cmpri(GEN x, GEN y) compares the `t_REAL`  $x$  to the `t_INT`  $y$ .

int cmprr(GEN x, GEN y) compares the `t_REAL`  $x$  to the `t_REAL`  $y$ .

int cmprrs(GEN x, long s) compares the `t_REAL`  $x$  to the `long`  $s$ .

int equalii(GEN x, GEN y) compares the `t_INT`s  $x$  and  $y$ . The result is 1 if  $x = y$ , 0 otherwise.

int equalrr(GEN x, GEN y) compares the `t_REAL`s  $x$  and  $y$ . The result is 1 if  $x = y$ , 0 otherwise. Equality is decided according to the following rules: all real zeroes are equal, and different from a non-zero real; two non-zero reals are equal if all their digits coincide up to the length of the shortest of the two, and the remaining words in the mantissa of the longest are all 0.

int equalis(GEN x, long s) compare the `t_INT`  $x$  and the `long`  $s$ . The result is 1 if  $x = y$ , 0 otherwise.

int equalsi(long s, GEN x)



`int equaliu(GEN x, ulong s)` compare the `t_INT` `x` and the `ulong` `s`. The result is 1 if `x = y`, 0 otherwise.

`int equalui(ulong s, GEN x)`

The remaining comparison operators disregard the sign of their operands

`int absequaliu(GEN x, ulong u)` compare the absolute value of the `t_INT` `x` and the `ulong` `s`. The result is 1 if  $|x| = y$ , 0 otherwise. This is marginally more efficient than `equalis` even when `x` is known to be non-negative.

`int absequalui(ulong u, GEN x)`

`int absncmpiu(GEN x, ulong u)` compare the absolute value of the `t_INT` `x` and the `ulong` `u`.

`int absncmpui(ulong u, GEN x)`

`int absncmpii(GEN x, GEN y)` compares the `t_INT`s `x` and `y`. The result is the sign of  $|x| - |y|$ .

`int absequalii(GEN x, GEN y)` compares the `t_INT`s `x` and `y`. The result is 1 if  $|x| = |y|$ , 0 otherwise.

`int absncmprr(GEN x, GEN y)` compares the `t_REAL`s `x` and `y`. The result is the sign of  $|x| - |y|$ .

`int absrnz_equal2n(GEN x)` tests whether a non-zero `t_REAL` `x` is equal to  $\pm 2^e$  for some integer `e`.

`int absrnz_equal1(GEN x)` tests whether a non-zero `t_REAL` `x` is equal to  $\pm 1$ .

**6.2.11 Generic binary operators.** The operators in this section have arguments of C-type `GEN`, `long`, and `ulong`, and only `t_INT` and `t_REAL` `GEN`s are allowed. We say an argument is a real type if it is a `t_REAL` `GEN`, and an integer type otherwise. The result is always a `t_REAL` unless both `x` and `y` are integer types.

Let “*op*” be a binary operation among

- **add**: addition (`x + y`).
- **sub**: subtraction (`x - y`).
- **mul**: multiplication (`x * y`).
- **div**: division (`x / y`). In the case where `x` and `y` are both integer types, the result is the Euclidean quotient, where the remainder has the same sign as the dividend `x`. It is the ordinary division otherwise. A division-by-0 error occurs if `y` is equal to 0.

The last two generic operations are defined only when arguments have integer types; and the result is a `t_INT`:

- **rem**: remainder (“`x % y`”). The result is the Euclidean remainder corresponding to **div**, i.e. its sign is that of the dividend `x`.
- **mod**: true remainder (`x % y`). The result is the true Euclidean remainder, i.e. non-negative and less than the absolute value of `y`.

**Important technical note.** The rules given above fixing the output type (to `t_REAL` unless both inputs are integer types) are subtly incompatible with the general rules obeyed by PARI's generic functions, such as `gmul` or `gdiv` for instance: the latter return a result containing as much information as could be deduced from the inputs, so it is not true that if  $x$  is a `t_INT` and  $y$  a `t_REAL`, then `gmul(x,y)` is always the same as `mulir(x,y)`. The exception is  $x = 0$ , in that case we can deduce that the result is an exact 0, so `gmul` returns `gen_0`, while `mulir` returns a `t_REAL` 0. Specifically, the one resulting from the conversion of `gen_0` to a `t_REAL` of precision `precision(y)`, multiplied by  $y$ ; this determines the exponent of the real 0 we obtain.

The reason for the discrepancy between the two rules is that we use the two sets of functions in different contexts: generic functions allow to write high-level code forgetting about types, letting PARI return results which are sensible and as simple as possible; type specific functions are used in kernel programming, where we do care about types and need to maintain strict consistency: it is much easier to compute the types of results when they are determined from the types of the inputs only (without taking into account further arithmetic properties, like being non-0).

The names and prototypes of the low-level functions corresponding to *op* are as follows. In this section, the *z* argument in the *z*-functions must be of type `t_INT` when no *r* or *mp* appears in the argument code (no `t_REAL` operand is involved, only integer types), and of type `t_REAL` otherwise.

`GEN mpop[z](GEN x, GEN y[, GEN z])` applies *op* to the `t_INT` or `t_REAL` *x* and *y*. The function `mpdivz` does not exist (its semantic would change drastically depending on the type of the *z* argument), and neither do `mprem[z]` nor `mpmod[z]` (specific to integers).

`GEN opsi[z](long s, GEN x[, GEN z])` applies *op* to the `long` *s* and the `t_INT` *x*. These functions always return the global constant `gen_0` (not a copy) when the sign of the result is 0.

`GEN opsr[z](long s, GEN x[, GEN z])` applies *op* to the `long` *s* and the `t_REAL` *x*.

`GEN opss[z](long s, long t[, GEN z])` applies *op* to the `long`s *s* and *t*. These functions always return the global constant `gen_0` (not a copy) when the sign of the result is 0.

`GEN opii[z](GEN x, GEN y[, GEN z])` applies *op* to the `t_INT`s *x* and *y*. These functions always return the global constant `gen_0` (not a copy) when the sign of the result is 0.

`GEN opir[z](GEN x, GEN y[, GEN z])` applies *op* to the `t_INT` *x* and the `t_REAL` *y*.

`GEN opis[z](GEN x, long s[, GEN z])` applies *op* to the `t_INT` *x* and the `long` *s*. These functions always return the global constant `gen_0` (not a copy) when the sign of the result is 0.

`GEN opri[z](GEN x, GEN y[, GEN z])` applies *op* to the `t_REAL` *x* and the `t_INT` *y*.

`GEN oprr[z](GEN x, GEN y[, GEN z])` applies *op* to the `t_REAL`s *x* and *y*.

`GEN oprs[z](GEN x, long s[, GEN z])` applies *op* to the `t_REAL` *x* and the `long` *s*.

Some miscellaneous routines:

`long expu(ulong x)` assuming  $x > 0$ , returns the binary exponent of the real number equal to  $x$ . This is a special case of `gexpo`.

`GEN adduu(ulong x, ulong y)`

`GEN addiu(GEN x, ulong y)`

`GEN addui(ulong x, GEN y)` adds *x* and *y*.

`GEN subuu(ulong x, ulong y)`

`GEN subiu(GEN x, ulong y)`  
`GEN subui(ulong x, GEN y)` subtracts  $x$  by  $y$ .  
`GEN muluu(ulong x, ulong y)` multiplies  $x$  by  $y$ .  
`ulong umuluu_le(ulong x, ulong y, ulong n)` multiplies  $x$  by  $y$ . Return  $xy$  if  $xy \leq n$  and 0 otherwise (in particular if  $xy$  does not fit in an `ulong`).  
`ulong umuluu_or_0(ulong x, ulong y)` multiplies  $x$  by  $y$ . Return 0 if  $xy$  does not fit in an `ulong`.  
`GEN mului(ulong x, GEN y)` multiplies  $x$  by  $y$ .  
`GEN muluui(ulong x, ulong y, GEN z)` return  $xyz$ .  
`GEN muliu(GEN x, ulong y)` multiplies  $x$  by  $y$ .  
`void addumului(ulong a, ulong b, GEN x)` return  $a + b|X|$ .  
`GEN addmuliu(GEN x, GEN y, ulong u)` returns  $x + yu$ .  
`GEN addmulii(GEN x, GEN y, GEN z)` returns  $x + yz$ .  
`GEN addmulii_inplace(GEN x, GEN y, GEN z)` returns  $x + yz$ , but returns  $x$  itself and not a copy if  $yz = 0$ . Not suitable for `gerepile` or `gerepileupto`.  
`GEN addmuliu_inplace(GEN x, GEN y, ulong u)` returns  $x + yu$ , but returns  $x$  itself and not a copy if  $yu = 0$ . Not suitable for `gerepile` or `gerepileupto`.  
`GEN submuliu_inplace(GEN x, GEN y, ulong u)` returns  $x - yu$ , but returns  $x$  itself and not a copy if  $yu = 0$ . Not suitable for `gerepile` or `gerepileupto`.  
`GEN lincombii(GEN u, GEN v, GEN x, GEN y)` returns  $ux + vy$ .  
`GEN mulsubii(GEN y, GEN z, GEN x)` returns  $yz - x$ .  
`GEN submulii(GEN x, GEN y, GEN z)` returns  $x - yz$ .  
`GEN submuliu(GEN x, GEN y, ulong u)` returns  $x - yu$ .  
`GEN mulu_interval(ulong a, ulong b)` returns  $a(a+1) \cdots b$ , assuming that  $a \leq b$ .  
`GEN muls_interval(long a, long b)` returns  $a(a+1) \cdots b$ , assuming that  $a \leq b$ .  
`GEN invr(GEN x)` returns the inverse of the non-zero `t_REAL`  $x$ .  
`GEN truedivii(GEN x, GEN y)` returns the true Euclidean quotient (with non-negative remainder less than  $|y|$ ).  
`GEN truedivis(GEN x, long y)` returns the true Euclidean quotient (with non-negative remainder less than  $|y|$ ).  
`GEN truedivsi(long x, GEN y)` returns the true Euclidean quotient (with non-negative remainder less than  $|y|$ ).  
`GEN centermodii(GEN x, GEN y, GEN y2)`, given `t_INTs`  $x, y$ , returns  $z$  congruent to  $x$  modulo  $y$ , such that  $-y/2 \leq z < y/2$ . The function requires an extra argument  $y2$ , such that  $y2 = \text{shifti}(y, -1)$ . (In most cases,  $y$  is constant for many reductions and  $y2$  need only be computed once.)  
`GEN remi2n(GEN x, long n)` returns  $x \bmod 2^n$ .

GEN addii\_sign(GEN x, long sx, GEN y, long sy) add the  $t\_INT$ s  $x$  and  $y$  as if their signs were  $sx$  and  $sy$ .

GEN addir\_sign(GEN x, long sx, GEN y, long sy) add the  $t\_INT$   $x$  and the  $t\_REAL$   $y$  as if their signs were  $sx$  and  $sy$ .

GEN addrr\_sign(GEN x, long sx, GEN y, long sy) add the  $t\_REAL$ s  $x$  and  $y$  as if their signs were  $sx$  and  $sy$ .

GEN addsi\_sign(long x, GEN y, long sy) add  $x$  and the  $t\_INT$   $y$  as if its sign was  $sy$ .

GEN addui\_sign(ulong x, GEN y, long sy) add  $x$  and the  $t\_INT$   $y$  as if its sign was  $sy$ .

### 6.2.12 Exact division and divisibility.

GEN diviexact(GEN x, GEN y) returns the Euclidean quotient  $x/y$ , assuming  $y$  divides  $x$ . Uses Jebelean algorithm (Jebelean-Krandick bidirectional exact division is not implemented).

GEN diviuexact(GEN x, ulong y) returns the Euclidean quotient  $x/y$ , assuming  $y$  divides  $x$  and  $y$  is non-zero.

GEN diviuuexact(GEN x, ulong y, ulong z) returns the Euclidean quotient  $x/(yz)$ , assuming  $yz$  divides  $x$  and  $yz \neq 0$ .

The following routines return 1 (true) if  $y$  divides  $x$ , and 0 otherwise. (Error if  $y$  is 0, even if  $x$  is 0.) All GEN are assumed to be  $t\_INT$ s:

```
int dvdi(GEN x, GEN y), int dvdis(GEN x, long y), int dvdiu(GEN x, ulong y),
int dvdsi(long x, GEN y), int dvdui(ulong x, GEN y).
```

The following routines return 1 (true) if  $y$  divides  $x$ , and in that case assign the quotient to  $z$ ; otherwise they return 0. All GEN are assumed to be  $t\_INT$ s:

```
int dvdiiz(GEN x, GEN y, GEN z), int dvdisz(GEN x, long y, GEN z).
int dvdiuz(GEN x, ulong y, GEN z) if  $y$  divides  $x$ , assigns the quotient  $|x|/y$  to  $z$  and returns
1 (true), otherwise returns 0 (false).
```

### 6.2.13 Division with integral operands and $t\_REAL$ result.

GEN rdivii(GEN x, GEN y, long prec), assuming  $x$  and  $y$  are both of type  $t\_INT$ , return the quotient  $x/y$  as a  $t\_REAL$  of precision  $prec$ .

GEN rdiviiz(GEN x, GEN y, GEN z), assuming  $x$  and  $y$  are both of type  $t\_INT$ , and  $z$  is a  $t\_REAL$ , assign the quotient  $x/y$  to  $z$ .

GEN rdivis(GEN x, long y, long prec), assuming  $x$  is of type  $t\_INT$ , return the quotient  $x/y$  as a  $t\_REAL$  of precision  $prec$ .

GEN rdivsi(long x, GEN y, long prec), assuming  $y$  is of type  $t\_INT$ , return the quotient  $x/y$  as a  $t\_REAL$  of precision  $prec$ .

GEN rdivss(long x, long y, long prec), return the quotient  $x/y$  as a  $t\_REAL$  of precision  $prec$ .

**6.2.14 Division with remainder.** The following functions return two objects, unless specifically asked for only one of them — a quotient and a remainder. The quotient is returned and the remainder is returned through the variable whose address is passed as the `r` argument. The term *true Euclidean remainder* refers to the non-negative one (`mod`), and *Euclidean remainder* by itself to the one with the same sign as the dividend (`rem`). All GENs, whether returned directly or through a pointer, are created on the stack.

GEN `dvmdii`(GEN `x`, GEN `y`, GEN `*r`) returns the Euclidean quotient of the `t_INT` `x` by a `t_INT` `y` and puts the remainder into `*r`. If `r` is equal to `NULL`, the remainder is not created, and if `r` is equal to `ONLY_REM`, only the remainder is created and returned. In the generic case, the remainder is created after the quotient and can be disposed of individually with a `cgiv(r)`. The remainder is always of the sign of the dividend `x`. If the remainder is 0 set `r = gen_0`.

void `dvmdiiz`(GEN `x`, GEN `y`, GEN `z`, GEN `t`) assigns the Euclidean quotient of the `t_INT`s `x` and `y` into the `t_INT` `z`, and the Euclidean remainder into the `t_INT` `t`.

Analogous routines `dvmdis[z]`, `dvmdsi[z]`, `dvmdss[z]` are available, where `s` denotes a long argument. But the following routines are in general more flexible:

long `sdivss_rem`(long `s`, long `t`, long `*r`) computes the Euclidean quotient and remainder of the longs `s` and `t`. Puts the remainder into `*r`, and returns the quotient. The remainder is of the sign of the dividend `s`, and has strictly smaller absolute value than `t`.

long `sdivsi_rem`(long `s`, GEN `x`, long `*r`) computes the Euclidean quotient and remainder of the long `s` by the `t_INT` `x`. As `sdivss_rem` otherwise.

long `sdivsi`(long `s`, GEN `x`) as `sdivsi_rem`, without remainder.

GEN `divis_rem`(GEN `x`, long `s`, long `*r`) computes the Euclidean quotient and remainder of the `t_INT` `x` by the long `s`. As `sdivss_rem` otherwise.

GEN `absdiviu_rem`(GEN `x`, ulong `s`, ulong `*r`) computes the Euclidean quotient and remainder of *absolute value* of the `t_INT` `x` by the ulong `s`. As `sdivss_rem` otherwise.

ulong `uabsdiviu_rem`(GEN `n`, ulong `d`, ulong `*r`) as `absdiviu_rem`, assuming that  $|n|/d$  fits into an ulong.

ulong `uabsdivui_rem`(ulong `x`, GEN `y`, ulong `*rem`) computes the Euclidean quotient and remainder of `x` by  $|y|$ . As `sdivss_rem` otherwise.

ulong `udivuu_rem`(ulong `x`, ulong `y`, ulong `*rem`) computes the Euclidean quotient and remainder of `x` by `y`. As `sdivss_rem` otherwise.

ulong `ceildivuu`(ulong `x`, ulong `y`) return the ceiling of  $x/y$ .

GEN `divsi_rem`(long `s`, GEN `y`, long `*r`) computes the Euclidean quotient and remainder of the long `s` by the GEN `y`. As `sdivss_rem` otherwise.

GEN `divss_rem`(long `x`, long `y`, long `*r`) computes the Euclidean quotient and remainder of the long `x` by the long `y`. As `sdivss_rem` otherwise.

GEN `truedvmdii`(GEN `x`, GEN `y`, GEN `*r`), as `dvmdii` but with a non-negative remainder.

GEN `truedvmdis`(GEN `x`, long `y`, GEN `*z`), as `dvmdis` but with a non-negative remainder.

GEN `truedvmdsi`(long `x`, GEN `y`, GEN `*z`), as `dvmdsi` but with a non-negative remainder.

**6.2.15 Modulo to longs.** The following variants of `modii` do not clutter the stack:

`long smodis(GEN x, long y)` computes the true Euclidean remainder of the `t_INT` `x` by the `long` `y`. This is the non-negative remainder, not the one whose sign is the sign of `x` as in the `div` functions.

`long smodss(long x, long y)` computes the true Euclidean remainder of the `long` `x` by a `long` `y`.

`ulong umodsu(long x, ulong y)` computes the true Euclidean remainder of the `long` `x` by a `ulong` `y`.

`ulong umodiu(GEN x, ulong y)` computes the true Euclidean remainder of the `t_INT` `x` by the `ulong` `y`.

`ulong umodui(ulong x, GEN y)` computes the true Euclidean remainder of the `ulong` `x` by the `t_INT` `|y|`.

The routine `smodsi` does not exist, since it would not always be defined: for a *negative* `x`, if the quotient is  $\pm 1$ , the result `x + |y|` would in general not fit into a `long`. Use either `umodui` or `modsi`.

These functions directly access the binary data and are thus much faster than the generic modulo functions:

`int mpodd(GEN x)` which is 1 if `x` is odd, and 0 otherwise.

`ulong Mod2(GEN x)`

`ulong Mod4(GEN x)`

`ulong Mod8(GEN x)`

`ulong Mod16(GEN x)`

`ulong Mod32(GEN x)`

`ulong Mod64(GEN x)` give the residue class of `x` modulo the corresponding power of 2.

`ulong umodi2n(GEN x, long n)` give the residue class of `x` modulo  $2^n$ ,  $0 \leq n < BITS\_IN\_LONG$ .

The following functions assume that  $x \neq 0$  and in fact disregard the sign of `x`. There are about 10% faster than the safer variants above:

`long mod2(GEN x)`

`long mod4(GEN x)`

`long mod8(GEN x)`

`long mod16(GEN x)`

`long mod32(GEN x)`

`long mod64(GEN x)` give the residue class of  $|x|$  modulo the corresponding power of 2, for *non-zero* `x`. As well,

`ulong mod2BIL(GEN x)` returns the least significant word of  $|x|$ , still assuming that  $x \neq 0$ .

### 6.2.16 Powering, Square root.

GEN `powii`(GEN `x`, GEN `n`), assumes  $x$  and  $n$  are `t_INT`s and returns  $x^n$ .

GEN `powuu`(ulong `x`, ulong `n`), returns  $x^n$ .

GEN `powiu`(GEN `x`, ulong `n`), assumes  $x$  is a `t_INT` and returns  $x^n$ .

GEN `powis`(GEN `x`, long `n`), assumes  $x$  is a `t_INT` and returns  $x^n$  (possibly a `t_FRAC` if  $n < 0$ ).

GEN `pows`(GEN `x`, long `n`), assumes  $x$  is a `t_REAL` and returns  $x^n$ . This is considered as a sequence of `mulrr`, possibly empty: as such the result has type `t_REAL`, even if  $n = 0$ . Note that the generic function `gpows(x,0)` would return `gen_1`, see the technical note in Section 6.2.11.

GEN `powru`(GEN `x`, ulong `n`), assumes  $x$  is a `t_REAL` and returns  $x^n$  (always a `t_REAL`, even if  $n = 0$ ).

GEN `powersr`(GEN `e`, long `n`). Given a `t_REAL`  $e$ , return the vector  $v$  of all  $e^i$ ,  $0 \leq i \leq n$ , where  $v[i] = e^{i-1}$ .

GEN `powrshalf`(GEN `x`, long `n`), assumes  $x$  is a `t_REAL` and returns  $x^{n/2}$  (always a `t_REAL`, even if  $n = 0$ ).

GEN `powruhalf`(GEN `x`, ulong `n`), assumes  $x$  is a `t_REAL` and returns  $x^{n/2}$  (always a `t_REAL`, even if  $n = 0$ ).

GEN `powfrac`(GEN `x`, long `n`, long `d`), assumes  $x$  is a `t_REAL` and returns  $x^{n/d}$  (always a `t_REAL`, even if  $n = 0$ ).

GEN `powIs`(long `n`) returns  $I^n \in \{1, I, -1, -I\}$  (`t_INT` for even  $n$ , `t_COMPLEX` otherwise).

ulong `upowuu`(ulong `x`, ulong `n`), returns  $x^n$  when  $< 2^{\text{BITS\_IN\_LONG}}$ , and 0 otherwise (overflow).

GEN `sqrtemi`(GEN `N`, GEN `*r`), returns the integer square root  $S$  of the non-negative `t_INT`  $N$  (rounded towards 0) and puts the remainder  $R$  into `*r`. Precisely,  $N = S^2 + R$  with  $0 \leq R < 2S$ . If `r` is equal to `NULL`, the remainder is not created. In the generic case, the remainder is created after the quotient and can be disposed of individually with `cgiv(R)`. If the remainder is 0 set `R = gen_0`.

Uses a divide and conquer algorithm (discrete variant of Newton iteration) due to Paul Zimmermann ("Karatsuba Square Root", INRIA Research Report 3805 (1999)).

GEN `sqrte`(GEN `N`), returns the integer square root  $S$  of the non-negative `t_INT`  $N$  (rounded towards 0). This is identical to `sqrtemi(N, NULL)`.

long `logintall`(GEN `B`, GEN `y`, GEN `*ptq`) returns the floor  $e$  of  $\log_y B$ , where  $B > 0$  and  $y > 1$  are integers. If `ptq` is not `NULL`, set it to  $y^e$ . (Analogous to `logint0`, without sanity checks.)

ulong `ulongintall`(ulong `B`, ulong `y`, ulong `*ptq`) as `logintall` for `ulong` arguments.

long `logint`(GEN `B`, GEN `y`) returns the floor  $e$  of  $\log_y B$ , where  $B > 0$  and  $y > 1$  are integers.

ulong `ulongint`(ulong `B`, ulong `y`) as `logint` for `ulong` arguments.

GEN `vecpowuu`(long `N`, ulong `a`) return the vector of  $n^a$ ,  $n = 1, \dots, N$ . Not memory clean.

GEN `vecpowug`(long `N`, GEN `a`, long `prec`) return the vector of  $n^a$ ,  $n = 1, \dots, N$ , where the powers are computed at precision `prec`. Not memory clean.

### 6.2.17 GCD, extended GCD and LCM.

`long cgcd(long x, long y)` returns the GCD of `x` and `y`.

`ulong ugcd(ulong x, ulong y)` returns the GCD of `x` and `y`.

`ulong ugcdiu(GEN x, ulong y)` returns the GCD of `x` and `y`.

`ulong ugcdui(ulong x, GEN y)` returns the GCD of `x` and `y`.

`GEN coprimes_zv(ulong N)` return a `t_VECSMALL` `T` with `N` entries such that  $T[i] = 1$  iff  $(i, N) = 1$  and 0 otherwise.

`long clcm(long x, long y)` returns the LCM of `x` and `y`, provided it fits into a `long`. Silently overflows otherwise.

`ulong ulcm(ulong x, ulong y)` returns the LCM of `x` and `y`, provided it fits into an `ulong`. Silently overflows otherwise.

`GEN gcdii(GEN x, GEN y)`, returns the GCD of the `t_INTs` `x` and `y`.

`GEN lcmii(GEN x, GEN y)`, returns the LCM of the `t_INTs` `x` and `y`.

`GEN bezout(GEN a, GEN b, GEN *u, GEN *v)`, returns the GCD  $d$  of `t_INTs` `a` and `b` and sets `u`, `v` to the Bezout coefficients such that  $au + bv = d$ .

`long cbezout(long a, long b, long *u, long *v)`, returns the GCD  $d$  of `a` and `b` and sets `u`, `v` to the Bezout coefficients such that  $au + bv = d$ .

`GEN ZV_extgcd(GEN A)` given a vector of  $n$  integers `A`, returns  $[d, U]$ , where  $d$  is the GCD of the  $A[i]$  and  $U$  is a matrix in  $\text{GL}_n(\mathbf{Z})$  such that  $AU = [0, \dots, 0, D]$ .

### 6.2.18 Continued fractions and convergents.

`GEN ZV_allpnqn(GEN x)` given  $x = [a_0, \dots, a_n]$  a continued fraction from `gboundcf`,  $n \geq 0$ , return all convergents as  $[P, Q]$ , where  $P = [p_0, \dots, p_n]$  and  $Q = [q_0, \dots, q_n]$ .

**6.2.19 Pseudo-random integers.** These routine return pseudo-random integers uniformly distributed in some interval. They all use the same underlying generator which can be seeded and restarted using `getrand` and `setrand`.

`void setrand(GEN seed)` reseeds the random number generator using the seed  $n$ . The seed is either a technical array output by `getrand` or a small positive integer, used to generate deterministically a suitable state array. For instance, running a randomized computation starting by `setrand(1)` twice will generate the exact same output.

`GEN getrand(void)` returns the current value of the seed used by the pseudo-random number generator `random`. Useful mainly for debugging purposes, to reproduce a specific chain of computations. The returned value is technical (reproduces an internal state array of type `t_VECSMALL`), and can only be used as an argument to `setrand`.

`ulong pari_rand(void)` returns a random  $0 \leq x < 2^{\text{BITS\_IN\_LONG}}$ .

`long random_bits(long k)` returns a random  $0 \leq x < 2^k$ . Assumes that  $0 \leq k \leq \text{BITS\_IN\_LONG}$ .

`ulong random_Fl(ulong p)` returns a pseudo-random integer in  $0, 1, \dots, p-1$ .

`GEN randomi(GEN n)` returns a random `t_INT` between 0 and `n` - 1.

`GEN randomr(long prec)` returns a random `t_REAL` in  $[0, 1[$ , with precision `prec`.



**6.2.20 Modular operations.** In this subsection, all GENs are `t_INT`.

`GEN Fp_red(GEN a, GEN m)` returns  $a$  modulo  $m$  (smallest non-negative residue). (This is identical to `modii`).

`GEN Fp_neg(GEN a, GEN m)` returns  $-a$  modulo  $m$  (smallest non-negative residue).

`GEN Fp_add(GEN a, GEN b, GEN m)` returns the sum of  $a$  and  $b$  modulo  $m$  (smallest non-negative residue).

`GEN Fp_sub(GEN a, GEN b, GEN m)` returns the difference of  $a$  and  $b$  modulo  $m$  (smallest non-negative residue).

`GEN Fp_center(GEN a, GEN p, GEN pov2)` assuming that `pov2` is `shifti(p,-1)` and that  $-p/2 < a < p$ , returns the representative of  $a$  in the symmetric residue system  $] -p/2, p/2]$ .

`GEN Fp_center_i(GEN a, GEN p, GEN pov2)` internal variant of `Fp_center`, not `gerepile-safe`: when  $a$  is already in the proper interval, it is returned as is, without a copy.

`GEN Fp_mul(GEN a, GEN b, GEN m)` returns the product of  $a$  by  $b$  modulo  $m$  (smallest non-negative residue).

`GEN Fp_addmul(GEN x, GEN y, GEN z, GEN p)` returns  $x + yz$ .

`GEN Fp_mulu(GEN a, ulong b, GEN m)` returns the product of  $a$  by  $b$  modulo  $m$  (smallest non-negative residue).

`GEN Fp_muls(GEN a, long b, GEN m)` returns the product of  $a$  by  $b$  modulo  $m$  (smallest non-negative residue).

`GEN Fp_half(GEN x, GEN m)` returns  $z$  such that  $2z = x$  modulo  $m$  assuming such  $z$  exists.

`GEN Fp_sqr(GEN a, GEN m)` returns  $a^2$  modulo  $m$  (smallest non-negative residue).

`ulong Fp_powu(GEN x, ulong n, GEN m)` raises  $x$  to the  $n$ -th power modulo  $m$  (smallest non-negative residue). Not memory-clean, but suitable for `gerepileupto`.

`ulong Fp_pows(GEN x, long n, GEN m)` raises  $x$  to the  $n$ -th power modulo  $m$  (smallest non-negative residue). A negative  $n$  is allowed. Not memory-clean, but suitable for `gerepileupto`.

`GEN Fp_pow(GEN x, GEN n, GEN m)` returns  $x^n$  modulo  $m$  (smallest non-negative residue).

`GEN Fp_pow_init(GEN x, GEN n, long k, GEN p)` Return a table  $R$  that can be used with `Fp_pow_table` to compute the powers of  $x$  up to  $n$ . The table is of size  $2^k \log_2(n)$ .

`GEN Fp_pow_table(GEN R, GEN n, GEN p)` return  $x^n$ , where  $R$  is given by `Fp_pow_init(x,m,k,p)` for some integer  $m \geq n$ .

`GEN Fp_powers(GEN x, long n, GEN m)` returns  $[x^0, \dots, x^n]$  modulo  $m$  as a `t_VEC` (smallest non-negative residue).

`GEN Fp_inv(GEN a, GEN m)` returns an inverse of  $a$  modulo  $m$  (smallest non-negative residue). Raise an error if  $a$  is not invertible.

`GEN Fp_invsafe(GEN a, GEN m)` as `Fp_inv`, but return `NULL` if  $a$  is not invertible.

`GEN Fp_invgen(GEN x, GEN m, GEN *pg)` set `*pg` to  $g = \gcd(x, m)$  and return  $u$  in  $(\mathbf{Z}/m\mathbf{Z})^*$  such that  $xu = g$  modulo  $m$ . We have  $g = 1$  if and only if  $x$  is invertible, and in this case  $u$  is its inverse.

`GEN FpV_inv(GEN x, GEN m)`  $x$  being a vector of `t_INTs`, return the vector of inverses of the  $x[i]$  mod  $m$ . The routine uses Montgomery's trick, and involves a single inversion mod  $m$ , plus  $3(N-1)$

multiplications for  $N$  entries. The routine is not stack-clean:  $2N$  integers mod  $m$  are left on stack, besides the  $N$  in the result.

**GEN Fp\_div**(GEN  $a$ , GEN  $b$ , GEN  $m$ ) returns the quotient of  $a$  by  $b$  modulo  $m$  (smallest non-negative residue). Raise an error if  $b$  is not invertible.

**int invmod**(GEN  $a$ , GEN  $m$ , GEN  $*g$ ), return 1 if  $a$  modulo  $m$  is invertible, else return 0 and set  $g = \gcd(a, m)$ .

In the following three functions the integer parameter **ord** can be given either as a positive **t\_INT**  $N$ , or as its factorization matrix  $faN$ , or as a pair  $[N, faN]$ . The parameter may be omitted by setting it to **NULL** (the value is then  $p - 1$ ).

**GEN Fp\_log**(GEN  $a$ , GEN  $g$ , GEN **ord**, GEN  $p$ ) Let  $g$  such that  $g^{ord} \equiv 1 \pmod{p}$ . Return an integer  $e$  such that  $a^e \equiv g \pmod{p}$ . If  $e$  does not exist, the result is undefined.

**GEN Fp\_order**(GEN  $a$ , GEN **ord**, GEN  $p$ ) returns the order of the **Fp**  $a$ . Assume that **ord** is a multiple of the order of  $a$ .

**GEN Fp\_factored\_order**(GEN  $a$ , GEN **ord**, GEN  $p$ ) returns  $[o, F]$ , where  $o$  is the multiplicative order of the **Fp**  $a$  in  $\mathbf{F}_p^*$ , and  $F$  is the factorization of  $o$ . Assume that **ord** is a multiple of the order of  $a$ .

**int Fp\_issquare**(GEN  $x$ , GEN  $p$ ) returns 1 if  $x$  is a square modulo  $p$ , and 0 otherwise.

**int Fp\_ispower**(GEN  $x$ , GEN  $n$ , GEN  $p$ ) returns 1 if  $x$  is an  $n$ -th power modulo  $p$ , and 0 otherwise.

**GEN Fp\_sqrt**(GEN  $x$ , GEN  $p$ ) returns a square root of  $x$  modulo  $p$  (the smallest non-negative residue), where  $x, p$  are **t\_INT**s, and  $p$  is assumed to be prime. Return **NULL** if  $x$  is not a quadratic residue modulo  $p$ .

**GEN Fp\_2gener**(GEN  $p$ ) return a generator of the 2-Sylow subgroup of  $\mathbf{F}_p^*$ . To use with **Fp\_sqrt\_i**.

**GEN Fp\_sqrt\_i**(GEN  $x$ , GEN  $s2$ , GEN  $p$ ) as **Fp\_sqrt** where  $s2$  is the element returned by **Fp\_2gener**.

**GEN Fp\_sqrtn**(GEN  $a$ , GEN  $n$ , GEN  $p$ , GEN  $*zn$ ) returns **NULL** if  $a$  is not an  $n$ -th power residue mod  $p$ . Otherwise, returns an  $n$ -th root of  $a$ ; if  $zn$  is non-**NULL** set it to a primitive  $m$ -th root of 1,  $m = \gcd(p - 1, n)$  allowing to compute all  $m$  solutions in  $\mathbf{F}_p$  of the equation  $x^n = a$ .

**GEN Zn\_sqrt**(GEN  $x$ , GEN  $n$ ) returns one of the square roots of  $x$  modulo  $n$  (possibly not prime), where  $x$  is a **t\_INT** and  $n$  is either a **t\_INT** or is given by its factorization matrix. Return **NULL** if no such square root exist.

**long kross**(**long**  $x$ , **long**  $y$ ) returns the Kronecker symbol  $(x|y)$ , i.e.  $-1, 0$  or  $1$ . If  $y$  is an odd prime, this is the Legendre symbol. (Contrary to **krouu**, **kross** also supports  $y = 0$ )

**long krouu**(**ulong**  $x$ , **ulong**  $y$ ) returns the Kronecker symbol  $(x|y)$ , i.e.  $-1, 0$  or  $1$ . Assumes  $y$  is non-zero. If  $y$  is an odd prime, this is the Legendre symbol.

**long krois**(GEN  $x$ , **long**  $y$ ) returns the Kronecker symbol  $(x|y)$  of **t\_INT**  $x$  and **long**  $y$ . As **kross** otherwise.

**long kroiu**(GEN  $x$ , **ulong**  $y$ ) returns the Kronecker symbol  $(x|y)$  of **t\_INT**  $x$  and non-zero **ulong**  $y$ . As **krouu** otherwise.

**long krosi**(**long**  $x$ , GEN  $y$ ) returns the Kronecker symbol  $(x|y)$  of **long**  $x$  and **t\_INT**  $y$ . As **kross** otherwise.

`long kroui(ulong x, GEN y)` returns the Kronecker symbol  $(x|y)$  of `long`  $x$  and `t_INT`  $y$ . As `kross` otherwise.

`long kronecker(GEN x, GEN y)` returns the Kronecker symbol  $(x|y)$  of `t_INTs`  $x$  and  $y$ . As `kross` otherwise.

`GEN pgener_Fp(GEN p)` returns the smallest primitive root modulo  $p$ , assuming  $p$  is prime.

`GEN pgener_Zp(GEN p)` returns the smallest primitive root modulo  $p^k$ ,  $k > 1$ , assuming  $p$  is an odd prime.

`long Zp_issquare(GEN x, GEN p)` returns 1 if the `t_INT`  $x$  is a  $p$ -adic square, 0 otherwise.

`long Zn_issquare(GEN x, GEN n)` returns 1 if `t_INT`  $x$  is a square modulo  $n$  (possibly not prime), where  $n$  is either a `t_INT` or is given by its factorization matrix. Return 0 otherwise.

`long Zn_ispower(GEN x, GEN n, GEN K, GEN *py)` returns 1 if `t_INT`  $x$  is a  $K$ -th power modulo  $n$  (possibly not prime), where  $n$  is either a `t_INT` or is given by its factorization matrix. Return 0 otherwise. If `py` is not `NULL`, set it to  $y$  such that  $y^K = x$  modulo  $n$ .

`GEN pgener_Fp_local(GEN p, GEN L)`,  $L$  being a vector of primes dividing  $p - 1$ , returns the smallest integer  $x > 1$  which is a generator of the  $\ell$ -Sylow of  $\mathbf{F}_p^*$  for every  $\ell$  in  $L$ . In other words,  $x^{(p-1)/\ell} \neq 1$  for all such  $\ell$ . In particular, returns `pgener_Fp(p)` if  $L$  contains all primes dividing  $p - 1$ . It is not necessary, and in fact slightly inefficient, to include  $\ell = 2$ , since 2 is treated separately in any case, i.e. the generator obtained is never a square.

`GEN rootsof1_Fp(GEN n, GEN p)` returns a primitive  $n$ -th root modulo the prime  $p$ .

`GEN rootsof1u_Fp(ulong n, GEN p)` returns a primitive  $n$ -th root modulo the prime  $p$ .

`ulong rootsof1_Fl(ulong n, ulong p)` returns a primitive  $n$ -th root modulo the prime  $p$ .

### 6.2.21 Extending functions to vector inputs.

The following functions apply  $f$  to the given arguments, recursively if they are of vector / matrix type:

`GEN map_proto_G(GEN (*f)(GEN), GEN x)` For instance, if  $x$  is a `t_VEC`, return a `t_VEC` whose components are the  $f(x[i])$ .

`GEN map_proto_lG(long (*f)(GEN), GEN x)` As above, applying the function `stoi( f() )`.

`GEN map_proto_GL(GEN (*f)(GEN, long), GEN x, long y)`

`GEN map_proto_lGL(long (*f)(GEN, long), GEN x, long y)`

In the last function,  $f$  implements an associative binary operator, which we extend naturally to an  $n$ -ary operator  $f_n$  for any  $n$ : by convention,  $f_0() = 1$ ,  $f_1(x) = x$ , and

$$f_n(x_1, \dots, x_n) = f(f_{n-1}(x_1, \dots, x_{n-1}), x_n),$$

for  $n \geq 2$ .

`GEN gassoc_proto(GEN (*f)(GEN, GEN), GEN x, GEN y)` If  $y$  is not `NULL`, return  $f(x, y)$ . Otherwise,  $x$  must be of vector type, and we return the result of  $f$  applied to its components, computed using a divide-and-conquer algorithm. More precisely, return

$$f(f(x_1, \text{NULL}), f(x_2, \text{NULL})),$$

where  $x_1, x_2$  are the two halves of  $x$ .

### 6.2.22 Miscellaneous arithmetic functions.

`long bigomegau(ulong n)` returns the number of prime divisors of  $n > 0$ , counted with multiplicity.

`ulong coreu(ulong n)`, unique squarefree integer  $d$  dividing  $n$  such that  $n/d$  is a square.

`ulong coreu_fact(GEN fa)` same, where `fa` is `factoru(n)`.

`ulong corediscs(long d, ulong *pt_f)`,  $d$  (possibly negative) being congruent to 0 or 1 modulo 4, return the fundamental discriminant  $D$  such that  $d = D * f^2$  and set `*pt_f` to  $f$  (if `*pt_f` not NULL).

`ulong eulerphiu(ulong n)`, Euler's totient function of  $n$ .

`ulong eulerphiu_fact(GEN fa)` same, where `fa` is `factoru(n)`.

`long moebiusu(ulong n)`, Moebius  $\mu$ -function of  $n$ .

`long moebiusu_fact(GEN fa)` same, where `fa` is `factoru(n)`.

`GEN divisorsu(ulong n)`, returns the divisors of  $n$  in a `t_VECSMALL`, sorted by increasing order.

`GEN divisorsu_fact(GEN fa)` same, where `fa` is `factoru(n)`.

`long numdivu(ulong n)`, returns the number of positive divisors of  $n > 0$ .

`long numdivu_fact(GEN fa)` same, where `fa` is `factoru(n)`.

`long omegau(ulong n)` returns the number of prime divisors of  $n > 0$ .

`long uissquarefree(ulong n)` returns 1 if  $n$  is square-free, and 0 otherwise.

`long uissquarefree_fact(GEN fa)` same, where `fa` is `factoru(n)`.

`long uposisfundamental(ulong x)` return 1 if  $x$  is a fundamental discriminant, and 0 otherwise.

`long unegisfundamental(ulong x)` return 1 if  $-x$  is a fundamental discriminant, and 0 otherwise.

`long sisfundamental(long x)` return 1 if  $x$  is a fundamental discriminant, and 0 otherwise.

`int uis_357_power(ulong x, ulong *pt, ulong *mask)` as `is_357_power` for `ulong x`.

`int uis_357_powermod(ulong x, ulong *mask)` as `uis_357_power`, but only check for 3rd, 5th or 7th powers modulo  $211 \times 209 \times 61 \times 203 \times 117 \times 31 \times 43 \times 71$ .

`long uisprimepower(ulong n, ulong *p)` as `isprimepower`, for `ulong n`.

`int uislucasp(ulong n)` returns 1 if the `ulong n` fails Lucas compositeness test (it thus may be prime or composite), and 0 otherwise (proving that  $n$  is composite).

`ulong sumdigitsu(ulong n)` returns the sum of decimal digits of  $u$ .

`GEN usumdiv_fact(GEN fa)`, sum of divisors of `ulong n`, where `fa` is `factoru(n)`.

`GEN usumdivk_fact(GEN fa, ulong k)`, sum of  $k$ -th powers of divisors of `ulong n`, where `fa` is `factoru(n)`.

`GEN hilbertii(GEN x, GEN y, GEN p)`, returns the Hilbert symbol  $(x, y)$  at the prime  $p$  (NULL for the place at infinity);  $x$  and  $y$  are `t_INTs`.

`GEN sumdedekind(GEN h, GEN k)` returns the Dedekind sum attached to the `t_INT`  $h$  and  $k$ ,  $k > 0$ .

GEN `sumdedekind_coprime`(GEN `h`, GEN `k`) as `sumdedekind`, except that  $h$  and  $k$  are assumed to be coprime `t_INT`s.

GEN `u_sumdedekind_coprime`(long `h`, long `k`) Let  $k > 0$ ,  $0 \leq h < k$ ,  $(h, k) = 1$ . Returns  $[s_1, s_2]$  in a `t_VECSMALL`, such that  $s(h, k) = (s_2 + ks_1)/(12k)$ . Requires  $\max(h + k/2, k) < \text{LONG\_MAX}$  to avoid overflow, in particular  $k \leq (2/3)\text{LONG\_MAX}$  is fine.



## Chapter 7:

### Level 2 kernel

These functions deal with modular arithmetic, linear algebra and polynomials where assumptions can be made about the types of the coefficients.

#### 7.1 Naming scheme.

A function name is built in the following way:  $A_1 \dots A_n fun$  for an operation  $fun$  with  $n$  arguments of class  $A_1, \dots, A_n$ . A class name is given by a base ring followed by a number of code letters. Base rings are among

**F1**:  $\mathbf{Z}/l\mathbf{Z}$  where  $l < 2^{\text{BITS\_IN\_LONG}}$  is not necessarily prime. Implemented using **ulongs**

**Fp**:  $\mathbf{Z}/p\mathbf{Z}$  where  $p$  is a **t\_INT**, not necessarily prime. Implemented as **t\_INTs**  $z$ , preferably satisfying  $0 \leq z < p$ . More precisely, any **t\_INT** can be used as an **Fp**, but reduced inputs are treated more efficiently. Outputs from **Fpxxx** routines are reduced.

**Fq**:  $\mathbf{Z}[X]/(p, T(X))$ ,  $p$  a **t\_INT**,  $T$  a **t\_POL** with **Fp** coefficients or **NULL** (in which case no reduction modulo  $T$  is performed). Implemented as **t\_POLs**  $z$  with **Fp** coefficients,  $\deg(z) < \deg T$ , although  $z$  a **t\_INT** is allowed for elements in the prime field.

**Z**: the integers  $\mathbf{Z}$ , implemented as **t\_INTs**.

**Zp**: the  $p$ -adic integers  $\mathbf{Z}_p$ , implemented as **t\_INTs**, for arbitrary  $p$

**Z1**: the  $p$ -adic integers  $\mathbf{Z}_p$ , implemented as **t\_INTs**, for  $p < 2^{\text{BITS\_IN\_LONG}}$

**z**: the integers  $\mathbf{Z}$ , implemented using (signed) **longs**.

**Q**: the rational numbers  $\mathbf{Q}$ , implemented as **t\_INTs** and **t\_FRACs**.

**Rg**: a commutative ring, whose elements can be **gadd**-ed, **gmul**-ed, etc.

Possible letters are:

**X**: polynomial in  $X$  (**t\_POL** in a fixed variable), e.g. **FpX** means  $\mathbf{Z}/p\mathbf{Z}[X]$

**Y**: polynomial in  $Y \neq X$ . This is used to resolve ambiguities. E.g. **FpXY** means  $((\mathbf{Z}/p\mathbf{Z}[X])[Y])$ .

**V**: vector (**t\_VEC** or **t\_COL**), treated as a line vector (independently of the actual type). E.g. **ZV** means  $\mathbf{Z}^k$  for some  $k$ .

**C**: vector (**t\_VEC** or **t\_COL**), treated as a column vector (independently of the actual type). The difference with **V** is purely semantic: if the result is a vector, it will be of type **t\_COL** unless mentioned otherwise. For instance the function **ZC.add** receives two integral vectors (**t\_COL** or **t\_VEC**, possibly different types) of the same length and returns a **t\_COL** whose entries are the sums of the input coefficients.

**M**: matrix (**t\_MAT**). E.g. **QM** means a matrix with rational entries

**T**: Trees. Either a leaf or a **t\_VEC** of trees.

**E**: point over an elliptic curve, represented as two-component vectors  $[x, y]$ , except for the represented by the one-component vector  $[0]$ . Not all curve models are supported.

**Q**: representative (**t\_POL**) of a class in a polynomial quotient ring. E.g. an **FpXQ** belongs to  $(\mathbf{Z}/p\mathbf{Z})[X]/(T(X))$ , **FpXQV** means a vector of such elements, etc.

**n**: a polynomial representative (**t\_POL**) for a truncated power series modulo  $X^n$ . E.g. an **FpXn** belongs to  $(\mathbf{Z}/p\mathbf{Z})[X]/(X^n)$ , **FpXnV** means a vector of such elements, etc.

**x**, **y**, **m**, **v**, **c**, **q**: as their uppercase counterpart, but coefficient arrays are implemented using **t\_VECSMALLs**, which coefficient understood as **ulongs**.

**x** and **y** (and **q**) are implemented by a **t\_VECSMALL** whose first coefficient is used as a code-word and the following are the coefficients, similarly to a **t\_POL**. This is known as a 'POLSMALL'.

**m** are implemented by a **t\_MAT** whose components (columns) are **t\_VECSMALLs**. This is known as a 'MATSMALL'.

**v** and **c** are regular **t\_VECSMALLs**. Difference between the two is purely semantic.

Omitting the letter means the argument is a scalar in the base ring. Standard functions *fun* are

**add**: add

**sub**: subtract

**mul**: multiply

**sqr**: square

**div**: divide (Euclidean quotient)

**rem**: Euclidean remainder

**divrem**: return Euclidean quotient, store remainder in a pointer argument. Three special values of that pointer argument modify the default behavior: **NULL** (do not store the remainder, used to implement **div**), **ONLY\_REM** (return the remainder, used to implement **rem**), **ONLY\_DIVIDES** (return the quotient if the division is exact, and **NULL** otherwise).

**gcd**: GCD

**extgcd**: return GCD, store Bezout coefficients in pointer arguments

**pow**: exponentiate

**eval**: evaluation / composition



## 7.2 Coefficient ring.

`long Rg_type(GEN x, GEN *ptp, GEN *ptpol, long *ptprec)` returns the “natural” base ring over which the object  $x$  is defined.

Raise an error if it detects consistency problems in modular objects: incompatible rings (e.g.  $\mathbf{F}_p$  and  $\mathbf{F}_q$  for primes  $p \neq q$ ,  $\mathbf{F}_p[X]/(T)$  and  $\mathbf{F}_p[X]/(U)$  for  $T \neq U$ ). Minor discrepancies are supported if they make general sense (e.g.  $\mathbf{F}_p$  and  $\mathbf{F}_{p^k}$ , but not  $\mathbf{F}_p$  and  $\mathbf{Q}_p$ ); `t_FFELT` and `t_POLMOD` of `t_INTMODs` are considered inconsistent, even if they define the same field: if you need to use simultaneously these different finite field implementations, multiply the polynomial by a `t_FFELT` equal to 1 first.

- 0: none of the others (presumably multivariate, possibly inconsistent).
- `t_INT`: defined over  $\mathbf{Z}$ .
- `t_FRAC`: defined over  $\mathbf{Q}$ .
- `t_INTMOD`: defined over  $\mathbf{Z}/p\mathbf{Z}$ , where `*ptp` is set to  $p$ . It is not checked whether  $p$  is prime.
- `t_COMPLEX`: defined over  $\mathbf{C}$  (at least one `t_COMPLEX` with at least one inexact floating point `t_REAL` component). Set `*ptprec` to the minimal accuracy (as per `precision`) of inexact components.
- `t_REAL`: defined over  $\mathbf{R}$  (at least one inexact floating point `t_REAL` component). Set `*ptprec` to the minimal accuracy (as per `precision`) of inexact components.
- `t_PADIC`: defined over  $\mathbf{Q}_p$ , where `*ptp` is set to  $p$  and `*ptprec` to the  $p$ -adic accuracy.
- `t_FFELT`: defined over a finite field  $\mathbf{F}_{p^k}$ , where `*ptp` is set to the field characteristic  $p$  and `*ptpol` is set to a `t_FFELT` belonging to the field.
- `t_POL`: defined over a polynomial ring.
- other values are composite corresponding to quotients  $R[X]/(T)$ , with one primary type `t1`, describing the form of the quotient, and a secondary type `t2`, describing  $R$ . If `t` is the `RgX_type`, `t1` and `t2` are recovered using

```
void RgX_type_decode(long t, long *t1, long *t2)
```

`t1` is one of

`t_POLMOD`: at least one `t_POLMOD` component, set `*ppol` to the modulus,

`t_QUAD`: no `t_POLMOD`, at least one `t_QUAD` component, set `*ppol` to the modulus (`-.pol`) of the `t_QUAD`,

`t_COMPLEX`: no `t_POLMOD` or `t_QUAD`, at least one `t_COMPLEX` component, set `*ppol` to  $y^2 + 1$ .

and the underlying base ring  $R$  is given by `t2`, which is one of `t_INT`, `t_INTMOD` (set `*ptp`) or `t_PADIC` (set `*ptp` and `*ptprec`), with the same meaning as above.

`int RgX_type_is_composite(long t)`  $t$  as returned by `RgX_type`, return 1 if  $t$  is a composite type, and 0 otherwise.

`GEN Rg_get_0(GEN x)` returns 0 in the base ring over which  $x$  is defined, to the proper accuracy (e.g. 0, Mod(0,3), 0(5^10)).

`GEN Rg_get_1(GEN x)` returns 1 in the base ring over which  $x$  is defined, to the proper accuracy (e.g. 0, Mod(0,3),

`long RgX_type(GEN x, GEN *ptp, GEN *ptpol, long *ptprec)` returns the “natural” base ring over which the polynomial  $x$  is defined, otherwise as `Rg_type`.

`long RgX_Rg_type(GEN x, GEN y, GEN *ptp, GEN *ptpol, long *ptprec)` returns the “natural” base ring over which the polynomial  $x$  and the element  $y$  are defined, otherwise as `Rg_type`.

`long RgX_type2(GEN x, GEN y, GEN *ptp, GEN *ptpol, long *ptprec)` returns the “natural” base ring over which the polynomials  $x$  and  $y$  are defined, otherwise as `Rg_type`.

`long RgX_type3(GEN x, GEN y, GEN z, GEN *ptp, GEN *ptpol, long *ptprec)` returns the “natural” base ring over which the polynomials  $x$ ,  $y$  and  $z$  are defined, otherwise as `Rg_type`.

`long RgM_type(GEN x, GEN *ptp, GEN *ptpol, long *ptprec)` returns the “natural” base ring over which the matrix  $x$  is defined, otherwise as `Rg_type`.

`long RgM_type2(GEN x, GEN y, GEN *ptp, GEN *ptpol, long *ptprec)` returns the “natural” base ring over which the matrices  $x$  and  $y$  are defined, otherwise as `Rg_type`.

`long RgM_RgC_type(GEN x, GEN y, GEN *ptp, GEN *ptpol, long *ptprec)` returns the “natural” base ring over which the matrix  $x$  and the vector  $y$  are defined, otherwise as `Rg_type`.

### 7.3 Modular arithmetic.

These routines implement univariate polynomial arithmetic and linear algebra over finite fields, in fact over finite rings of the form  $(\mathbf{Z}/p\mathbf{Z})[X]/(T)$ , where  $p$  is not necessarily prime and  $T \in (\mathbf{Z}/p\mathbf{Z})[X]$  is possibly reducible; and finite extensions thereof. All this can be emulated with `t_INTMOD` and `t_POLMOD` coefficients and using generic routines, at a considerable loss of efficiency. Also, specialized routines are available that have no obvious generic equivalent.

**7.3.1 FpC / FpV, FpM.** A ZV (resp. a ZM) is a `t_VEC` or `t_COL` (resp. `t_MAT`) with `t_INT` coefficients. An FpV or FpM, with respect to a given `t_INT p`, is the same with Fp coordinates; operations are understood over  $\mathbf{Z}/p\mathbf{Z}$ .

#### 7.3.1.1 Conversions.

`int Rg_is_Fp(GEN z, GEN *p)`, checks if  $z$  can be mapped to  $\mathbf{Z}/p\mathbf{Z}$ : a `t_INT` or a `t_INTMOD` whose modulus is equal to  $*p$ , (if  $*p$  not NULL), in that case return 1, else 0. If a modulus is found it is put in  $*p$ , else  $*p$  is left unchanged.

`int RgV_is_FpV(GEN z, GEN *p)`,  $z$  a `t_VEC` (resp. `t_COL`), checks if it can be mapped to a FpV (resp. FpC), by checking `Rg_is_Fp` coefficientwise.

`int RgM_is_FpM(GEN z, GEN *p)`,  $z$  a `t_MAT`, checks if it can be mapped to a FpM, by checking `RgV_is_FpV` columnwise.

`GEN Rg_to_Fp(GEN z, GEN p)`,  $z$  a scalar which can be mapped to  $\mathbf{Z}/p\mathbf{Z}$ : a `t_INT`, a `t_INTMOD` whose modulus is divisible by  $p$ , a `t_FRAC` whose denominator is coprime to  $p$ , or a `t_PADIC` with underlying prime  $\ell$  satisfying  $p = \ell^n$  for some  $n$  (less than the accuracy of the input). Returns `lift(z * Mod(1,p))`, normalized.

`GEN padic_to_Fp(GEN x, GEN p)` special case of `Rg_to_Fp`, for a  $x$  a `t_PADIC`.

`GEN RgV_to_FpV(GEN z, GEN p)`,  $z$  a `t_VEC` or `t_COL`, returns the FpV (as a `t_VEC`) obtained by applying `Rg_to_Fp` coefficientwise.

GEN RgC\_to\_FpC(GEN z, GEN p),  $z$  a  $t\_VEC$  or  $t\_COL$ , returns the FpC (as a  $t\_COL$ ) obtained by applying Rg\_to\_Fp coefficientwise.

GEN RgM\_to\_FpM(GEN z, GEN p),  $z$  a  $t\_MAT$ , returns the FpM obtained by applying RgC\_to\_FpC columnwise.

GEN RgM\_Fp\_init(GEN z, GEN p, ulong \*pp), given an RgM  $z$ , whose entries can be mapped to  $F_p$  (as per Rg\_to\_Fp), and a prime number  $p$ . This routine returns a normal form of  $z$ : either an F2m ( $p = 2$ ), an Flm ( $p$  fits into an ulong) or an FpM. In the first two cases,  $pp$  is set to  $itou(p)$ , and to 0 in the last.

The functions above are generally used as follow:

```
GEN add(GEN x, GEN y)
{
    GEN p = NULL;
    if (Rg_is_Fp(x, &p) && Rg_is_Fp(y, &p) && p)
    {
        x = Rg_to_Fp(x, p); y = Rg_to_Fp(y, p);
        z = Fp_add(x, y, p);
        return Fp_to_mod(z);
    }
    else return gadd(x, y);
}
```

GEN FpC\_red(GEN z, GEN p),  $z$  a ZC. Returns  $\text{lift}(\text{Col}(z) * \text{Mod}(1,p))$ , hence a  $t\_COL$ .

GEN FpV\_red(GEN z, GEN p),  $z$  a ZV. Returns  $\text{lift}(\text{Vec}(z) * \text{Mod}(1,p))$ , hence a  $t\_VEC$

GEN FpM\_red(GEN z, GEN p),  $z$  a ZM. Returns  $\text{lift}(z * \text{Mod}(1,p))$ , which is an FpM.

### 7.3.1.2 Basic operations.

GEN random\_FpC(long n, GEN p) returns a random FpC with  $n$  components.

GEN random\_FpV(long n, GEN p) returns a random FpV with  $n$  components.

GEN FpC\_center(GEN z, GEN p, GEN pov2) returns a  $t\_COL$  whose entries are the Fp\_center of the  $\text{gel}(z,i)$ .

GEN FpM\_center(GEN z, GEN p, GEN pov2) returns a matrix whose entries are the Fp\_center of the  $\text{gcoeff}(z,i,j)$ .

void FpC\_center\_inplace(GEN z, GEN p, GEN pov2) in-place version of FpC\_center, using affii.

void FpM\_center\_inplace(GEN z, GEN p, GEN pov2) in-place version of FpM\_center, using affii.

GEN FpC\_add(GEN x, GEN y, GEN p) adds the ZC  $x$  and  $y$  and reduce modulo  $p$  to obtain an FpC.

GEN FpV\_add(GEN x, GEN y, GEN p) same as FpC\_add, returning and FpV.

GEN FpM\_add(GEN x, GEN y, GEN p) adds the two ZMs  $x$  and  $y$  (assumed to have compatible dimensions), and reduce modulo  $p$  to obtain an FpM.

GEN FpC\_sub(GEN x, GEN y, GEN p) subtracts the ZC  $y$  to the ZC  $x$  and reduce modulo  $p$  to obtain an FpC.

GEN FpV\_sub(GEN x, GEN y, GEN p) same as FpC\_sub, returning and FpV.

GEN FpM\_sub(GEN x, GEN y, GEN p) subtracts the two ZMs x and y (assumed to have compatible dimensions), and reduce modulo p to obtain an FpM.

GEN FpC\_Fp\_mul(GEN x, GEN y, GEN p) multiplies the ZC x (seen as a column vector) by the t\_INT y and reduce modulo p to obtain an FpC.

GEN FpM\_Fp\_mul(GEN x, GEN y, GEN p) multiplies the ZM x (seen as a column vector) by the t\_INT y and reduce modulo p to obtain an FpM.

GEN FpC\_FpV\_mul(GEN x, GEN y, GEN p) multiplies the ZC x (seen as a column vector) by the ZV y (seen as a row vector, assumed to have compatible dimensions), and reduce modulo p to obtain an FpM.

GEN FpM\_mul(GEN x, GEN y, GEN p) multiplies the two ZMs x and y (assumed to have compatible dimensions), and reduce modulo p to obtain an FpM.

GEN FpM\_powu(GEN x, ulong n, GEN p) computes  $x^n$  where x is a square FpM.

GEN FpM\_FpC\_mul(GEN x, GEN y, GEN p) multiplies the ZM x by the ZC y (seen as a column vector, assumed to have compatible dimensions), and reduce modulo p to obtain an FpC.

GEN FpM\_FpC\_mul\_FpX(GEN x, GEN y, GEN p, long v) is a memory-clean version of

```
GEN tmp = FpM_FpC_mul(x,y,p);
return RgV_to_RgX(tmp, v);
```

GEN FpV\_FpC\_mul(GEN x, GEN y, GEN p) multiplies the ZV x (seen as a row vector) by the ZC y (seen as a column vector, assumed to have compatible dimensions), and reduce modulo p to obtain an Fp.

GEN FpV\_dotproduct(GEN x, GEN y, GEN p) scalar product of x and y (assumed to have the same length).

GEN FpV\_dotsquare(GEN x, GEN p) scalar product of x with itself. has t\_INT entries.

GEN FpV\_factorback(GEN L, GEN e, GEN p) given an FpV L and a ZV e of the same length, return  $\prod_i L_i^{e_i}$  modulo p.

**7.3.1.3 Fp-linear algebra.** The implementations are not asymptotically efficient ( $O(n^3)$  standard algorithms).

GEN FpM\_deplin(GEN x, GEN p) returns a non-trivial kernel vector, or NULL if none exist.

GEN FpM\_det(GEN x, GEN p) as det

GEN FpM\_gauss(GEN a, GEN b, GEN p) as gauss, where a and b are FpM.

GEN FpM\_FpC\_gauss(GEN a, GEN b, GEN p) as gauss, where a is a FpM and b a FpC.

GEN FpM\_image(GEN x, GEN p) as image

GEN FpM\_intersect(GEN x, GEN y, GEN p) as intersect

GEN FpM\_inv(GEN x, GEN p) returns a left inverse of x (the inverse if x is square), or NULL if x is not invertible.

GEN FpM\_FpC\_invimage(GEN A, GEN y, GEN p) given an FpM A and an FpC y, returns an x such that  $Ax = y$ , or NULL if no such vector exist.

GEN FpM\_invimage(GEN A, GEN y, GEN p) given two FpM  $A$  and  $y$ , returns  $x$  such that  $Ax = y$ , or NULL if no such matrix exist.

GEN FpM\_ker(GEN x, GEN p) as ker

long FpM\_rank(GEN x, GEN p) as rank

GEN FpM\_indexrank(GEN x, GEN p) as indexrank

GEN FpM\_suppl(GEN x, GEN p) as suppl

GEN FpM\_hess(GEN x, GEN p) upper Hessenberg form of  $x$  over  $\mathbf{F}_p$ .

GEN FpM\_charpoly(GEN x, GEN p) characteristic polynomial of  $x$ .

#### 7.3.1.4 FqC, FqM and Fq-linear algebra.

An FqM (resp. FqC) is a matrix (resp a t\_COL) with Fq coefficients (with respect to given T, p), not necessarily reduced (i.e arbitrary t\_INTs and ZXs in the same variable as T).

GEN RgC\_to\_FqC(GEN z, GEN T, GEN p)

GEN RgM\_to\_FqM(GEN z, GEN T, GEN p)

GEN FqC\_add(GEN a, GEN b, GEN T, GEN p)

GEN FqC\_sub(GEN a, GEN b, GEN T, GEN p)

GEN FqC\_Fq\_mul(GEN a, GEN b, GEN T, GEN p)

GEN FqM\_FqC\_gauss(GEN a, GEN b, GEN T, GEN p) as gauss, where  $b$  is a FqC.

GEN FqM\_FqC\_invimage(GEN a, GEN b, GEN T, GEN p)

GEN FqM\_FqC\_mul(GEN a, GEN b, GEN T, GEN p)

GEN FqM\_deplin(GEN x, GEN T, GEN p) returns a non-trivial kernel vector, or NULL if none exist.

GEN FqM\_det(GEN x, GEN T, GEN p) as det

GEN FqM\_gauss(GEN a, GEN b, GEN T, GEN p) as gauss, where  $b$  is a FqM.

GEN FqM\_image(GEN x, GEN T, GEN p) as image

GEN FqM\_indexrank(GEN x, GEN T, GEN p) as indexrank

GEN FqM\_inv(GEN x, GEN T, GEN p) returns the inverse of  $x$ , or NULL if  $x$  is not invertible.

GEN FqM\_invimage(GEN a, GEN b, GEN T, GEN p) as invimage

GEN FqM\_ker(GEN x, GEN T, GEN p) as ker

GEN FqM\_mul(GEN a, GEN b, GEN T, GEN p)

long FqM\_rank(GEN x, GEN T, GEN p) as rank

GEN FqM\_suppl(GEN x, GEN T, GEN p) as suppl

**7.3.2 Flc / Flv, Flm.** See FpV, FpM operations.

GEN Flv\_copy(GEN x) returns a copy of x.

GEN Flv\_center(GEN z, ulong p, ulong ps2)

GEN random\_Flv(long n, ulong p) returns a random Flv with  $n$  components.

GEN Flm\_copy(GEN x) returns a copy of x.

GEN matid\_Flm(long n) returns an Flm which is an  $n \times n$  identity matrix.

GEN scalar\_Flm(long s, long n) returns an Flm which is  $s$  times the  $n \times n$  identity matrix.

GEN Flm\_center(GEN z, ulong p, ulong ps2)

GEN Flm\_Fl\_add(GEN x, ulong y, ulong p) returns  $x + y * \text{Id}$  ( $x$  must be square).

GEN Flm\_Flc\_mul(GEN x, GEN y, ulong p) multiplies  $x$  and  $y$  (assumed to have compatible dimensions).

GEN Flm\_Flc\_mul\_pre(GEN x, GEN y, ulong p, ulong pi) multiplies  $x$  and  $y$  (assumed to have compatible dimensions), assuming  $pi$  is the pseudo inverse of  $p$ .

GEN Flc\_Flv\_mul(GEN x, GEN y, ulong p) multiplies the column vector  $x$  by the row vector  $y$ . The result is a matrix.

GEN Flm\_Flc\_mul\_pre\_Flx(GEN x, GEN y, ulong p, ulong pi, long sv) return Flv\_to\_Flx(Flm\_Flc\_mul\_pre(x, y, p, pi), sv).

GEN Flm\_Fl\_mul(GEN x, ulong y, ulong p) multiplies the Flm  $x$  by  $y$ .

GEN Flm\_neg(GEN x, ulong p) negates the Flm  $x$ .

void Flm\_Fl\_mul\_inplace(GEN x, ulong y, ulong p) replaces the Flm  $x$  by  $x * y$ .

GEN Flv\_Fl\_mul(GEN x, ulong y, ulong p) multiplies the Flv  $x$  by  $y$ .

void Flv\_Fl\_mul\_inplace(GEN x, ulong y, ulong p) replaces the Flc  $x$  by  $x * y$ .

void Flv\_Fl\_mul\_part\_inplace(GEN x, ulong y, ulong p, long l) multiplies  $x[1..l]$  by  $y$  modulo  $p$ . In place.

GEN Flv\_Fl\_div(GEN x, ulong y, ulong p) divides the Flv  $x$  by  $y$ .

void Flv\_Fl\_div\_inplace(GEN x, ulong y, ulong p) replaces the Flv  $x$  by  $x/y$ .

void Flc\_lincomb1\_inplace(GEN X, GEN Y, ulong v, ulong q) sets  $X \leftarrow X + vY$ , where  $X, Y$  are Flc. Memory efficient (e.g. no-op if  $v = 0$ ), and gerepile-safe.

GEN Flv\_add(GEN x, GEN y, ulong p) adds two Flv.

void Flv\_add\_inplace(GEN x, GEN y, ulong p) replaces  $x$  by  $x + y$ .

GEN Flv\_neg(GEN x, ulong p) returns  $-x$ .

void Flv\_neg\_inplace(GEN x, ulong p) replaces  $x$  by  $-x$ .

GEN Flv\_sub(GEN x, GEN y, ulong p) subtracts  $y$  to  $x$ .

void Flv\_sub\_inplace(GEN x, GEN y, ulong p) replaces  $x$  by  $x - y$ .

ulong Flv\_dotproduct(GEN x, GEN y, ulong p) returns the scalar product of  $x$  and  $y$

`ulong Flv_dotproduct_pre(GEN x, GEN y, ulong p, ulong pi)` returns the scalar product of  $x$  and  $y$  assuming  $pi$  is the pseudo inverse of  $p$ .

`ulong Flv_sum(GEN x, ulong p)` returns the sum of the components of  $x$ .

`ulong Flv_prod(GEN x, ulong p)` returns the product of the components of  $x$ .

`ulong Flv_prod_pre(GEN x, ulong p, ulong pi)` as `Flv_prod` assuming  $pi$  is the pseudo inverse of  $p$ .

`GEN Flv_inv(GEN x, ulong p)` returns the vector of inverses of the elements of  $x$  (as a `Flv`). Use Montgomery trick.

`void Flv_inv_inplace(GEN x, ulong p)` in place variant of `Flv_inv`.

`GEN Flv_inv_pre(GEN x, ulong p, ulong pi)` as `Flv_inv` assuming  $pi$  is the pseudo inverse of  $p$ .

`void Flv_inv_pre_inplace(GEN x, ulong p, ulong pi)` in place variant of `Flv_inv`.

`GEN Flc_FpV_mul(GEN x, GEN y, GEN p)` multiplies  $x$  (seen as a column vector) by  $y$  (seen as a row vector, assumed to have compatible dimensions) to obtain an `Flm`.

`GEN zero_Flm(long m, long n)` creates a `Flm` with  $m \times n$  components set to 0. Note that the result allocates a *single* column, so modifying an entry in one column modifies it in all columns.

`GEN zero_Flm_copy(long m, long n)` creates a `Flm` with  $m \times n$  components set to 0.

`GEN zero_Flv(long n)` creates a `Flv` with  $n$  components set to 0.

`GEN Flm_row(GEN A, long x0)` return  $A[i,]$ , the  $i$ -th row of the `Flm`  $A$ .

`GEN Flm_add(GEN x, GEN y, ulong p)` adds  $x$  and  $y$  (assumed to have compatible dimensions).

`GEN Flm_sub(GEN x, GEN y, ulong p)` subtracts  $x$  and  $y$  (assumed to have compatible dimensions).

`GEN Flm_mul(GEN x, GEN y, ulong p)` multiplies  $x$  and  $y$  (assumed to have compatible dimensions).

`GEN Flm_powers(GEN x, ulong n, ulong p)` returns  $[x^0, \dots, x^n]$  as a `t_VEC` of `Flms`.

`GEN Flm_powu(GEN x, ulong n, ulong p)` computes  $x^n$  where  $x$  is a square `Flm`.

`GEN Flm_charpoly(GEN x, ulong p)` return the characteristic polynomial of the square `Flm`  $x$ , as a `Flx`.

`GEN Flm_deplin(GEN x, ulong p)`

`ulong Flm_det(GEN x, ulong p)`

`ulong Flm_det_sp(GEN x, ulong p)`, as `Flm_det`, in place (destroys  $x$ ).

`GEN Flm_gauss(GEN a, GEN b, ulong p)` as `gauss`, where  $b$  is a `Flm`.

`GEN Flm_Flc_gauss(GEN a, GEN b, ulong p)` as `gauss`, where  $b$  is a `Flc`.

`GEN Flm_indexrank(GEN x, ulong p)`

`GEN Flm_inv(GEN x, ulong p)`

`GEN Flm_adjoint(GEN x, ulong p)` as `matadjoint`.

GEN Flm\_Flc\_invimage(GEN A, GEN y, ulong p) given an Flm A and an Flc y, returns an x such that  $Ax = y$ , or NULL if no such vector exist.

GEN Flm\_invimage(GEN A, GEN y, ulong p) given two Flm A and y, returns x such that  $Ax = y$ , or NULL if no such matrix exist.

GEN Flm\_ker(GEN x, ulong p)

GEN Flm\_ker\_sp(GEN x, ulong p, long deplin), as Flm\_ker (if deplin=0) or Flm\_deplin (if deplin=1), in place (destroys x).

long Flm\_rank(GEN x, ulong p)

long Flm\_suppl(GEN x, ulong p)

GEN Flm\_image(GEN x, ulong p)

GEN Flm\_intersect(GEN x, GEN y, ulong p)

GEN Flm\_transpose(GEN x)

GEN Flm\_hess(GEN x, ulong p) upper Hessenberg form of x over  $\mathbf{F}_p$ .

**7.3.3 F2c / F2v, F2m.** An F2v v is a t\_VECSMALL representing a vector over  $\mathbf{F}_2$ . Specifically z[0] is the usual codeword, z[1] is the number of components of v and the coefficients are given by the bits of remaining words by increasing indices.

ulong F2v\_coeff(GEN x, long i) returns the coefficient  $i \geq 1$  of x.

void F2v\_clear(GEN x, long i) sets the coefficient  $i \geq 1$  of x to 0.

void F2v\_flip(GEN x, long i) adds 1 to the coefficient  $i \geq 1$  of x.

void F2v\_set(GEN x, long i) sets the coefficient  $i \geq 1$  of x to 1.

void F2v\_copy(GEN x) returns a copy of x.

GEN F2v\_slice(GEN x, long a, long b) returns the F2v with entries  $x[a], \dots, x[b]$ . Assumes  $a \leq b$ .

ulong F2m\_coeff(GEN x, long i, long j) returns the coefficient  $(i, j)$  of x.

void F2m\_clear(GEN x, long i, long j) sets the coefficient  $(i, j)$  of x to 0.

void F2m\_flip(GEN x, long i, long j) adds 1 to the coefficient  $(i, j)$  of x.

void F2m\_set(GEN x, long i, long j) sets the coefficient  $(i, j)$  of x to 1.

void F2m\_copy(GEN x) returns a copy of x.

GEN F2m\_rowslice(GEN x, long a, long b) returns the F2m built from the a-th to b-th rows of the F2m x. Assumes  $a \leq b$ .

GEN F2m\_F2c\_mul(GEN x, GEN y) multiplies x and y (assumed to have compatible dimensions).

GEN F2m\_image(GEN x) gives a subset of the columns of x that generate the image of x.

GEN F2m\_invimage(GEN A, GEN B)

GEN F2m\_F2c\_invimage(GEN A, GEN y)

GEN F2m\_gauss(GEN a, GEN b) as gauss, where b is a F2m.



GEN F2m\_F2c\_gauss(GEN a, GEN b) as **gauss**, where *b* is a F2c.

GEN F2m\_indexrank(GEN x) *x* being a matrix of rank *r*, returns a vector with two **t\_VECSMALL** components *y* and *z* of length *r* giving a list of rows and columns respectively (starting from 1) such that the extracted matrix obtained from these two vectors using **vecextract**(*x, y, z*) is invertible.

GEN F2m\_mul(GEN x, GEN y) multiplies *x* and *y* (assumed to have compatible dimensions).

GEN F2m\_powu(GEN x, ulong n) computes  $x^n$  where *x* is a square F2m.

long F2m\_rank(GEN x) as **rank**.

long F2m\_suppl(GEN x) as **suppl**.

GEN matid\_F2m(long n) returns an F2m which is an  $n \times n$  identity matrix.

GEN zero\_F2v(long n) creates a F2v with *n* components set to 0.

GEN const\_F2v(long n) creates a F2v with *n* components set to 1.

GEN F2v\_ei(long n, long i) creates a F2v with *n* components set to 0, but for the *i*-th one, which is set to 1 (*i*-th vector in the canonical basis).

GEN zero\_F2m(long m, long n) creates a F2m with *m* x *n* components set to 0. Note that the result allocates a *single* column, so modifying an entry in one column modifies it in all columns.

GEN zero\_F2m\_copy(long m, long n) creates a F2m with *m* x *n* components set to 0.

GEN F2v\_to\_Flv(GEN x)

GEN F2c\_to\_ZC(GEN x)

GEN ZV\_to\_F2v(GEN x)

GEN RgV\_to\_F2v(GEN x)

GEN F2m\_to\_Flm(GEN x)

GEN F2m\_to\_ZM(GEN x)

GEN Flv\_to\_F2v(GEN x)

GEN Flm\_to\_F2m(GEN x)

GEN ZM\_to\_F2m(GEN x)

GEN RgM\_to\_F2m(GEN x)

void F2v\_add\_inplace(GEN x, GEN y) replaces *x* by  $x + y$ . It is allowed for *y* to be shorter than *x*.

ulong F2m\_det(GEN x)

ulong F2m\_det\_sp(GEN x), as **F2m\_det**, in place (destroys *x*).

GEN F2m\_deplin(GEN x)

ulong F2v\_dotproduct(GEN x, GEN y) returns the scalar product of *x* and *y*

GEN F2m\_inv(GEN x)

GEN F2m\_ker(GEN x)

GEN F2m\_ker\_sp(GEN x, long deplin), as **F2m\_ker** (if *deplin*=0) or **F2m\_deplin** (if *deplin*=1), in place (destroys *x*).

**7.3.4** FlxqV, FlxqC, FlxqM. See FqV, FqC, FqM operations.

```

GEN FlxqV_dotproduct(GEN x, GEN y, GEN T, ulong p) as FpV_dotproduct.
GEN FlxM_Flx_add_shallow(GEN x, GEN y, ulong p) as RgM_Rg_add_shallow.
GEN FlxqC_Flxq_mul(GEN x, GEN y, GEN T, ulong p)
GEN FlxqM_Flxq_mul(GEN x, GEN y, GEN T, ulong p)
GEN FlxqM_FlxqC_gauss(GEN a, GEN b, GEN T, ulong p)
GEN FlxqM_FlxqC_invimage(GEN a, GEN b, GEN T, ulong p)
GEN FlxqM_FlxqC_mul(GEN a, GEN b, GEN T, ulong p)
GEN FlxqM_deplin(GEN x, GEN T, ulong p)
GEN FlxqM_det(GEN x, GEN T, ulong p)
GEN FlxqM_gauss(GEN a, GEN b, GEN T, ulong p)
GEN FlxqM_image(GEN x, GEN T, ulong p)
GEN FlxqM_indexrank(GEN x, GEN T, ulong p)
GEN FlxqM_inv(GEN x, GEN T, ulong p)
GEN FlxqM_invimage(GEN a, GEN b, GEN T, ulong p)
GEN FlxqM_ker(GEN x, GEN T, ulong p)
GEN FlxqM_mul(GEN a, GEN b, GEN T, ulong p)
long FlxqM_rank(GEN x, GEN T, ulong p)
GEN FlxqM_suppl(GEN x, GEN T, ulong p)
GEN matid_FlxqM(long n, GEN T, ulong p)

```

**7.3.5** FpX. Let  $p$  an understood  $t\_INT$ , to be given in the function arguments; in practice  $p$  is not assumed to be prime, but be wary. Recall that an Fp object is a  $t\_INT$ , preferably belonging to  $[0, p - 1]$ ; an FpX is a  $t\_POL$  in a fixed variable whose coefficients are Fp objects. Unless mentioned otherwise, all outputs in this section are FpXs. All operations are understood to take place in  $(\mathbf{Z}/p\mathbf{Z})[X]$ .

**7.3.5.1 Conversions.** In what follows  $p$  is always a  $t\_INT$ , not necessarily prime.

`int RgX_is_FpX(GEN z, GEN *p)`,  $z$  a  $t\_POL$ , checks if it can be mapped to a FpX, by checking `Rg_is_Fp` coefficientwise.

`GEN RgX_to_FpX(GEN z, GEN p)`,  $z$  a  $t\_POL$ , returns the FpX obtained by applying `Rg_to_Fp` coefficientwise.

`GEN FpX_red(GEN z, GEN p)`,  $z$  a ZX, returns `lift(z * Mod(1,p))`, normalized.

`GEN FpXV_red(GEN z, GEN p)`,  $z$  a  $t\_VEC$  of ZX. Applies `FpX_red` componentwise and returns the result (and we obtain a vector of FpXs).

`GEN FpXT_red(GEN z, GEN p)`,  $z$  a tree of ZX. Applies `FpX_red` to each leaf and returns the result (and we obtain a tree of FpXs).

**7.3.5.2 Basic operations.** In what follows  $p$  is always a  $\mathbf{t\_INT}$ , not necessarily prime.

Now, except for  $p$ , the operands and outputs are all  $\mathbf{FpX}$  objects. Results are undefined on other inputs.

$\mathbf{GEN\ FpX\_add}(\mathbf{GEN\ x}, \mathbf{GEN\ y}, \mathbf{GEN\ p})$  adds  $x$  and  $y$ .

$\mathbf{GEN\ FpX\_neg}(\mathbf{GEN\ x}, \mathbf{GEN\ p})$  returns  $-x$ , the components are between 0 and  $p$  if this is the case for the components of  $x$ .

$\mathbf{GEN\ FpX\_renormalize}(\mathbf{GEN\ x}, \mathbf{long\ l})$ , as  $\mathbf{normalizepol}$ , where  $l = \lg(x)$ , in place.

$\mathbf{GEN\ FpX\_sub}(\mathbf{GEN\ x}, \mathbf{GEN\ y}, \mathbf{GEN\ p})$  returns  $x - y$ .

$\mathbf{GEN\ FpX\_halve}(\mathbf{GEN\ x}, \mathbf{GEN\ p})$  returns  $z$  such that  $2z = x$  modulo  $p$  assuming such  $z$  exists.

$\mathbf{GEN\ FpX\_mul}(\mathbf{GEN\ x}, \mathbf{GEN\ y}, \mathbf{GEN\ p})$  returns  $xy$ .

$\mathbf{GEN\ FpX\_mulspec}(\mathbf{GEN\ a}, \mathbf{GEN\ b}, \mathbf{GEN\ p}, \mathbf{long\ na}, \mathbf{long\ nb})$  see  $\mathbf{ZX\_mulspec}$

$\mathbf{GEN\ FpX\_sqr}(\mathbf{GEN\ x}, \mathbf{GEN\ p})$  returns  $x^2$ .

$\mathbf{GEN\ FpX\_powu}(\mathbf{GEN\ x}, \mathbf{ulong\ n}, \mathbf{GEN\ p})$  returns  $x^n$ .

$\mathbf{GEN\ FpX\_convol}(\mathbf{GEN\ x}, \mathbf{GEN\ y}, \mathbf{GEN\ p})$  return the-term by-term product of  $x$  and  $y$ .

$\mathbf{GEN\ FpX\_divrem}(\mathbf{GEN\ x}, \mathbf{GEN\ y}, \mathbf{GEN\ p}, \mathbf{GEN\ *pr})$  returns the quotient of  $x$  by  $y$ , and sets  $\mathbf{pr}$  to the remainder.

$\mathbf{GEN\ FpX\_div}(\mathbf{GEN\ x}, \mathbf{GEN\ y}, \mathbf{GEN\ p})$  returns the quotient of  $x$  by  $y$ .

$\mathbf{GEN\ FpX\_div\_by\_X\_x}(\mathbf{GEN\ A}, \mathbf{GEN\ a}, \mathbf{GEN\ p}, \mathbf{GEN\ *r})$  returns the quotient of the  $\mathbf{FpX}$   $A$  by  $(X - a)$ , and sets  $\mathbf{r}$  to the remainder  $A(a)$ .

$\mathbf{GEN\ FpX\_rem}(\mathbf{GEN\ x}, \mathbf{GEN\ y}, \mathbf{GEN\ p})$  returns the remainder  $x \bmod y$ .

$\mathbf{long\ FpX\_valrem}(\mathbf{GEN\ x}, \mathbf{GEN\ t}, \mathbf{GEN\ p}, \mathbf{GEN\ *r})$  The arguments  $x$  and  $e$  being non-zero  $\mathbf{FpX}$  returns the highest exponent  $e$  such that  $\mathbf{t}^e$  divides  $x$ . The quotient  $x/\mathbf{t}^e$  is returned in  $\mathbf{*r}$ . In particular, if  $\mathbf{t}$  is irreducible, this returns the valuation at  $\mathbf{t}$  of  $x$ , and  $\mathbf{*r}$  is the prime-to- $\mathbf{t}$  part of  $x$ .

$\mathbf{GEN\ FpX\_deriv}(\mathbf{GEN\ x}, \mathbf{GEN\ p})$  returns the derivative of  $x$ . This function is not memory-clean, but nevertheless suitable for  $\mathbf{gerepileupto}$ .

$\mathbf{GEN\ FpX\_integ}(\mathbf{GEN\ x}, \mathbf{GEN\ p})$  returns the primitive of  $x$  whose constant term is 0.

$\mathbf{GEN\ FpX\_digits}(\mathbf{GEN\ x}, \mathbf{GEN\ B}, \mathbf{GEN\ p})$  returns a vector of  $\mathbf{FpX}$   $[c_0, \dots, c_n]$  of degree less than the degree of  $B$  and such that  $x = \sum_{i=0}^n c_i B^i$ .

$\mathbf{GEN\ FpXV\_FpX\_fromdigits}(\mathbf{GEN\ v}, \mathbf{GEN\ B}, \mathbf{GEN\ p})$  where  $v = [c_0, \dots, c_n]$  is a vector of  $\mathbf{FpX}$ , returns  $\sum_{i=0}^n c_i B^i$ .

$\mathbf{GEN\ FpX\_translate}(\mathbf{GEN\ P}, \mathbf{GEN\ c}, \mathbf{GEN\ p})$  let  $c$  be an  $\mathbf{Fp}$  and let  $P$  be an  $\mathbf{FpX}$ ; returns the translated  $\mathbf{FpX}$  of  $P(X + c)$ .

$\mathbf{GEN\ FpX\_gcd}(\mathbf{GEN\ x}, \mathbf{GEN\ y}, \mathbf{GEN\ p})$  returns a (not necessarily monic) greatest common divisor of  $x$  and  $y$ .

$\mathbf{GEN\ FpX\_halfgcd}(\mathbf{GEN\ x}, \mathbf{GEN\ y}, \mathbf{GEN\ p})$  returns a two-by-two  $\mathbf{FpXM}$   $M$  with determinant  $\pm 1$  such that the image  $(a, b)$  of  $(x, y)$  by  $M$  has the property that  $\deg a \geq \frac{\deg x}{2} > \deg b$ .

GEN FpX\_extgcd(GEN x, GEN y, GEN p, GEN \*u, GEN \*v) returns  $d = \text{GCD}(x, y)$  (not necessarily monic), and sets \*u, \*v to the Bezout coefficients such that  $*ux + *vy = d$ . If \*u is set to NULL, it is not computed which is a bit faster. This is useful when computing the inverse of  $y$  modulo  $x$ .

GEN FpX\_center(GEN z, GEN p, GEN pov2) returns the polynomial whose coefficient belong to the symmetric residue system. Assumes the coefficients already belong to  $] -p/2, p[$  and that pov2 is shifti(p, -1).

GEN FpX\_center\_i(GEN z, GEN p, GEN pov2) internal variant of FpX\_center, not gerepile-safe.

GEN FpX\_Frobenius(GEN T, GEN p) returns  $X^p \pmod{T(X)}$ .

GEN FpX\_matFrobenius(GEN T, GEN p) returns the matrix of the Frobenius automorphism  $x \mapsto x^p$  over the power basis of  $\mathbf{F}_p[X]/(T)$ .

**7.3.5.3 Mixed operations.** The following functions implement arithmetic operations between FpX and Fp operands, the result being of type FpX. The integer p need not be prime.

GEN Z\_to\_FpX(GEN x, GEN p, long v) converts a t\_INT to a scalar polynomial in variable v, reduced modulo p.

GEN FpX\_Fp\_add(GEN y, GEN x, GEN p) add the Fp x to the FpX y.

GEN FpX\_Fp\_add\_shallow(GEN y, GEN x, GEN p) add the Fp x to the FpX y, using a shallow copy (result not suitable for gerepileupto)

GEN FpX\_Fp\_sub(GEN y, GEN x, GEN p) subtract the Fp x from the FpX y.

GEN FpX\_Fp\_sub\_shallow(GEN y, GEN x, GEN p) subtract the Fp x from the FpX y, using a shallow copy (result not suitable for gerepileupto)

GEN Fp\_FpX\_sub(GEN x, GEN y, GEN p) returns  $x - y$ , where x is a t\_INT and y an FpX.

GEN FpX\_Fp\_mul(GEN x, GEN y, GEN p) multiplies the FpX x by the Fp y.

GEN FpX\_Fp\_mulspec(GEN x, GEN y, GEN p, long lx) see ZX\_mulspec

GEN FpX\_mulu(GEN x, ulong y, GEN p) multiplies the FpX x by y.

GEN FpX\_Fp\_mul\_to\_monic(GEN y, GEN x, GEN p) returns  $yx$  assuming the result is monic of the same degree as y (in particular  $x \neq 0$ ).

**7.3.5.4 Miscellaneous operations.**

GEN FpX\_normalize(GEN z, GEN p) divides the FpX z by its leading coefficient. If the latter is 1, z itself is returned, not a copy. If not, the inverse remains uncollected on the stack.

GEN FpX\_invBarrett(GEN T, GEN p), returns the Barrett inverse  $M$  of  $T$  defined by  $M(x)x^n \times T(1/x) \equiv 1 \pmod{x^{n-1}}$  where  $n$  is the degree of  $T$ .

GEN FpX\_rescale(GEN P, GEN h, GEN p) returns  $h^{\deg(P)}P(x/h)$ . P is an FpX and h is a non-zero Fp (the routine would work with any non-zero t\_INT but is not efficient in this case).

GEN FpX\_eval(GEN x, GEN y, GEN p) evaluates the FpX x at the Fp y. The result is an Fp.

GEN FpX\_FpV\_multieval(GEN P, GEN v, GEN p) returns the vector  $[P(v[1]), \dots, P(v[n])]$  as a FpV.

GEN FpX\_dotproduct(GEN x, GEN y, GEN p) return the scalar product  $\sum_{i \geq 0} x_i y_i$  of the coefficients of x and y.

GEN FpXV\_FpC\_mul(GEN V, GEN W, GEN p) multiplies a non-empty line vector of FpX by a column vector of Fp of compatible dimensions. The result is an FpX.

GEN FpXV\_prod(GEN V, GEN p), V being a vector of FpX, returns their product.

GEN FpV\_roots\_to\_pol(GEN V, GEN p, long v), V being a vector of INTs, returns the monic FpX  $\prod_i (\text{pol\_x}[v] - V[i])$ .

GEN FpX\_chinese\_coprime(GEN x, GEN y, GEN Tx, GEN Ty, GEN Tz, GEN p): returns an FpX, congruent to x mod Tx and to y mod Ty. Assumes Tx and Ty are coprime, and Tz = Tx \* Ty or NULL (in which case it is computed within).

GEN FpV\_polint(GEN x, GEN y, GEN p, long v) returns the FpX interpolation polynomial with value y[i] at x[i]. Assumes lengths are the same, components are t\_INTs, and the x[i] are distinct modulo p.

GEN FpV\_FpM\_polint(GEN x, GEN V, GEN p, long v) equivalent (but faster) to applying FpV\_polint(x,...) to all the elements of the vector V (thus, returns a FpXV).

GEN FpV\_invVandermonde(GEN L, GEN d, GEN p) L being a FpV of length n, return the inverse M of the Vandermonde matrix attached to the elements of L, eventually multiplied by d if it is not NULL. If A is a FpV and B = MA, then the polynomial  $P = \sum_{i=1}^n B[i]X^{i-1}$  verifies  $P(L[i]) = dA[i]$  for  $1 \leq i \leq n$ .

int FpX\_is\_squarefree(GEN f, GEN p) returns 1 if the FpX f is squarefree, 0 otherwise.

int FpX\_is\_irred(GEN f, GEN p) returns 1 if the FpX f is irreducible, 0 otherwise. Assumes that p is prime. If f has few factors, FpX\_nbfact(f,p) == 1 is much faster.

int FpX\_is\_totally\_split(GEN f, GEN p) returns 1 if the FpX f splits into a product of distinct linear factors, 0 otherwise. Assumes that p is prime.

long FpX\_ispower(GEN f, ulong k, GEN p, GEN \*pt) return 1 if the FpX f is a k-th power, 0 otherwise. If pt is not NULL, set it to g such that  $g^k = f$ .

GEN FpX\_factor(GEN f, GEN p), factors the FpX f. Assumes that p is prime. The returned value v is a t\_VEC with two components: v[1] is a vector of distinct irreducible (FpX) factors, and v[2] is a t\_VECSMALL of corresponding exponents. The order of the factors is deterministic (the computation is not).

GEN FpX\_factor\_squarefree(GEN f, GEN p) returns the squarefree factorization of f modulo p. This is a vector  $[u_1, \dots, u_k]$  of pairwise coprime FpX such that  $u_k \neq 1$  and  $f = \prod u_i$ . Shallow function.

GEN FpX\_ddf(GEN f, GEN p) assuming that f is squarefree, returns the distinct degree factorization of f modulo p. The returned value v is a t\_VEC with two components: F=v[1] is a vector of (FpX) factors, and E=v[2] is a t\_VECSMALL, such that f is equal to the product of the F[i] and each F[i] is a product of irreducible factors of degree E[i].

long FpX\_ddf\_degree(GEN f, GEN XP, GEN p) assuming that f is squarefree and that all its factors have the same degree, return the common degree, where XP is FpX\_Frobenius(f, p).

long FpX\_nbfact(GEN f, GEN p), assuming the FpX f is squarefree, returns the number of its irreducible factors. Assumes that p is prime.

long FpX\_nbfact\_Frobenius(GEN f, GEN XP, GEN p), as FpX\_nbfact(f, p) but faster, where XP is FpX\_Frobenius(f, p).

`GEN FpX_degfact(GEN f, GEN p)`, as `FpX_factor`, but the degrees of the irreducible factors are returned instead of the factors themselves (as a `t_VECSMALL`). Assumes that `p` is prime.

`long FpX_nbroots(GEN f, GEN p)` returns the number of distinct roots in  $\mathbf{Z}/p\mathbf{Z}$  of the `FpX` `f`. Assumes that `p` is prime.

`GEN FpX_oneroot(GEN f, GEN p)` returns one root in  $\mathbf{Z}/p\mathbf{Z}$  of the `FpX` `f`. Return `NULL` if no root exists. Assumes that `p` is prime.

`GEN FpX_oneroot_split(GEN f, GEN p)` as `FpX_oneroot`. Faster when `f` is close to be totally split.

`GEN FpX_roots(GEN f, GEN p)` returns the roots in  $\mathbf{Z}/p\mathbf{Z}$  of the `FpX` `f` (without multiplicity, as a vector of `Fps`). Assumes that `p` is prime.

`GEN FpX_split_part(GEN f, GEN p)` returns the largest totally split squarefree factor of `f`.

`GEN random_FpX(long d, long v, GEN p)` returns a random `FpX` in variable `v`, of degree less than `d`.

`GEN FpX_resultant(GEN x, GEN y, GEN p)` returns the resultant of `x` and `y`, both `FpX`. The result is a `t_INT` belonging to  $[0, p-1]$ .

`GEN FpX_disc(GEN x, GEN p)` returns the discriminant of the `FpX` `x`. The result is a `t_INT` belonging to  $[0, p-1]$ .

`GEN FpX_FpXY_resultant(GEN a, GEN b, GEN p)`, `a` a `t_POL` of `t_INTs` (say in variable `X`), `b` a `t_POL` (say in variable `X`) whose coefficients are either `t_POLs` in  $\mathbf{Z}[Y]$  or `t_INTs`. Returns  $\text{Res}_X(a, b)$  in  $\mathbf{F}_p[Y]$  as an `FpY`. The function assumes that `X` has lower priority than `Y`.

`GEN FpX_Newton(GEN x, long n, GEN p)` return  $\sum i = 0^{n-1} \pi_i X^i$  where  $\pi_i$  is the sum of the  $i$ th-power of the roots of `x` in an algebraic closure.

`GEN FpX_fromNewton(GEN x, GEN p)` recover a polynomial from its Newton sums given by the coefficients of `x`. This function assumes that `p` and the accuracy of `x` as a `FpXn` is larger than the degree of the solution.

`GEN FpX_Laplace(GEN x, GEN p)` return  $\sum i = 0^{n-1} x_i i! X^i$ .

`GEN FpX_invLaplace(GEN x, GEN p)` return  $\sum i = 0^{n-1} x_i / i! X^i$ .

**7.3.6 FpXQ, Fq.** Let `p` a `t_INT` and `T` an `FpX` for `p`, both to be given in the function arguments; an `FpXQ` object is an `FpX` whose degree is strictly less than the degree of `T`. An `Fq` is either an `FpXQ` or an `Fp`. Both represent a class in  $(\mathbf{Z}/p\mathbf{Z})[X]/(T)$ , in which all operations below take place. In addition, `Fq` routines also allow `T = NULL`, in which case no reduction mod `T` is performed on the result.

For efficiency, the routines in this section may leave small unused objects behind on the stack (their output is still suitable for `gerepileupto`). Besides `T` and `p`, arguments are either `FpXQ` or `Fq` depending on the function name. (All `Fq` routines accept `FpXQs` by definition, not the other way round.)

### 7.3.6.1 Preconditioned reduction.

For faster reduction, the modulus  $T$  can be replaced by an extended modulus in all `FpXQ`- and `Fq`-classes functions, and in `FpX_rem` and `FpX_divrem`. An extended modulus (`FpXT`, which is a tree whose leaves are `FpX`) In current implementation, an extended modulus is either a plain modulus (an `FpX`) or a pair of polynomials, one being the plain modulus  $T$  and the other being `FpX_invBarret( $T, p$ )`.

`GEN FpX_get_red(GEN T, GEN p)` returns the extended modulus `eT`.

To write code that works both with plain and extended moduli, the following accessors are defined:

`GEN get_FpX_mod(GEN eT)` returns the underlying modulus  $T$ .

`GEN get_FpX_var(GEN eT)` returns the variable number `varn( $T$ )`.

`GEN get_FpX_degree(GEN eT)` returns the degree `degpol( $T$ )`.

### 7.3.6.2 Conversions.

`GEN Rg_is_FpXQ(GEN z, GEN *T, GEN *p)`, checks if  $z$  is a `GEN` which can be mapped to  $\mathbf{F}_p[X]/(T)$ : anything for which `Rg_is_Fp` return 1, a `t_POL` for which `RgX_to_FpX` return 1, a `t_POLMOD` whose modulus is equal to  $*T$  if  $*T$  is not `NULL` (once mapped to a `FpX`), or a `t_FFELT`  $z$  with the same definition field as  $*T$  if  $*T$  is not `NULL` and is a `t_FFELT`.

If an integer modulus is found it is put in  $*p$ , else  $*p$  is left unchanged. If a polynomial modulus is found it is put in  $*T$ , if a `t_FFELT`  $z$  is found,  $z$  is put in  $*T$ , else  $*T$  is left unchanged.

`int RgX_is_FpXQX(GEN z, GEN *T, GEN *p)`,  $z$  a `t_POL`, checks if it can be mapped to a `FpXQX`, by checking `Rg_is_FpXQ` coefficientwise.

`GEN Rg_to_FpXQ(GEN z, GEN T, GEN p)`,  $z$  a `GEN` which can be mapped to  $\mathbf{F}_p[X]/(T)$ : anything `Rg_to_Fp` can be applied to, a `t_POL` to which `RgX_to_FpX` can be applied to, a `t_POLMOD` whose modulus is divisible by  $T$  (once mapped to a `FpX`), a suitable `t_RFRAC`. Returns  $z$  as an `FpXQ`, normalized.

`GEN Rg_to_Fq(GEN z, GEN T, GEN p)`, applies `Rg_to_Fp` if  $T$  is `NULL` and `Rg_to_FpXQ` otherwise.

`GEN RgX_to_FpXQX(GEN z, GEN T, GEN p)`,  $z$  a `t_POL`, returns the `FpXQ` obtained by applying `Rg_to_FpXQ` coefficientwise.

`GEN RgX_to_FqX(GEN z, GEN T, GEN p)`: let  $z$  be a `t_POL`; returns the `FqX` obtained by applying `Rg_to_Fq` coefficientwise.

`GEN Fq_to_FpXQ(GEN z, GEN T, GEN p /*unused*/)` if  $z$  is a `t_INT`, convert it to a constant polynomial in the variable of  $T$ , otherwise return  $z$  (shallow function).

`GEN Fq_red(GEN x, GEN T, GEN p)`,  $x$  a `ZX` or `t_INT`, reduce it to an `Fq` ( $T = \text{NULL}$  is allowed iff  $x$  is a `t_INT`).

`GEN FqX_red(GEN x, GEN T, GEN p)`,  $x$  a `t_POL` whose coefficients are `ZXs` or `t_INTs`, reduce them to `Fqs`. (If  $T = \text{NULL}$ , as `FpXX_red(x, p)`.)

`GEN FqV_red(GEN x, GEN T, GEN p)`,  $x$  a vector of `ZXs` or `t_INTs`, reduce them to `Fqs`. (If  $T = \text{NULL}$ , only reduce components mod  $p$  to `FpXs` or `Fps`.)

`GEN FpXQ_red(GEN x, GEN T, GEN p)`  $x$  a `t_POL` whose coefficients are `t_INTs`, reduce them to `FpXQs`.

### 7.3.7 FpXQ.

GEN FpXQ\_add(GEN x, GEN y, GEN T, GEN p)

GEN FpXQ\_sub(GEN x, GEN y, GEN T, GEN p)

GEN FpXQ\_mul(GEN x, GEN y, GEN T, GEN p)

GEN FpXQ\_sqr(GEN x, GEN T, GEN p)

GEN FpXQ\_div(GEN x, GEN y, GEN T, GEN p)

GEN FpXQ\_inv(GEN x, GEN T, GEN p) computes the inverse of x

GEN FpXQ\_invsafe(GEN x, GEN T, GEN p), as FpXQ\_inv, returning NULL if x is not invertible.

GEN FpXQ\_pow(GEN x, GEN n, GEN T, GEN p) computes  $x^n$ .

GEN FpXQ\_powu(GEN x, ulong n, GEN T, GEN p) computes  $x^n$  for small n.

In the following three functions the integer parameter `ord` can be given either as a positive  $t\_INT$   $N$ , or as its factorization matrix  $faN$ , or as a pair  $[N, faN]$ . The parameter may be omitted by setting it to NULL (the value is then  $p^d - 1$ ,  $d = \deg T$ ).

GEN FpXQ\_log(GEN a, GEN g, GEN ord, GEN T, GEN p) Let  $g$  be of order `ord` in the finite field  $\mathbf{F}_p[X]/(T)$ , return  $e$  such that  $a^e = g$ . If  $e$  does not exists, the result is undefined. Assumes that  $T$  is irreducible mod  $p$ .

GEN Fp\_FpXQ\_log(GEN a, GEN g, GEN ord, GEN T, GEN p) As FpXQ\_log,  $a$  being a Fp.

GEN FpXQ\_order(GEN a, GEN ord, GEN T, GEN p) returns the order of the FpXQ  $a$ . Assume that `ord` is a multiple of the order of  $a$ . Assume that  $T$  is irreducible mod  $p$ .

int FpXQ\_issquare(GEN x, GEN T, GEN p) returns 1 if  $x$  is a square and 0 otherwise. Assumes that  $T$  is irreducible mod  $p$ .

GEN FpXQ\_sqrt(GEN x, GEN T, GEN p) returns a square root of  $x$ . Return NULL if  $x$  is not a square.

GEN FpXQ\_sqrtn(GEN x, GEN n, GEN T, GEN p, GEN \*zn) Let  $T$  be irreducible mod  $p$  and  $q = p^{\deg T}$ ; returns NULL if  $a$  is not an  $n$ -th power residue mod  $p$ . Otherwise, returns an  $n$ -th root of  $a$ ; if `zn` is non-NULL set it to a primitive  $m$ -th root of 1 in  $\mathbf{F}_q$ ,  $m = \gcd(q - 1, n)$  allowing to compute all  $m$  solutions in  $\mathbf{F}_q$  of the equation  $x^n = a$ .

### 7.3.8 Fq.

GEN Fq\_add(GEN x, GEN y, GEN T/\*unused\*/, GEN p)

GEN Fq\_sub(GEN x, GEN y, GEN T/\*unused\*/, GEN p)

GEN Fq\_mul(GEN x, GEN y, GEN T, GEN p)

GEN Fq\_Fp\_mul(GEN x, GEN y, GEN T, GEN p) multiplies the Fq  $x$  by the  $t\_INT$   $y$ .

GEN Fq\_mulu(GEN x, ulong y, GEN T, GEN p) multiplies the Fq  $x$  by the scalar  $y$ .

GEN Fq\_halve(GEN x, GEN T, GEN p) returns  $z$  such that  $2z = x$  assuming such  $z$  exists.

GEN Fq\_sqr(GEN x, GEN T, GEN p)

GEN Fq\_neg(GEN x, GEN T, GEN p)



`GEN Fq_neg_inv(GEN x, GEN T, GEN p)` computes  $-x^{-1}$   
`GEN Fq_inv(GEN x, GEN pol, GEN p)` computes  $x^{-1}$ , raising an error if  $x$  is not invertible.  
`GEN Fq_invsafe(GEN x, GEN pol, GEN p)` as `Fq_inv`, but returns NULL if  $x$  is not invertible.  
`GEN Fq_div(GEN x, GEN y, GEN T, GEN p)`  
`GEN FqV_inv(GEN x, GEN T, GEN p)`  $x$  being a vector of Fqs, return the vector of inverses of the  $x[i]$ . The routine uses Montgomery's trick, and involves a single inversion, plus  $3(N - 1)$  multiplications for  $N$  entries. The routine is not stack-clean:  $2N$  FpXQ are left on stack, besides the  $N$  in the result.  
`GEN Fq_pow(GEN x, GEN n, GEN pol, GEN p)` returns  $x^n$ .  
`GEN Fq_powu(GEN x, ulong n, GEN pol, GEN p)` returns  $x^n$  for small  $n$ .  
`GEN Fq_log(GEN a, GEN g, GEN ord, GEN T, GEN p)` as `Fp_log` or `FpXQ_log`.  
`int Fq_issquare(GEN x, GEN T, GEN p)` returns 1 if  $x$  is a square and 0 otherwise. Assumes that  $T$  is irreducible mod  $p$  and that  $p$  is prime;  $T = \text{NULL}$  is forbidden unless  $x$  is an Fp.  
`long Fq_ispower(GEN x, GEN n, GEN T, GEN p)` returns 1 if  $x$  is a  $n$ -th power and 0 otherwise. Assumes that  $T$  is irreducible mod  $p$  and that  $p$  is prime;  $T = \text{NULL}$  is forbidden unless  $x$  is an Fp.  
`GEN Fq_sqrt(GEN x, GEN T, GEN p)` returns a square root of  $x$ . Return NULL if  $x$  is not a square.  
`GEN Fq_sqrtn(GEN a, GEN n, GEN T, GEN p, GEN *zn)` as `FpXQ_sqrtn`.  
`GEN FpXQ_charpoly(GEN x, GEN T, GEN p)` returns the characteristic polynomial of  $x$   
`GEN FpXQ_minpoly(GEN x, GEN T, GEN p)` returns the minimal polynomial of  $x$   
`GEN FpXQ_norm(GEN x, GEN T, GEN p)` returns the norm of  $x$   
`GEN FpXQ_trace(GEN x, GEN T, GEN p)` returns the trace of  $x$   
`GEN FpXQ_conjvec(GEN x, GEN T, GEN p)` returns the vector of conjugates  $[x, x^p, x^{p^2}, \dots, x^{p^{n-1}}]$  where  $n$  is the degree of  $T$ .  
`GEN gener_FpXQ(GEN T, GEN p, GEN *po)` returns a primitive root modulo  $(T, p)$ .  $T$  is an FpX assumed to be irreducible modulo the prime  $p$ . If  $po$  is not NULL it is set to  $[o, fa]$ , where  $o$  is the order of the multiplicative group of the finite field, and  $fa$  is its factorization.  
`GEN gener_FpXQ_local(GEN T, GEN p, GEN L)`,  $L$  being a vector of primes dividing  $p^{\deg T} - 1$ , returns an element of  $G := \mathbf{F}_p[x]/(T)$  which is a generator of the  $\ell$ -Sylow of  $G$  for every  $\ell$  in  $L$ . It is not necessary, and in fact slightly inefficient, to include  $\ell = 2$ , since 2 is treated separately in any case, i.e. the generator obtained is never a square if  $p$  is odd.  
`GEN gener_Fq_local(GEN T, GEN p, GEN L)` as `pgener_Fp_local(p, L)` if  $T$  is NULL, or `gener_FpXQ_local` (otherwise).  
`GEN FpXQ_powers(GEN x, long n, GEN T, GEN p)` returns  $[x^0, \dots, x^n]$  as a `t_VEC` of FpXQs.  
`GEN FpXQ_matrix_pow(GEN x, long m, long n, GEN T, GEN p)`, as `FpXQ_powers(x, n - 1, T, p)`, but returns the powers as a  $m \times n$  matrix. Usually, we have  $m = n = \deg T$ .  
`GEN FpXQ_autpow(GEN a, ulong n, GEN T, GEN p)` computes  $\sigma^n(X)$  assuming  $a = \sigma(X)$  where  $\sigma$  is an automorphism of the algebra  $\mathbf{F}_p[X]/T(X)$ .

`GEN FpXQ_autsum(GEN a, ulong n, GEN T, GEN p)`  $a$  being a two-component vector,  $\sigma$  being the automorphism defined by  $\sigma(X) = a[1] \pmod{T(X)}$ , returns the vector  $[\sigma^n(X), b\sigma(b) \dots \sigma^{n-1}(b)]$  where  $b = a[2]$ .

`GEN FpXQ_auttrace(GEN a, ulong n, GEN T, GEN p)`  $a$  being a two-component vector,  $\sigma$  being the automorphism defined by  $\sigma(X) = a[1] \pmod{T(X)}$ , returns the vector  $[\sigma^n(X), b + \sigma(b) + \dots + \sigma^{n-1}(b)]$  where  $b = a[2]$ .

`GEN FpXQ_outpowers(GEN S, long n, GEN T, GEN p)` returns  $[x, S(x), S(S(x)), \dots, S^{(n)}(x)]$  as a `t_VEC` of `FpXQs`.

`GEN FpXQM_autsum(GEN a, long n, GEN T, GEN p)`  $\sigma$  being the automorphism defined by  $\sigma(X) = a[1] \pmod{T(X)}$ , returns the vector  $[\sigma^n(X), b\sigma(b) \dots \sigma^{n-1}(b)]$  where  $b = a[2]$  is a square matrix.

`GEN FpX_FpXQ_eval(GEN f, GEN x, GEN T, GEN p)` returns  $f(x)$ .

`GEN FpX_FpXQV_eval(GEN f, GEN V, GEN T, GEN p)` returns  $f(x)$ , assuming that  $V$  was computed by `FpXQ_powers(x, n, T, p)`.

`GEN FpXC_FpXQV_eval(GEN C, GEN V, GEN T, GEN p)` applies `FpX_FpXQV_eval` to all elements of the vector  $C$  and returns a `t_COL`.

`GEN FpXM_FpXQV_eval(GEN M, GEN V, GEN T, GEN p)` applies `FpX_FpXQV_eval` to all elements of the matrix  $M$ .

**7.3.9 FpXn.** Let  $p$  a `t_INT` and  $T$  an `FpX` for  $p$ , both to be given in the function arguments; an `FpXn` object is an `FpX` whose degree is strictly less than  $n$ . They represent a class in  $(\mathbf{Z}/p\mathbf{Z})[X]/(X^n)$ , in which all operations below take place. They can be seen as truncated power series.

`GEN FpXn_mul(GEN x, GEN y, long n, GEN p)` return  $xy \pmod{X^n}$ .

`GEN FpXn_sqr(GEN x, long n, GEN p)` return  $x^2 \pmod{X^n}$ .

`GEN FpXn_inv(GEN x, long n, GEN p)` return  $1/x \pmod{X^n}$ .

`GEN FpXn_exp(GEN x, long n, GEN p)` return  $\exp(x)$  as a composition of formal power series. It is required that the valuation of  $x$  is positive and that  $p > n$ .

**7.3.10 FpXC, FpXM.**

`GEN FpXC_center(GEN C, GEN p, GEN pov2)`

`GEN FpXM_center(GEN M, GEN p, GEN pov2)`

**7.3.11 FpXX, FpXY.** Contrary to what the name implies, an FpXX is a  $\mathbf{t\_POL}$  whose coefficients are either  $\mathbf{t\_INTs}$  or FpXs. This reduces memory overhead at the expense of consistency. The prefix FpXY is an alias for FpXX when variables matters.

GEN FpXX\_red(GEN z, GEN p), z a  $\mathbf{t\_POL}$  whose coefficients are either ZXs or  $\mathbf{t\_INTs}$ . Returns the  $\mathbf{t\_POL}$  equal to z with all components reduced modulo p.

GEN FpXX\_renormalize(GEN x, long l), as `normalizepol`, where  $l = \lg(x)$ , in place.

GEN FpXX\_add(GEN x, GEN y, GEN p) adds x and y.

GEN FpXX\_sub(GEN x, GEN y, GEN p) returns  $x - y$ .

GEN FpXX\_neg(GEN x, GEN p) returns  $-x$ .

GEN FpXX\_Fp\_mul(GEN x, GEN y, GEN p) multiplies the FpXX x by the Fp y.

GEN FpXX\_FpX\_mul(GEN x, GEN y, GEN p) multiplies the coefficients of the FpXX x by the FpX y.

GEN FpXX\_mulu(GEN x, GEN y, GEN p) multiplies the FpXX x by the scalar y.

GEN FpXX\_halve(GEN x, GEN p) returns z such that  $2z = x$  assuming such z exists.

GEN FpXX\_deriv(GEN P, GEN p) differentiates P with respect to the main variable.

GEN FpXX\_integ(GEN P, GEN p) returns the primitive of P with respect to the main variable whose constant term is 0.

GEN FpXY\_eval(GEN Q, GEN y, GEN x, GEN p) Q being an FpXY, i.e. a  $\mathbf{t\_POL}$  with Fp or FpX coefficients representing an element of  $\mathbf{F}_p[X][Y]$ . Returns the Fp  $Q(x, y)$ .

GEN FpXY\_evalx(GEN Q, GEN x, GEN p) Q being an FpXY, returns the FpX  $Q(x, Y)$ , where Y is the main variable of Q.

GEN FpXY\_evaly(GEN Q, GEN y, GEN p, long vx) Q an FpXY, returns the FpX  $Q(X, y)$ , where X is the second variable of Q, and vx is the variable number of X.

GEN FpXY\_Fq\_evaly(GEN Q, GEN y, GEN T, GEN p, long vx) Q an FpXY and y being an Fq, returns the FqX  $Q(X, y)$ , where X is the second variable of Q, and vx is the variable number of X.

GEN FpXY\_FpXQ\_evalx(GEN Q, GEN x, ulong p) Q an FpXY and x being an FpXQ, returns the FpXQX  $Q(x, Y)$ , where Y is the first variable of Q.

GEN FpXY\_FpXQV\_evalx(GEN Q, GEN V, ulong p) Q an FpXY and x being an FpXQ, returns the FpXQX  $Q(x, Y)$ , where Y is the first variable of Q, assuming that V was computed by `FpXQ_powers(x, n, T, p)`.

GEN FpXYQQ\_pow(GEN x, GEN n, GEN S, GEN T, GEN p), x being a FpXY, T being a FpX and S being a FpY, return  $x^n \pmod{S, T, p}$ .

**7.3.12 FpXQX, FqX.** Contrary to what the name implies, an FpXQX is a  $\mathbf{t\_POL}$  whose coefficients are Fqs. So the only difference between FqX and FpXQX routines is that  $T = \mathbf{NULL}$  is not allowed in the latter. (It was thought more useful to allow  $\mathbf{t\_INT}$  components than to enforce strict consistency, which would not imply any efficiency gain.)

### 7.3.12.1 Basic operations.

GEN FqX\_add(GEN x, GEN y, GEN T, GEN p)

GEN FqX\_Fq\_add(GEN x, GEN y, GEN T, GEN p) adds the Fq y to the FqX x.

GEN FqX\_Fq\_sub(GEN x, GEN y, GEN T, GEN p) subtracts the Fq y to the FqX x.

GEN FqX\_neg(GEN x, GEN T, GEN p)

GEN FqX\_sub(GEN x, GEN y, GEN T, GEN p)

GEN FqX\_mul(GEN x, GEN y, GEN T, GEN p)

GEN FqX\_Fq\_mul(GEN x, GEN y, GEN T, GEN p) multiplies the FqX x by the Fq y.

GEN FqX\_mulu(GEN x, ulong y, GEN T, GEN p) multiplies the FqX x by the scalar y.

GEN FqX\_half(GEN x, GEN T, GEN p) returns  $z$  such that  $2z = x$  assuming such  $z$  exists.

GEN FqX\_Fp\_mul(GEN x, GEN y, GEN T, GEN p) multiplies the FqX x by the  $t\_INT$  y.

GEN FqX\_Fq\_mul\_to\_monic(GEN x, GEN y, GEN T, GEN p) returns  $xy$  assuming the result is monic of the same degree as  $x$  (in particular  $y \neq 0$ ).

GEN FpXQX\_normalize(GEN z, GEN T, GEN p)

GEN FqX\_normalize(GEN z, GEN T, GEN p) divides the FqX z by its leading term. The leading coefficient becomes 1 as a  $t\_INT$ .

GEN FqX\_sqr(GEN x, GEN T, GEN p)

GEN FqX\_powu(GEN x, ulong n, GEN T, GEN p)

GEN FqX\_divrem(GEN x, GEN y, GEN T, GEN p, GEN \*z)

GEN FqX\_div(GEN x, GEN y, GEN T, GEN p)

GEN FqX\_div\_by\_X\_x(GEN a, GEN x, GEN T, GEN p, GEN \*r)

GEN FqX\_rem(GEN x, GEN y, GEN T, GEN p)

GEN FqX\_deriv(GEN x, GEN T, GEN p) returns the derivative of x. (This function is suitable for gerepilupto but not memory-clean.)

GEN FqX\_integ(GEN x, GEN T, GEN p) returns the primitive of x. whose constant term is 0.

GEN FqX\_translate(GEN P, GEN c, GEN T, GEN p) let  $c$  be an Fq defined modulo  $(p, T)$ , and let  $P$  be an FqX; returns the translated FqX of  $P(X + c)$ .

GEN FqX\_gcd(GEN P, GEN Q, GEN T, GEN p) returns a (not necessarily monic) greatest common divisor of  $x$  and  $y$ .

GEN FqX\_extgcd(GEN x, GEN y, GEN T, GEN p, GEN \*ptu, GEN \*ptv) returns  $d = \text{GCD}(x, y)$  (not necessarily monic), and sets  $*u, *v$  to the Bezout coefficients such that  $*ux + *vy = d$ .

GEN FqX\_halfgcd(GEN x, GEN y, GEN T, GEN p) returns a two-by-two FqXM  $M$  with determinant  $\pm 1$  such that the image  $(a, b)$  of  $(x, y)$  by  $M$  has the property that  $\deg a \geq \frac{\deg x}{2} > \deg b$ .

GEN FqX\_eval(GEN x, GEN y, GEN T, GEN p) evaluates the FqX x at the Fq y. The result is an Fq.

`GEN FqXY_eval(GEN Q, GEN y, GEN x, GEN T, GEN p)`  $Q$  an  $F_qXY$ , i.e. a  $t\_POL$  with  $F_q$  or  $F_qX$  coefficients representing an element of  $\mathbf{F}_q[X][Y]$ . Returns the  $F_q$   $Q(x, y)$ .

`GEN FqXY_evalx(GEN Q, GEN x, GEN T, GEN p)`  $Q$  being an  $F_qXY$ , returns the  $F_qX$   $Q(x, Y)$ , where  $Y$  is the main variable of  $Q$ .

`GEN random_FpXQX(long d, long v, GEN T, GEN p)` returns a random  $FpXQX$  in variable  $v$ , of degree less than  $d$ .

`GEN FpXQX_renormalize(GEN x, long lx)`

`GEN FpXQX_red(GEN z, GEN T, GEN p)`  $z$  a  $t\_POL$  whose coefficients are  $ZXs$  or  $t\_INTs$ , reduce them to  $FpXQs$ .

`GEN FpXQX_to_mod(GEN P, GEN T, GEN p)`  $P$  being a  $FpXQX$ , converts each coefficient to a  $t\_POLMOD$  with  $t\_INTMOD$  coefficients.

`GEN FqX_to_mod(GEN P, GEN T, GEN p)` same but allow  $T = NULL$ .

`GEN FpXQX_mul(GEN x, GEN y, GEN T, GEN p)`

`GEN Kronecker_to_FpXQX(GEN z, GEN T, GEN p)`. Let  $n = \deg T$  and let  $P(X, Y) \in \mathbf{Z}[X, Y]$  lift a polynomial in  $K[Y]$ , where  $K := \mathbf{F}_p[X]/(T)$  and  $\deg_X P < 2n - 1$  — such as would result from multiplying minimal degree lifts of two polynomials in  $K[Y]$ . Let  $z = P(t, t^{2^{*n}-1})$  be a Kronecker form of  $P$ , this function returns  $Q \in \mathbf{Z}[X, t]$  such that  $Q$  is congruent to  $P(X, t) \bmod (p, T(X))$ ,  $\deg_X Q < n$ , and all coefficients are in  $[0, p[$ . Not stack-clean. Note that  $t$  need not be the same variable as  $Y$ !

`GEN FpXQX_FpXQ_mul(GEN x, GEN y, GEN T, GEN p)`

`GEN FpXQX_sqr(GEN x, GEN T, GEN p)`

`GEN FpXQX_divrem(GEN x, GEN y, GEN T, GEN p, GEN *pr)`

`GEN FpXQX_div(GEN x, GEN y, GEN T, GEN p)`

`GEN FpXQX_div_by_X_x(GEN a, GEN x, GEN T, GEN p, GEN *r)`

`GEN FpXQX_rem(GEN x, GEN y, GEN T, GEN p)`

`GEN FpXQX_powu(GEN x, ulong n, GEN T, GEN p)` returns  $x^n$ .

`GEN FpXQX_digits(GEN x, GEN B, GEN T, GEN p)`

`GEN FpXQX_dotproduct(GEN x, GEN y, GEN T, GEN p)` returns the scalar product of the coefficients of  $x$  and  $y$ .

`GEN FpXQXV_FpXQX_fromdigits(GEN v, GEN B, GEN T, GEN p)`

`GEN FpXQX_invBarrett(GEN y, GEN T, GEN p)` returns the Barrett inverse of the  $FpXQX$   $y$ , namely a lift of  $1/\text{polrecip}(y) + O(x^{\deg(y)-1})$ .

`GEN FpXQXV_prod(GEN V, GEN T, GEN p)`,  $V$  being a vector of  $FpXQX$ , returns their product.

`GEN FpXQX_gcd(GEN x, GEN y, GEN T, GEN p)`

`GEN FpXQX_extgcd(GEN x, GEN y, GEN T, GEN p, GEN *ptu, GEN *ptv)`

`GEN FpXQX_halfgcd(GEN x, GEN y, GEN T, GEN p)`

`GEN FpXQX_FpXQXQ_eval(GEN f, GEN x, GEN S, GEN T, GEN p)` returns  $f(x)$ .

### 7.3.13 FpXQXn, FqXn.

A FpXQXn is a t\_FpXQX which represents an element of the ring  $(Fp[X]/T(X))[Y]/(Y^n)$ , where  $T$  is a FpX.

GEN FpXQXn\_sqr(GEN x, long n, GEN T, GEN p)

GEN FqXn\_sqr(GEN x, long n, GEN T, GEN p)

GEN FpXQXn\_mul(GEN x, GEN y, long n, GEN T, GEN p)

GEN FqXn\_mul(GEN x, GEN y, long n, GEN T, GEN p)

GEN FpXQXn\_inv(GEN x, long n, GEN T, GEN p)

GEN FqXn\_inv(GEN x, long n, GEN T, GEN p)

GEN FpXQXn\_exp(GEN x, long n, GEN T, GEN p) return  $\exp(x)$  as a composition of formal power series. It is required that the valuation of  $x$  is positive and that  $p > n$ .

GEN FqXn\_exp(GEN x, long n, GEN T, GEN p)

### 7.3.14 FpXQXQ, FqXQ.

A FpXQXQ is a t\_FpXQX which represents an element of the ring  $(Fp[X]/T(X))[Y]/S(X, Y)$ , where  $T$  is a FpX and  $S$  a FpXQX modulo  $T$ . A FqXQ is identical except that  $T$  is allowed to be NULL in which case  $S$  must be a FpX.

#### 7.3.14.1 Preconditioned reduction.

For faster reduction, the modulus  $S$  can be replaced by an extended modulus, which is an FpXQXT, in all FpXQXQ- and FqXQ-classes functions, and in FpXQX\_rem and FpXQX\_divrem.

GEN FpXQX\_get\_red(GEN S, GEN T, GEN p) returns the extended modulus eS.

GEN FqX\_get\_red(GEN S, GEN T, GEN p) identical, but allow  $T$  to be NULL, in which case it returns FpX\_get\_red(S, p).

To write code that works both with plain and extended moduli, the following accessors are defined:

GEN get\_FpXQX\_mod(GEN eS) returns the underlying modulus  $S$ .

GEN get\_FpXQX\_var(GEN eS) returns the variable number of the modulus.

GEN get\_FpXQX\_degree(GEN eS) returns the degree of the modulus.

Furthermore, ZXXT\_to\_FlxXT allows to convert an extended modulus for a FpXQX to an extended modulus for the corresponding FlxqX.

### 7.3.14.2 basic operations.

GEN FpXQX\_FpXQXQV\_eval(GEN f, GEN V, GEN S, GEN T, GEN p) returns  $f(x)$ , assuming that V was computed by FpXQXQ\_powers(x, n, S, T, p).

GEN FpXQXQ\_div(GEN x, GEN y, GEN S, GEN T, GEN p), x, y and S being FpXQXs, returns  $x * y^{-1}$  modulo S.

GEN FpXQXQ\_inv(GEN x, GEN S, GEN T, GEN p), x and S being FpXQXs, returns  $x^{-1}$  modulo S.

GEN FpXQXQ\_invsafe(GEN x, GEN S, GEN T, GEN p), as FpXQXQ\_inv, returning NULL if x is not invertible.

GEN FpXQXQ\_mul(GEN x, GEN y, GEN S, GEN T, GEN p), x, y and S being FpXQXs, returns  $xy$  modulo S.

GEN FpXQXQ\_sqr(GEN x, GEN S, GEN T, GEN p), x and S being FpXQXs, returns  $x^2$  modulo S.

GEN FpXQXQ\_pow(GEN x, GEN n, GEN S, GEN T, GEN p), x and S being FpXQXs, returns  $x^n$  modulo S.

GEN FpXQXQ\_powers(GEN x, long n, GEN S, GEN T, GEN p), x and S being FpXQXs, returns  $[x^0, \dots, x^n]$  as a t\_VEC of FpXQXs.

GEN FpXQXQ\_halfFrobenius(GEN A, GEN S, GEN T, GEN p) returns  $A(X)^{(q-1)/2} \pmod{S(X)}$  over the finite field  $\mathbf{F}_q$  defined by T and p, thus  $q = p^n$  where n is the degree of T.

GEN FpXQXQ\_minpoly(GEN x, GEN S, GEN T, GEN p), as FpXQ\_minpoly

GEN FpXQXQ\_matrix\_pow(GEN x, long m, long n, GEN S, GEN T, GEN p) returns the same powers of x as FpXQXQ\_powers(x, n - 1, S, T, p), but as an  $m \times n$  matrix.

GEN FpXQXQ\_autpow(GEN a, long n, GEN S, GEN T, GEN p)  $\sigma$  being the automorphism defined by  $\sigma(X) = a[1] \pmod{T(X)}$ ,  $\sigma(Y) = a[2] \pmod{S(X, Y), T(X)}$ , returns  $[\sigma^n(X), \sigma^n(Y)]$ .

GEN FpXQXQ\_autsum(GEN a, long n, GEN S, GEN T, GEN p)  $\sigma$  being the automorphism defined by  $\sigma(X) = a[1] \pmod{T(X)}$ ,  $\sigma(Y) = a[2] \pmod{S(X, Y), T(X)}$ , returns the vector  $[\sigma^n(X), \sigma^n(Y), b\sigma(b) \dots \sigma^{n-1}(b)]$  where  $b = a[3]$ .

GEN FpXQXQ\_auttrace(GEN a, long n, GEN S, GEN T, GEN p)  $\sigma$  being the automorphism defined by  $\sigma(X) = X \pmod{T(X)}$ ,  $\sigma(Y) = a[1] \pmod{S(X, Y), T(X)}$ , returns the vector  $[\sigma^n(X), \sigma^n(Y), b + \sigma(b) + \dots + \sigma^{n-1}(b)]$  where  $b = a[2]$ .

GEN FqXQ\_add(GEN x, GEN y, GEN S, GEN T, GEN p), x, y and S being FqXs, returns  $x + y$  modulo S.

GEN FqXQ\_sub(GEN x, GEN y, GEN S, GEN T, GEN p), x, y and S being FqXs, returns  $x - y$  modulo S.

GEN FqXQ\_mul(GEN x, GEN y, GEN S, GEN T, GEN p), x, y and S being FqXs, returns  $xy$  modulo S.

GEN FqXQ\_div(GEN x, GEN y, GEN S, GEN T, GEN p), x and S being FqXs, returns  $x/y$  modulo S.

GEN FqXQ\_inv(GEN x, GEN S, GEN T, GEN p), x and S being FqXs, returns  $x^{-1}$  modulo S.

GEN FqXQ\_invsafe(GEN x, GEN S, GEN T, GEN p), as FqXQ\_inv, returning NULL if x is not invertible.

GEN FqXQ\_sqr(GEN x, GEN S, GEN T, GEN p), x and S being FqXs, returns  $x^2$  modulo S.

GEN FqXQ\_pow(GEN x, GEN n, GEN S, GEN T, GEN p), x and S being FqXs, returns  $x^n$  modulo S.

GEN FqXQ\_powers(GEN x, long n, GEN S, GEN T, GEN p), x and S being FqXs, returns  $[x^0, \dots, x^n]$  as a t\_VEC of FqXs.

GEN FqXQ\_matrix\_pow(GEN x, long m, long n, GEN S, GEN T, GEN p) returns the same powers of x as FqXQ\_powers(x, n-1, S, T, p), but as an  $m \times n$  matrix.

GEN FqV\_roots\_to\_pol(GEN V, GEN T, GEN p, long v), V being a vector of Fqs, returns the monic FqX  $\prod_i (\text{pol\_x}[v] - V[i])$ .

### 7.3.14.3 Miscellaneous operations.

GEN init\_Fq(GEN p, long n, long v) returns an irreducible polynomial of degree n > 0 over  $\mathbf{F}_p$ , in variable v.

int FqX\_is\_squarefree(GEN P, GEN T, GEN p)

GEN FpXQX\_roots(GEN f, GEN T, GEN p) return the roots of f in  $\mathbf{F}_p[X]/(T)$ . Assumes p is prime and T irreducible in  $\mathbf{F}_p[X]$ .

GEN FqX\_roots(GEN f, GEN T, GEN p) same but allow T = NULL.

GEN FpXQX\_factor(GEN f, GEN T, GEN p) same output convention as FpX\_factor. Assumes p is prime and T irreducible in  $\mathbf{F}_p[X]$ .

GEN FqX\_factor(GEN f, GEN T, GEN p) same but allow T = NULL.

GEN FpXQX\_factor\_squarefree(GEN f, GEN T, GEN p) squarefree factorization of f modulo (T, p); same output convention as FpX\_factor\_squarefree. Assumes p is prime and T irreducible in  $\mathbf{F}_p[X]$ .

GEN FqX\_factor\_squarefree(GEN f, GEN T, GEN p) same but allow T = NULL.

GEN FpXQX\_ddf(GEN f, GEN T, GEN p) as FpX\_ddf.

GEN FqX\_ddf(GEN f, GEN T, GEN p) same but allow T = NULL.

long FpXQX\_ddf\_degree(GEN f, GEN XP, GEN T, GEN p), as FpX\_ddf\_degree.

GEN FpXQX\_degfact(GEN f, GEN T, GEN p), as FpX\_degfact.

GEN FqX\_degfact(GEN f, GEN T, GEN p) same but allow T = NULL.

GEN FpXQX\_split\_part(GEN f, GEN T, GEN p) returns the largest totally split squarefree factor of f.

long FpXQX\_ispower(GEN f, ulong k, GEN T, GEN p, GEN \*pt) return 1 if the FpXQX f is a k-th power, 0 otherwise. If pt is not NULL, set it to g such that  $g^k = f$ .

long FqX\_ispower(GEN f, ulong k, GEN T, GEN p, GEN \*pt) same but allow T = NULL.

GEN FpX\_factorff(GEN P, GEN T, GEN p). Assumes p prime and T irreducible in  $\mathbf{F}_p[X]$ . Factor the FpX P over the finite field  $\mathbf{F}_p[Y]/(T(Y))$ . See FpX\_factorff\_irred if P is known to be irreducible of  $\mathbf{F}_p$ .

GEN FpX\_rootsff(GEN P, GEN T, GEN p). Assumes p prime and T irreducible in  $\mathbf{F}_p[X]$ . Returns the roots of the FpX P belonging to the finite field  $\mathbf{F}_p[Y]/(T(Y))$ .



`GEN FpX_factorff_irred(GEN P, GEN T, GEN p)`. Assumes  $p$  prime and  $T$  irreducible in  $\mathbf{F}_p[X]$ . Factors the *irreducible*  $\mathbf{F}_p[X]$   $P$  over the finite field  $\mathbf{F}_p[Y]/(T(Y))$  and returns the vector of irreducible  $\mathbf{F}_q[X]$ s factors (the exponents, being all equal to 1, are not included).

`GEN FpX_ffisom(GEN P, GEN Q, GEN p)`. Assumes  $p$  prime,  $P, Q$  are ZXs, both irreducible mod  $p$ , and  $\deg(P) \mid \deg(Q)$ . Outputs a monomorphism between  $\mathbf{F}_p[X]/(P)$  and  $\mathbf{F}_p[X]/(Q)$ , as a polynomial  $R$  such that  $Q \mid P(R)$  in  $\mathbf{F}_p[X]$ . If  $P$  and  $Q$  have the same degree, it is of course an isomorphism.

`void FpX_ffintersect(GEN P, GEN Q, long n, GEN p, GEN *SP, GEN *SQ, GEN MA, GEN MB)`  
Assumes  $p$  is prime,  $P, Q$  are ZXs, both irreducible mod  $p$ , and  $n$  divides both the degree of  $P$  and  $Q$ . Compute  $SP$  and  $SQ$  such that the subfield of  $\mathbf{F}_p[X]/(P)$  generated by  $SP$  and the subfield of  $\mathbf{F}_p[X]/(Q)$  generated by  $SQ$  are isomorphic of degree  $n$ . The polynomials  $P$  and  $Q$  do not need to be of the same variable. If  $MA$  (resp.  $MB$ ) is not NULL, it must be the matrix of the Frobenius map in  $\mathbf{F}_p[X]/(P)$  (resp.  $\mathbf{F}_p[X]/(Q)$ ).

`GEN FpXQ_ffisom_inv(GEN S, GEN T, GEN p)`. Assumes  $p$  is prime,  $T$  a ZX, which is irreducible modulo  $p$ ,  $S$  a ZX representing an automorphism of  $\mathbf{F}_q := \mathbf{F}_p[X]/(T)$ . ( $S(X)$  is the image of  $X$  by the automorphism.) Returns the inverse automorphism of  $S$ , in the same format, i.e. an  $\mathbf{F}_p[X]$   $H$  such that  $H(S) \equiv X$  modulo  $(T, p)$ .

`long FpXQX_nbfact(GEN S, GEN T, GEN p)` returns the number of irreducible factors of the polynomial  $S$  over the finite field  $\mathbf{F}_q$  defined by  $T$  and  $p$ .

`long FpXQX_nbfact_Frobenius(GEN S, GEN Xq, GEN T, GEN p)` as `FpXQX_nbfact` where  $Xq$  is `FpXQX_Frobenius(S, T, p)`.

`long FqX_nbfact(GEN S, GEN T, GEN p)` as above but accept  $T=NULL$ .

`long FpXQX_nbroots(GEN S, GEN T, GEN p)` returns the number of roots of the polynomial  $S$  over the finite field  $\mathbf{F}_q$  defined by  $T$  and  $p$ .

`long FqX_nbroots(GEN S, GEN T, GEN p)` as above but accept  $T=NULL$ .

`GEN FpXQX_Frobenius(GEN S, GEN T, GEN p)` returns  $X^q \pmod{S(X)}$  over the finite field  $\mathbf{F}_q$  defined by  $T$  and  $p$ , thus  $q = p^n$  where  $n$  is the degree of  $T$ .

**7.3.15 Flx.** Let  $p$  an understood `ulong`, assumed to be prime, to be given the function arguments; an  $\mathbf{Fl}$  is an `ulong` belonging to  $[0, p-1]$ , an  $\mathbf{Flx}$   $z$  is a `t_VECSMALL` representing a polynomial with small integer coefficients. Specifically  $z[0]$  is the usual codeword,  $z[1] = \text{evalvarn}(v)$  for some variable  $v$ , then the coefficients by increasing degree. An  $\mathbf{FlxX}$  is a `t_POL` whose coefficients are  $\mathbf{Flx}$ s.

In the following, an argument called `sv` is of the form `evalvarn(v)` for some variable number  $v$ .

#### 7.3.15.1 Preconditioned reduction.

For faster reduction, the modulus  $T$  can be replaced by an extended modulus ( $\mathbf{FlxT}$ ) in all  $\mathbf{Flxq}$ -classes functions, and in `Flx_divrem`.

`GEN Flx_get_red(GEN T, ulong p)` returns the extended modulus `eT`.

To write code that works both with plain and extended moduli, the following accessors are defined:

`GEN get_Flx_mod(GEN eT)` returns the underlying modulus  $T$ .

`GEN get_Flx_var(GEN eT)` returns the variable number of the modulus.

GEN `get_Flx_degree`(GEN `eT`) returns the degree of the modulus.

Furthermore, `ZXT_to_FlxT` allows to convert an extended modulus for a `FpX` to an extended modulus for the corresponding `Flx`.

### 7.3.15.2 Basic operations.

ulong `Flx_lead`(GEN `x`) returns the leading coefficient of  $x$  as a `ulong` (return 0 for the zero polynomial).

ulong `Flx_constant`(GEN `x`) returns the constant coefficient of  $x$  as a `ulong` (return 0 for the zero polynomial).

GEN `Flx_red`(GEN `z`, ulong `p`) converts from `zx` with non-negative coefficients to `Flx` (by reducing them mod  $p$ ).

int `Flx_equal1`(GEN `x`) returns 1 (true) if the `Flx`  $x$  is equal to 1, 0 (false) otherwise.

int `Flx_equal`(GEN `x`, GEN `y`) returns 1 (true) if the `Flx`  $x$  and  $y$  are equal, and 0 (false) otherwise.

GEN `Flx_copy`(GEN `x`) returns a copy of  $x$ .

GEN `Flx_add`(GEN `x`, GEN `y`, ulong `p`)

GEN `Flx_Fl_add`(GEN `y`, ulong `x`, ulong `p`)

GEN `Flx_neg`(GEN `x`, ulong `p`)

GEN `Flx_neg_inplace`(GEN `x`, ulong `p`), same as `Flx_neg`, in place ( $x$  is destroyed).

GEN `Flx_sub`(GEN `x`, GEN `y`, ulong `p`)

GEN `Flx_half`(GEN `x`, ulong `p`) returns  $z$  such that  $2z = x$  modulo  $p$  assuming such  $z$  exists.

GEN `Flx_mul`(GEN `x`, GEN `y`, ulong `p`)

GEN `Flxn_mul`(GEN `a`, GEN `b`, long `n`, ulong `p`) returns  $ab$  modulo  $X^n$ .

GEN `Flxn_inv`(GEN `a`, long `n`, ulong `p`) returns  $1/a$  modulo  $X^n$ .

GEN `Flx_Fl_mul`(GEN `y`, ulong `x`, ulong `p`)

GEN `Flx_double`(GEN `y`, ulong `p`) returns  $2y$ .

GEN `Flx_triple`(GEN `y`, ulong `p`) returns  $3y$ .

GEN `Flx_mulu`(GEN `y`, ulong `x`, ulong `p`) as `Flx_Fl_mul` but do not assume that  $x < p$ .

GEN `Flx_Fl_mul_to_monic`(GEN `y`, ulong `x`, ulong `p`) returns  $yx$  assuming the result is monic of the same degree as  $y$  (in particular  $x \neq 0$ ).

GEN `Flx_sqr`(GEN `x`, ulong `p`)

GEN `Flx_powu`(GEN `x`, ulong `n`, ulong `p`) return  $x^n$ .

GEN `Flx_divrem`(GEN `x`, GEN `y`, ulong `p`, GEN `*pr`)

GEN `Flx_div`(GEN `x`, GEN `y`, ulong `p`)

GEN `Flx_rem`(GEN `x`, GEN `y`, ulong `p`)

GEN `Flx_deriv`(GEN `z`, ulong `p`)

GEN `Flx_translate1`(GEN `P`, ulong `p`) return  $P(x+1)$

GEN Flx\_diff1(GEN P, ulong p) return  $P(x+1) - P(x)$

GEN Flx\_digits(GEN x, GEN B, ulong p) returns a vector of Flx  $[c_0, \dots, c_n]$  of degree less than the degree of  $B$  and such that  $x = \sum_{i=0}^n c_i B^i$ .

GEN FlxV\_Flx\_fromdigits(GEN v, GEN B, ulong p) where  $v = [c_0, \dots, c_n]$  is a vector of Flx, returns  $\sum_{i=0}^n c_i B^i$ .

GEN Flx\_Frobenius(GEN T, ulong p)

GEN Flx\_matFrobenius(GEN T, ulong p)

GEN Flx\_gcd(GEN a, GEN b, ulong p) returns a (not necessarily monic) greatest common divisor of  $x$  and  $y$ .

GEN Flx\_halfgcd(GEN x, GEN y, ulong p) returns a two-by-two FlxM  $M$  with determinant  $\pm 1$  such that the image  $(a, b)$  of  $(x, y)$  by  $M$  has the property that  $\deg a \geq \frac{\deg x}{2} > \deg b$ .

GEN Flx\_extgcd(GEN a, GEN b, ulong p, GEN \*ptu, GEN \*ptv)

GEN Flx\_roots(GEN f, ulong p) returns the vector of roots of  $f$  (without multiplicity, as a t\_VECSMALL). Assumes that  $p$  is prime.

ulong Flx\_oneroot(GEN f, ulong p) returns one root  $0 \leq r < p$  of the Flx  $f$  in  $\mathbf{Z}/p\mathbf{Z}$ . Return 0 if no root exists. Assumes that  $p$  is prime.

ulong Flx\_oneroot\_split(GEN f, ulong p) as Flx\_oneroot but assume  $f$  is totally split.

long Flx\_ispower(GEN f, ulong k, ulong p, GEN \*pt) return 1 if the Flx  $f$  is a  $k$ -th power, 0 otherwise. If pt is not NULL, set it to  $g$  such that  $g^k = f$ .

GEN Flx\_factor(GEN f, ulong p)

GEN Flx\_ddf(GEN f, ulong p)

GEN Flx\_factor\_squarefree(GEN f, ulong p) returns the squarefree factorization of  $f$  modulo  $p$ . This is a vector  $[u_1, \dots, u_k]$  of pairwise coprime Flx such that  $u_k \neq 1$  and  $f = \prod u_i^i$ . Shallow function.

GEN Flx\_mod\_Xn1(GEN T, ulong n, ulong p) return  $T$  modulo  $(X^n + 1, p)$ . Shallow function.

GEN Flx\_mod\_Xnm1(GEN T, ulong n, ulong p) return  $T$  modulo  $(X^n - 1, p)$ . Shallow function.

GEN Flx\_degfact(GEN f, ulong p) as FpX\_degfact.

GEN Flx\_factorff\_irred(GEN P, GEN Q, ulong p) as FpX\_factorff\_irred.

GEN Flx\_rootsff(GEN P, GEN T, ulong p) as FpX\_rootsff.

GEN Flx\_ffisom(GEN P, GEN Q, ulong l) as FpX\_ffisom.

### 7.3.15.3 Miscellaneous operations.

GEN `pol0_Flx(long sv)` returns a zero Flx in variable  $v$ .

GEN `zero_Flx(long sv)` alias for `pol0_Flx`

GEN `pol1_Flx(long sv)` returns the unit Flx in variable  $v$ .

GEN `polx_Flx(long sv)` returns the variable  $v$  as degree 1 Flx.

GEN `monomial_Flx(ulong a, long d, long sv)` returns the Flx  $aX^d$  in variable  $v$ .

GEN `Flx_normalize(GEN z, ulong p)`, as `FpX_normalize`.

GEN `Flx_rescale(GEN P, ulong h, ulong p)` returns  $h^{\deg(P)}P(x/h)$ ,  $P$  is a Flx and  $h$  is a non-zero integer.

GEN `random_Flx(long d, long sv, ulong p)` returns a random Flx in variable  $v$ , of degree less than  $d$ .

GEN `Flx_recip(GEN x)`, returns the reciprocal polynomial

`ulong Flx_resultant(GEN a, GEN b, ulong p)`, returns the resultant of  $a$  and  $b$

`ulong Flx_extresultant(GEN a, GEN b, ulong p, GEN *ptU, GEN *ptV)` given two Flx  $a$  and  $b$ , returns their resultant and sets Bezout coefficients (if the resultant is 0, the latter are not set).

GEN `Flx_invBarrett(GEN T, ulong p)`, returns the Barrett inverse  $M$  of  $T$  defined by  $M(x) \times x^n T(1/x) \equiv 1 \pmod{x^{n-1}}$  where  $n$  is the degree of  $T$ .

GEN `Flx_renormalize(GEN x, long l)`, as `FpX_renormalize`, where  $l = \lg(x)$ , in place.

GEN `Flx_shift(GEN T, long n)` returns  $T * x^n$  if  $n \geq 0$ , and  $T \backslash x^{-n}$  otherwise.

`long Flx_val(GEN x)` returns the valuation of  $x$ , i.e. the multiplicity of the 0 root.

`long Flx_valrem(GEN x, GEN *Z)` as `RgX_valrem`, returns the valuation of  $x$ . In particular, if the valuation is 0, set  $*Z$  to  $x$ , not a copy.

GEN `Flx_div_by_X_x(GEN A, ulong a, ulong p, ulong *rem)`, returns the Euclidean quotient of the Flx  $A$  by  $X - a$ , and sets  $rem$  to the remainder  $A(a)$ .

`ulong Flx_eval(GEN x, ulong y, ulong p)`, as `FpX_eval`.

`ulong Flx_eval_pre(GEN x, ulong y, ulong p, ulong pi)`, as `Flx_eval`, assuming  $pi$  is the pseudo inverse of  $p$ .

`ulong Flx_eval_powers_pre(GEN P, GEN y, ulong p, ulong pi)`. Let  $y$  be the `t_VECSMALL`  $(1, a, \dots, a^n)$ , where  $n$  is the degree of the Flx  $P$ , return  $P(a)$ , assuming  $pi$  is the pseudo inverse of  $p$ .

GEN `Flx_Flv_multieval(GEN P, GEN v, ulong p)` returns the vector  $[P(v[1]), \dots, P(v[n])]$  as a Flv.

`ulong Flx_dotproduct(GEN x, GEN y, ulong p)` returns the scalar product of the coefficients of  $x$  and  $y$ .

GEN `Flx_deflate(GEN P, long d)` assuming  $P$  is a polynomial of the form  $Q(X^d)$ , return  $Q$ .

GEN `Flx_splitting(GEN P, long k)`, as `RgX_splitting`.

GEN `Flx_inflate(GEN P, long d)` returns  $P(X^d)$ .

```

int Flx_is_squarefree(GEN z, ulong p)

int Flx_is_irred(GEN f, ulong p), as FpX_is_irred.

int Flx_is_smooth(GEN f, long r, ulong p) return 1 if all irreducible factors of  $f$  are of degree
at most  $r$ , 0 otherwise.

long Flx_nbroots(GEN f, ulong p), as FpX_nbroots.

long Flx_nbfact(GEN z, ulong p), as FpX_nbfact.

long Flx_nbfact_Frobenius(GEN f, GEN XP, ulong p), as FpX_nbfact_Frobenius.

GEN Flx_degfact(GEN f, ulong p), as FpX_degfact.

GEN Flx_nbfact_by_degree(GEN z, long *nb, ulong p) Assume that the Flx  $z$  is squarefree
mod the prime  $p$ . Returns a t_VECSMALL  $D$  with  $\deg z$  entries, such that  $D[i]$  is the number of
irreducible factors of degree  $i$ . Set  $nb$  to the total number of irreducible factors (the sum of the
 $D[i]$ ).

void Flx_ffintersect(GEN P, GEN Q, long n, ulong p, GEN*SP, GEN*SQ, GEN MA, GEN MB)
,
as FpX_ffintersect

GEN Flv_polint(GEN x, GEN y, ulong p, long sv) as FpV_polint, returning an Flx in variable
 $v$ .

GEN Flv_Flm_polint(GEN x, GEN V, ulong p, long sv) equivalent (but faster) to applying
Flv_polint( $x, \dots$ ) to all the elements of the vector  $V$  (thus, returns a FlxV).

GEN Flv_invVandermonde(GEN L, ulong d, ulong p)  $L$  being a Flv of length  $n$ , return the
inverse  $M$  of the Vandermonde matrix attached to the elements of  $L$ , multiplied by  $d$ . If  $A$  is a Flv
and  $B = MA$ , then the polynomial  $P = \sum_{i=1}^n B[i]X^{i-1}$  verifies  $P(L[i]) = dA[i]$  for  $1 \leq i \leq n$ .

GEN Flv_roots_to_pol(GEN a, ulong p, long sv) as FpV_roots_to_pol returning an Flx in
variable  $v$ .

7.3.16 FlxV. See FpXV operations.

GEN FlxV_Flc_mul(GEN V, GEN W, ulong p), as FpXV_FpC_mul.

GEN FlxV_red(GEN V, ulong p) reduces each components with Flx_red.

GEN FlxV_prod(GEN V, ulong p),  $V$  being a vector of Flx, returns their product.

ulong FlxC_eval_powers_pre(GEN x, GEN y, ulong p, ulong pi) apply Flx_eval_powers_pre
to all elements of  $x$ .

GEN FlxC_neg(GEN x, ulong p)

GEN FlxC_sub(GEN x, GEN y, ulong p)

GEN zero_FlxC(long n, long sv)

```

**7.3.17** FlxM. See FpXM operations.

ulong FlxM\_eval\_powers\_pre(GEN M, GEN y, ulong p, ulong pi) this function applies FlxC\_eval\_powers\_pre to all entries of M.

GEN FlxM\_neg(GEN x, ulong p)

GEN FlxM\_sub(GEN x, GEN y, ulong p)

GEN zero\_FlxM(long r, long c, long sv)

**7.3.18** FlxT. See FpXT operations.

GEN FlxT\_red(GEN V, ulong p) reduces each leaf with Flx\_red.

**7.3.19** Flxq. See FpXQ operations.

GEN Flxq\_add(GEN x, GEN y, GEN T, ulong p)

GEN Flxq\_sub(GEN x, GEN y, GEN T, ulong p)

GEN Flxq\_mul(GEN x, GEN y, GEN T, ulong p)

GEN Flxq\_sqr(GEN y, GEN T, ulong p)

GEN Flxq\_inv(GEN x, GEN T, ulong p)

GEN Flxq\_invsafe(GEN x, GEN T, ulong p)

GEN Flxq\_div(GEN x, GEN y, GEN T, ulong p)

GEN Flxq\_pow(GEN x, GEN n, GEN T, ulong p)

GEN Flxq\_powu(GEN x, ulong n, GEN T, ulong p)

GEN Flxq\_pow\_init(GEN x, GEN n, long k, GEN T, ulong p)

GEN Flxq\_pow\_table(GEN R, GEN n, GEN T, ulong p)

GEN Flxq\_powers(GEN x, long n, GEN T, ulong p)

GEN Flxq\_matrix\_pow(GEN x, long m, long n, GEN T, ulong p), see FpXQ\_matrix\_pow.

GEN Flxq\_autpow(GEN a, long n, GEN T, ulong p) see FpXQ\_autpow.

GEN Flxq\_autsum(GEN a, long n, GEN T, ulong p) see FpXQ\_autsum.

GEN Flxq\_auttrace(GEN a, ulong n, GEN T, ulong p) see FpXQ\_auttrace.

GEN Flxq\_ffisom\_inv(GEN S, GEN T, ulong p), as FpXQ\_ffisom\_inv.

GEN Flx\_Flxq\_eval(GEN f, GEN x, GEN T, ulong p) returns  $f(x)$ .

GEN Flx\_FlxqV\_eval(GEN f, GEN x, GEN T, ulong p), see FpX\_FpXQV\_eval.

GEN FlxqV\_roots\_to\_pol(GEN V, GEN T, ulong p, long v) as FqV\_roots\_to\_pol returning an FlxqX in variable  $v$ .

int Flxq\_issquare(GEN x, GEN T, ulong p) returns 1 if  $x$  is a square and 0 otherwise. Assume that  $T$  is irreducible mod  $p$ .

int Flxq\_is2npower(GEN x, long n, GEN T, ulong p) returns 1 if  $x$  is a  $2^n$ -th power and 0 otherwise. Assume that  $T$  is irreducible mod  $p$ .

`GEN Flxq_order(GEN a, GEN ord, GEN T, ulong p)` as `FpXQ_order`.  
`GEN Flxq_log(GEN a, GEN g, GEN ord, GEN T, ulong p)` as `FpXQ_log`  
`GEN Flxq_sqrtn(GEN x, GEN n, GEN T, ulong p, GEN *zn)` as `FpXQ_sqrtn`.  
`GEN Flxq_sqrt(GEN x, GEN T, ulong p)` returns a square root of  $x$ . Return NULL if  $x$  is not a square.  
`GEN Flxq_lroot(GEN a, GEN T, ulong p)` returns  $x$  such that  $x^p = a$ .  
`GEN Flxq_lroot_fast(GEN a, GEN V, GEN T, ulong p)` assuming that  $V = \text{Flxq\_powers}(s, p-1, T, p)$  where  $s(x)^p \equiv x \pmod{T(x), p}$ , returns  $b$  such that  $b^p = a$ . Only useful if  $p$  is less than the degree of  $T$ .  
`GEN Flxq_charpoly(GEN x, GEN T, ulong p)` returns the characteristic polynomial of  $x$   
`GEN Flxq_minpoly(GEN x, GEN T, ulong p)` returns the minimal polynomial of  $x$   
`ulong Flxq_norm(GEN x, GEN T, ulong p)` returns the norm of  $x$   
`ulong Flxq_trace(GEN x, GEN T, ulong p)` returns the trace of  $x$   
`GEN Flxq_conjvec(GEN x, GEN T, ulong p)` returns the conjugates  $[x, x^p, x^{p^2}, \dots, x^{p^{n-1}}]$  where  $n$  is the degree of  $T$ .  
`GEN gener_Flxq(GEN T, ulong p, GEN *po)` returns a primitive root modulo  $(T, p)$ .  $T$  is an `Flx` assumed to be irreducible modulo the prime  $p$ . If `po` is not NULL it is set to  $[o, fa]$ , where  $o$  is the order of the multiplicative group of the finite field, and  $fa$  is its factorization.

### 7.3.20 FlxX. See FpXX operations.

`GEN pol1_FlxX(long vX, long sx)` returns the unit `FlxX` as a `t_POL` in variable  $vX$  which only coefficient is `pol1_Flx(sx)`.  
`GEN polx_FlxX(long vX, long sx)` returns the variable  $X$  as a degree 1 `t_POL` with `Flx` coefficients in the variable  $x$ .  
`long FlxY_degrees(GEN P)` return the degree of  $P$  with respect to the secondary variable.  
`GEN FlxX_add(GEN P, GEN Q, ulong p)`  
`GEN FlxX_sub(GEN P, GEN Q, ulong p)`  
`GEN FlxX_Fl_mul(GEN x, ulong y, ulong p)`  
`GEN FlxX_double(GEN x, ulong p)`  
`GEN FlxX_triple(GEN x, ulong p)`  
`GEN FlxX_neg(GEN x, ulong p)`  
`GEN FlxX_Flx_add(GEN x, GEN y, ulong p)`  
`GEN FlxX_Flx_sub(GEN x, GEN y, ulong p)`  
`GEN FlxX_Flx_mul(GEN x, GEN y, ulong p)`  
`GEN FlxY_Flx_div(GEN x, GEN y, ulong p)` divides the coefficients of  $x$  by  $y$  using `Flx_div`.  
`GEN FlxX_deriv(GEN P, ulong p)` returns the derivative of  $P$  with respect to the main variable.

`GEN FlxY_evalx(GEN P, ulong z, ulong p)`  $P$  being an `FlxY`, returns the `Flx`  $P(z, Y)$ , where  $Y$  is the main variable of  $P$ .

`GEN FlxY_Flx_translate(GEN P, GEN f, ulong p)`  $P$  being an `FlxY` and  $f$  being an `Flx`, return  $(P(x, Y + f(x)))$ , where  $Y$  is the main variable of  $P$ .

`ulong FlxY_evalx_powers_pre(GEN P, GEN xp, ulong p, ulong pi)`,  $xp$  being the vector  $[1, x, \dots, x^n]$ , where  $n$  is larger or equal to the degree of  $P$  in  $X$ , return  $P(x, Y)$ , where  $Y$  is the main variable of  $Q$ , assuming  $pi$  is the pseudo inverse of  $p$ .

`ulong FlxY_eval_powers_pre(GEN P, GEN xp, GEN yp, ulong p, ulong pi)`,  $xp$  being the vector  $[1, x, \dots, x^n]$ , where  $n$  is larger or equal to the degree of  $P$  in  $X$  and  $yp$  being the vector  $[1, y, \dots, y^m]$ , where  $m$  is larger or equal to the degree of  $P$  in  $Y$  return  $P(x, y)$ , assuming  $pi$  is the pseudo inverse of  $p$ .

`GEN FlxY_Flxq_evalx(GEN x, GEN y, GEN T, ulong p)` as `FpXY_FpXQ_evalx`.

`GEN FlxY_FlxqV_evalx(GEN x, GEN V, GEN T, ulong p)` as `FpXY_FpXQV_evalx`.

`GEN FlxX_renormalize(GEN x, long l)`, as `normalizepol`, where  $l = \lg(x)$ , in place.

`GEN FlxX_resultant(GEN u, GEN v, ulong p, long sv)` Returns  $\text{Res}_X(u, v)$ , which is an `Flx`. The coefficients of  $u$  and  $v$  are assumed to be in the variable  $v$ .

`GEN Flx_FlxY_resultant(GEN a, GEN b, ulong p)` Returns  $\text{Res}_x(a, b)$ , which is an `Flx` in the main variable of  $b$ .

`GEN FlxX_shift(GEN a, long n, long sv)`, as `RgX_shift_shallow`, where  $v$  is the secondary variable.

`GEN FlxX_swap(GEN x, long n, long ws)`, as `RgXY_swap`.

`GEN FlxYqq_pow(GEN x, GEN n, GEN S, GEN T, ulong p)`, as `FpXYQQ_pow`.

**7.3.21** `FlxqX`. See `FpXQX` operations.

#### 7.3.21.1 Preconditioned reduction.

For faster reduction, the modulus  $S$  can be replaced by an extended modulus, which is an `FlxqXT`, in all `FlxqXQ`-classes functions, and in `FlxqX_rem` and `FlxqX_divrem`.

`GEN FlxqX_get_red(GEN S, GEN T, ulong p)` returns the extended modulus `eS`.

To write code that works both with plain and extended moduli, the following accessors are defined:

`GEN get_FlxqX_mod(GEN eS)` returns the underlying modulus  $S$ .

`GEN get_FlxqX_var(GEN eS)` returns the variable number of the modulus.

`GEN get_FlxqX_degree(GEN eS)` returns the degree of the modulus.



### 7.3.21.2 basic functions.

GEN random\_FlxqX(long d, long v, GEN T, ulong p) returns a random FlxqX in variable v, of degree less than d.

GEN zzX\_to\_Kronecker(GEN P, GEN Q) assuming  $P(X, Y)$  is a polynomial of degree in  $X$  strictly less than  $n$ , returns  $P(X, X^{2^{*n-1}})$ , the Kronecker form of  $P$ .

GEN Kronecker\_to\_FlxqX(GEN z, GEN T, ulong p). Let  $n = \deg T$  and let  $P(X, Y) \in \mathbf{Z}[X, Y]$  lift a polynomial in  $K[Y]$ , where  $K := \mathbf{F}_p[X]/(T)$  and  $\deg_X P < 2n-1$  — such as would result from multiplying minimal degree lifts of two polynomials in  $K[Y]$ . Let  $z = P(t, t^{2^{*n-1}})$  be a Kronecker form of  $P$ , this function returns  $Q \in \mathbf{Z}[X, t]$  such that  $Q$  is congruent to  $P(X, t) \pmod{(p, T(X))}$ ,  $\deg_X Q < n$ , and all coefficients are in  $[0, p[$ . Not stack-clean. Note that  $t$  need not be the same variable as  $Y$ !

GEN FlxqX\_red(GEN z, GEN T, ulong p)

GEN FlxqX\_normalize(GEN z, GEN T, ulong p)

GEN FlxqX\_mul(GEN x, GEN y, GEN T, ulong p)

GEN FlxqX\_Flxq\_mul(GEN P, GEN U, GEN T, ulong p)

GEN FlxqX\_Flxq\_mul\_to\_monic(GEN P, GEN U, GEN T, ulong p) returns  $P * U$  assuming the result is monic of the same degree as  $P$  (in particular  $U \neq 0$ ).

GEN FlxqX\_sqr(GEN x, GEN T, ulong p)

GEN FlxqX\_powu(GEN x, ulong n, GEN T, ulong p)

GEN FlxqX\_divrem(GEN x, GEN y, GEN T, ulong p, GEN \*pr)

GEN FlxqX\_div(GEN x, GEN y, GEN T, ulong p)

GEN FlxqX\_rem(GEN x, GEN y, GEN T, ulong p)

GEN FlxqX\_invBarrett(GEN T, GEN Q, ulong p)

GEN FlxqX\_gcd(GEN x, GEN y, ulong p) returns a (not necessarily monic) greatest common divisor of  $x$  and  $y$ .

GEN FlxqX\_extgcd(GEN x, GEN y, GEN T, ulong p, GEN \*ptu, GEN \*ptv)

GEN FlxqX\_halfgcd(GEN x, GEN y, GEN T, ulong p), see FpX\_halfgcd.

GEN FlxqXV\_prod(GEN V, GEN T, ulong p)

GEN FlxqX\_safegcd(GEN P, GEN Q, GEN T, ulong p) Returns the *monic* GCD of  $P$  and  $Q$  if Euclid's algorithm succeeds and NULL otherwise. In particular, if  $p$  is not prime or  $T$  is not irreducible over  $\mathbf{F}_p[X]$ , the routine may still be used (but will fail if non-invertible leading terms occur).

GEN FlxqX\_dotproduct(GEN x, GEN y, GEN T, ulong p) returns the scalar product of the coefficients of  $x$  and  $y$ .

long FlxqX\_is\_squarefree(GEN S, GEN T, ulong p), as FpX\_is\_squarefree.

long FlxqX\_ispower(GEN f, ulong k, GEN T, ulong p, GEN \*pt) return 1 if the FlxqX  $f$  is a  $k$ -th power, 0 otherwise. If  $pt$  is not NULL, set it to  $g$  such that  $g^k = f$ .

GEN FlxqX\_Frobenius(GEN S, GEN T, ulong p), as FpXQX\_Frobenius

GEN FlxqX\_roots(GEN f, GEN T, ulong p) return the roots of  $f$  in  $\mathbf{F}_p[X]/(T)$ . Assumes  $p$  is prime and  $T$  irreducible in  $\mathbf{F}_p[X]$ .

GEN FlxqX\_factor(GEN f, GEN T, ulong p) return the factorization of  $f$  over  $\mathbf{F}_p[X]/(T)$ . Assumes  $p$  is prime and  $T$  irreducible in  $\mathbf{F}_p[X]$ .

GEN FlxqX\_factor\_squarefree(GEN f, GEN T, ulong p) returns the squarefree factorization of  $f$ , see FpX\_factor\_squarefree.

GEN FlxqX\_ddf(GEN f, GEN T, ulong p) as FpX\_ddf.

long FlxqX\_ddf\_degree(GEN f, GEN XP, GEN T, GEN p), as FpX\_ddf\_degree.

GEN FlxqX\_degfact(GEN f, GEN T, ulong p), as FpX\_degfact.

long FlxqX\_nbroots(GEN S, GEN T, ulong p), as FpX\_nbroots.

long FlxqX\_nbfact(GEN S, GEN T, ulong p), as FpX\_nbfact.

long FlxqX\_nbfact\_Frobenius(GEN S, GEN Xq, GEN T, ulong p), as FpX\_nbfact\_Frobenius.

GEN FlxqX\_FlxqXQ\_eval(GEN Q, GEN x, GEN S, GEN T, ulong p) as FpX\_FpXQ\_eval.

GEN FlxqX\_FlxqXQV\_eval(GEN P, GEN V, GEN S, GEN T, ulong p) as FpX\_FpXQV\_eval.

### 7.3.22 FlxqXQ. See FpXQXQ operations.

GEN FlxqXQ\_mul(GEN x, GEN y, GEN S, GEN T, ulong p)

GEN FlxqXQ\_sqr(GEN x, GEN S, GEN T, ulong p)

GEN FlxqXQ\_inv(GEN x, GEN S, GEN T, ulong p)

GEN FlxqXQ\_invsafe(GEN x, GEN S, GEN T, ulong p)

GEN FlxqXQ\_div(GEN x, GEN y, GEN S, GEN T, ulong p)

GEN FlxqXQ\_pow(GEN x, GEN n, GEN S, GEN T, ulong p)

GEN FlxqXQ\_powu(GEN x, ulong n, GEN S, GEN T, ulong p)

GEN FlxqXQ\_powers(GEN x, long n, GEN S, GEN T, ulong p)

GEN FlxqXQ\_matrix\_pow(GEN x, long n, long m, GEN S, GEN T, ulong p)

GEN FlxqXQ\_autpow(GEN a, long n, GEN S, GEN T, ulong p) as FpXQXQ\_autpow

GEN FlxqXQ\_autsum(GEN a, long n, GEN S, GEN T, ulong p) as FpXQXQ\_autsum

GEN FlxqXQ\_auttrace(GEN a, long n, GEN S, GEN T, ulong p) as FpXQXQ\_auttrace

GEN FlxqXQ\_halfFrobenius(GEN A, GEN S, GEN T, ulong p), as FpXQXQ\_halfFrobenius

GEN FlxqXQ\_minpoly(GEN x, GEN S, GEN T, ulong p), as FpXQ\_minpoly

**7.3.23 F2x.** An F2x  $z$  is a `t_VECSMALL` representing a polynomial over  $\mathbf{F}_2[X]$ . Specifically  $z[0]$  is the usual codeword,  $z[1] = \text{evalvarn}(v)$  for some variable  $v$  and the coefficients are given by the bits of remaining words by increasing degree.

### 7.3.23.1 Preconditioned reduction.

For faster reduction, the modulus  $T$  can be replaced by an extended modulus ( $\text{FlxT}$ ) in all  $\text{Flxq}$ -classes functions, and in  $\text{Flx\_divrem}$ .

`GEN F2x_get_red(GEN T)` returns the extended modulus  $eT$ .

To write code that works both with plain and extended moduli, the following accessors are defined:

`GEN get_F2x_mod(GEN eT)` returns the underlying modulus  $T$ .

`GEN get_F2x_var(GEN eT)` returns the variable number of the modulus.

`GEN get_F2x_degree(GEN eT)` returns the degree of the modulus.

### 7.3.23.2 Basic operations.

`ulong F2x_coeff(GEN x, long i)` returns the coefficient  $i \geq 0$  of  $x$ .

`void F2x_clear(GEN x, long i)` sets the coefficient  $i \geq 0$  of  $x$  to 0.

`void F2x_flip(GEN x, long i)` adds 1 to the coefficient  $i \geq 0$  of  $x$ .

`void F2x_set(GEN x, long i)` sets the coefficient  $i \geq 0$  of  $x$  to 1.

`GEN F2x_copy(GEN x)`

`GEN Flx_to_F2x(GEN x)`

`GEN Z_to_F2x(GEN x, long v)`

`GEN ZX_to_F2x(GEN x)`

`GEN F2v_to_F2x(GEN x, long sv)`

`GEN F2x_to_Flx(GEN x)`

`GEN F2x_to_F2xX(GEN x, long sv)`

`GEN F2x_to_ZX(GEN x)`

`GEN pol0_F2x(long sv)` returns a zero F2x in variable  $v$ .

`GEN zero_F2x(long sv)` alias for `pol0_F2x`.

`GEN pol1_F2x(long sv)` returns the F2x in variable  $v$  constant to 1.

`GEN polx_F2x(long sv)` returns the variable  $v$  as degree 1 F2x.

`GEN monomial_F2x(long d, long sv)` returns the F2x  $X^d$  in variable  $v$ .

`GEN random_F2x(long d, long sv)` returns a random F2x in variable  $v$ , of degree less than  $d$ .

`long F2x_degree(GEN x)` returns the degree of the F2x  $x$ . The degree of 0 is defined as  $-1$ .

`int F2x_equal1(GEN x)`

`int F2x_equal(GEN x, GEN y)`

GEN F2x\_1\_add(GEN y) returns  $y+1$  where  $y$  is a Flx.  
 GEN F2x\_add(GEN x, GEN y)  
 GEN F2x\_mul(GEN x, GEN y)  
 GEN F2x\_sqr(GEN x)  
 GEN F2x\_divrem(GEN x, GEN y, GEN \*pr)  
 GEN F2x\_rem(GEN x, GEN y)  
 GEN F2x\_div(GEN x, GEN y)  
 GEN F2x\_renormalize(GEN x, long lx)  
 GEN F2x\_deriv(GEN x)  
 GEN F2x\_deflate(GEN x, long d)  
 ulong F2x\_eval(GEN P, ulong u) returns  $P(u)$ .  
 void F2x\_shift(GEN x, long d) as RgX\_shift  
 void F2x\_even\_odd(GEN P, GEN \*pe, GEN \*po) as RgX\_even\_odd  
 long F2x\_valrem(GEN x, GEN \*Z)  
 GEN F2x\_extgcd(GEN a, GEN b, GEN \*ptu, GEN \*ptv)  
 GEN F2x\_gcd(GEN a, GEN b)  
 GEN F2x\_halfgcd(GEN a, GEN b)  
 int F2x\_issquare(GEN x) returns 1 if  $x$  is a square of a F2x and 0 otherwise.  
 int F2x\_is\_irred(GEN f), as FpX\_is\_irred.  
 GEN F2x\_degfact(GEN f) as FpX\_degfact.  
 GEN F2x\_sqrt(GEN x) returns the squareroot of  $x$ , assuming  $x$  is a square of a F2x.  
 GEN F2x\_Frobenius(GEN T)  
 GEN F2x\_matFrobenius(GEN T)  
 GEN F2x\_factor(GEN f)  
 GEN F2x\_factor\_squarefree(GEN f)  
 GEN F2x\_ddf(GEN f)

**7.3.24** F2xq. See FpXQ operations.

GEN F2xq\_mul(GEN x, GEN y, GEN T)

GEN F2xq\_sqr(GEN x, GEN T)

GEN F2xq\_div(GEN x, GEN y, GEN T)

GEN F2xq\_inv(GEN x, GEN T)

GEN F2xq\_invsafe(GEN x, GEN T)

GEN F2xq\_pow(GEN x, GEN n, GEN T)

GEN F2xq\_powu(GEN x, ulong n, GEN T)

GEN F2xq\_pow\_init(GEN x, GEN n, long k, GEN T)

GEN F2xq\_pow\_table(GEN R, GEN n, GEN T)

ulong F2xq\_trace(GEN x, GEN T)

GEN F2xq\_conjvec(GEN x, GEN T) returns the vector of conjugates  $[x, x^2, x^{2^2}, \dots, x^{2^{n-1}}]$  where  $n$  is the degree of  $T$ .

GEN F2xq\_log(GEN a, GEN g, GEN ord, GEN T)

GEN F2xq\_order(GEN a, GEN ord, GEN T)

GEN F2xq\_Artin\_Schreier(GEN a, GEN T) returns a solution of  $x^2 + x = a$ , assuming it exists.

GEN F2xq\_sqrt(GEN a, GEN T)

GEN F2xq\_sqrt\_fast(GEN a, GEN s, GEN T) assuming that  $s^2 \equiv x \pmod{T(x)}$ , computes  $b \equiv a(s) \pmod{T}$  so that  $b^2 = a$ .

GEN F2xq\_sqrtn(GEN a, GEN n, GEN T, GEN \*zeta)

GEN gener\_F2xq(GEN T, GEN \*po)

GEN F2xq\_powers(GEN x, long n, GEN T)

GEN F2xq\_matrix\_pow(GEN x, long m, long n, GEN T)

GEN F2x\_F2xq\_eval(GEN f, GEN x, GEN T)

GEN F2x\_F2xqV\_eval(GEN f, GEN x, GEN T), see FpX\_FpXQV\_eval.

GEN F2xq\_autpow(GEN a, long n, GEN T) computes  $\sigma^n(X)$  assuming  $a = \sigma(X)$  where  $\sigma$  is an automorphism of the algebra  $\mathbf{F}_2[X]/T(X)$ .

**7.3.25** F2xqV, F2xqM.. See FqV, FqM operations.

```

GEN F2xqM_F2xqC_gauss(GEN a, GEN b, GEN T)
GEN F2xqM_F2xqC_invimage(GEN a, GEN b, GEN T)
GEN F2xqM_F2xqC_mul(GEN a, GEN b, GEN T)
GEN F2xqM_deplin(GEN x, GEN T)
GEN F2xqM_det(GEN a, GEN T)
GEN F2xqM_gauss(GEN a, GEN b, GEN T)
GEN F2xqM_image(GEN x, GEN T)
GEN F2xqM_indexrank(GEN x, GEN T)
GEN F2xqM_inv(GEN a, GEN T)
GEN F2xqM_invimage(GEN a, GEN b, GEN T)
GEN F2xqM_ker(GEN x, GEN T)
GEN F2xqM_mul(GEN a, GEN b, GEN T)
long F2xqM_rank(GEN x, GEN T)
GEN F2xqM_suppl(GEN x, GEN T)
GEN matid_F2xqM(long n, GEN T)

```

**7.3.26** F2xX.. See FpXX operations.

```

GEN ZXX_to_F2xX(GEN x, long v)
GEN FlxX_to_F2xX(GEN x)
GEN F2xX_to_ZXX(GEN B)
GEN F2xX_renormalize(GEN x, long lx)
long F2xY_degreeex(GEN P) return the degree of  $P$  with respect to the secondary variable.
GEN pol1_F2xX(long v, long sv)
GEN polx_F2xX(long v, long sv)
GEN F2xX_add(GEN x, GEN y)
GEN F2xX_F2x_add(GEN x, GEN y)
GEN F2xX_F2x_mul(GEN x, GEN y)
GEN F2xX_deriv(GEN P) returns the derivative of  $P$  with respect to the main variable.
GEN Kronecker_to_F2xqX(GEN z, GEN T)
GEN F2xX_to_Kronecker(GEN z, GEN T)
GEN F2xY_F2xq_evalx(GEN x, GEN y, GEN T) as FpXY_FpXQ_evalx.
GEN F2xY_F2xqV_evalx(GEN x, GEN V, GEN T) as FpXY_FpXQV_evalx.

```

**7.3.27** F2xXV/F2xXC.. See FpXXV operations.

GEN FlxXC\_to\_F2xXC(GEN B)

GEN F2xXC\_to\_ZXXC(GEN B)

**7.3.28** F2xqX.. See FlxqX operations.

**7.3.28.1 Preconditioned reduction.**

For faster reduction, the modulus  $S$  can be replaced by an extended modulus, which is an F2xqXT, in all F2xqXQ-classes functions, and in F2xqX\_rem and F2xqX\_divrem.

GEN F2xqX\_get\_red(GEN S, GEN T) returns the extended modulus  $eS$ .

To write code that works both with plain and extended moduli, the following accessors are defined:

GEN get\_F2xqX\_mod(GEN eS) returns the underlying modulus  $S$ .

GEN get\_F2xqX\_var(GEN eS) returns the variable number of the modulus.

GEN get\_F2xqX\_degree(GEN eS) returns the degree of the modulus.

**7.3.28.2 basic functions.**

GEN random\_F2xqX(long d, long v, GEN T, ulong p) returns a random F2xqX in variable  $v$ , of degree less than  $d$ .

GEN F2xqX\_red(GEN z, GEN T)

GEN F2xqX\_normalize(GEN z, GEN T)

GEN F2xqX\_F2xq\_mul(GEN P, GEN U, GEN T)

GEN F2xqX\_F2xq\_mul\_to\_monic(GEN P, GEN U, GEN T)

GEN F2xqX\_mul(GEN x, GEN y, GEN T)

GEN F2xqX\_sqr(GEN x, GEN T)

GEN F2xqX\_powu(GEN x, ulong n, GEN T)

GEN F2xqX\_rem(GEN x, GEN y, GEN T)

GEN F2xqX\_div(GEN x, GEN y, GEN T)

GEN F2xqX\_divrem(GEN x, GEN y, GEN T, GEN \*pr)

GEN F2xqXQ\_inv(GEN x, GEN S, GEN T)

GEN F2xqXQ\_invsafe(GEN x, GEN S, GEN T)

GEN F2xqX\_invBarrett(GEN T, GEN Q)

GEN F2xqX\_extgcd(GEN x, GEN y, GEN T, GEN \*ptu, GEN \*ptv)

GEN F2xqX\_gcd(GEN x, GEN y, GEN T)

long F2xqX\_ispower(GEN f, ulong k, GEN T, GEN \*pt)

GEN F2xqX\_F2xqXQ\_eval(GEN Q, GEN x, GEN S, GEN T) as FpX\_FpXQ\_eval.

GEN F2xqX\_F2xqXQV\_eval(GEN P, GEN V, GEN S, GEN T) as FpX\_FpXQV\_eval.

GEN F2xqX\_roots(GEN f, GEN T) return the roots of  $f$  in  $\mathbf{F}_2[X]/(T)$ . Assumes  $T$  irreducible in  $\mathbf{F}_2[X]$ .

GEN F2xqX\_factor(GEN f, GEN T) return the factorization of  $f$  over  $\mathbf{F}_2[X]/(T)$ . Assumes  $T$  irreducible in  $\mathbf{F}_2[X]$ .

GEN F2xqX\_factor\_squarefree(GEN f, GEN T) as FlxqX\_factor\_squarefree.

GEN F2xqX\_ddf(GEN f, GEN T) as FpX\_ddf.

GEN F2xqX\_degfact(GEN f, GEN T) as FpX\_degfact.

**7.3.29** F2xqXQ.. See FlxqXQ operations.

GEN FlxqXQ\_inv(GEN x, GEN S, GEN T)

GEN FlxqXQ\_invsafe(GEN x, GEN S, GEN T)

GEN F2xqXQ\_mul(GEN x, GEN y, GEN S, GEN T)

GEN F2xqXQ\_sqr(GEN x, GEN S, GEN T)

GEN F2xqXQ\_pow(GEN x, GEN n, GEN S, GEN T)

GEN F2xqXQ\_powers(GEN x, long n, GEN S, GEN T)

GEN F2xqXQ\_autpow(GEN a, long n, GEN S, GEN T) as FpXQXQ\_autpow

GEN F2xqXQ\_auttrace(GEN a, long n, GEN S, GEN T). Let  $\sigma$  be the automorphism defined by  $\sigma(X) = a[1] \pmod{T(X)}$  and  $\sigma(Y) = a[2] \pmod{S(X,Y),T(X)}$ ; returns the vector  $[\sigma^n(X), \sigma^n(Y), b + \sigma(b) + \dots + \sigma^{n-1}(b)]$  where  $b = a[3]$ .

GEN F2xqXQV\_red(GEN x, GEN S, GEN T)

**7.3.30 Functions returning objects with t\_INTMOD coefficients.**

Those functions are mostly needed for interface reasons: `t_INTMOD`s should not be used in library mode since the modular kernel is more flexible and more efficient, but GP users do not have access to the modular kernel. We document them for completeness:

GEN Fp\_to\_mod(GEN z, GEN p),  $z$  a `t_INT`. Returns  $z * \text{Mod}(1,p)$ , normalized. Hence the returned value is a `t_INTMOD`.

GEN FpX\_to\_mod(GEN z, GEN p),  $z$  a `ZX`. Returns  $z * \text{Mod}(1,p)$ , normalized. Hence the returned value has `t_INTMOD` coefficients.

GEN FpC\_to\_mod(GEN z, GEN p),  $z$  a `ZC`. Returns  $\text{Col}(z) * \text{Mod}(1,p)$ , a `t_COL` with `t_INTMOD` coefficients.

GEN FpV\_to\_mod(GEN z, GEN p),  $z$  a `ZV`. Returns  $\text{Vec}(z) * \text{Mod}(1,p)$ , a `t_VEC` with `t_INTMOD` coefficients.

GEN FpVV\_to\_mod(GEN z, GEN p),  $z$  a `ZVV`. Returns  $\text{Vec}(z) * \text{Mod}(1,p)$ , a `t_VEC` of `t_VEC` with `t_INTMOD` coefficients.

GEN FpM\_to\_mod(GEN z, GEN p),  $z$  a `ZM`. Returns  $z * \text{Mod}(1,p)$ , with `t_INTMOD` coefficients.

GEN F2c\_to\_mod(GEN x)

GEN F2m\_to\_mod(GEN x)



`GEN Flc_to_mod(GEN z)`  
`GEN Flm_to_mod(GEN z)`  
`GEN FqM_to_mod(GEN z, GEN T, GEN p)`  
`GEN FpXQC_to_mod(GEN V, GEN T, GEN p)`  $V$  being a vector of `FpXQ`, converts each entry to a `t_POLMOD` with `t_INTMOD` coefficients, and return a `t_COL`.  
`GEN QXQV_to_mod(GEN V, GEN T)`  $V$  a vector of `QXQ`, which are lifted representatives of elements of  $\mathbf{Q}[X]/(T)$  (number field elements in most applications) and  $T$  is in  $\mathbf{Z}[X]$ . Return a vector where all non-rational entries are converted to `t_POLMOD` modulo  $T$ ; no reduction mod  $T$  is attempted: the representatives should be already reduced. Used to normalize the output of `nfroots`.  
`GEN QXQX_to_mod_shallow(GEN P, GEN T)`  $P$  a polynomial with `QXQ` coefficients; replace them by `mkpolmod(.,T)`. Shallow function.  
`GEN QXQC_to_mod_shallow(GEN V, GEN T)`  $V$  a vector with `QXQ` coefficients; replace them by `mkpolmod(.,T)`. Shallow function.  
`GEN QXQM_to_mod_shallow(GEN M, GEN T)`  $M$  a matrix with `QXQ` coefficients; replace them by `mkpolmod(.,T)`. Shallow function.  
`GEN QXQXV_to_mod(GEN V, GEN T)`  $V$  a vector of polynomials whose coefficients are `QXQ`. Analogous to `QXQV_to_mod`. Used to normalize the output of `nfactor`.

The following functions are obsolete and should not be used: they receive a polynomial with arbitrary coefficients, apply a conversion function to map them to a finite field, a function from the modular kernel, then `*to_mod`:

`GEN rootmod(GEN f, GEN p)`, applies `FpX_roots`.  
`GEN rootmod2(GEN f, GEN p)`, (now) identical to `rootmod`.  
`GEN rootmod0(GEN f, GEN p, long flag)`, calls either `rootmod` or `rootmod2` depending on `flag`.  
`GEN factmod(GEN f, GEN p)` applies `*_factor`.  
`GEN simplefactmod(GEN f, GEN p)` applies `*_degfact`.

**7.3.31 Slow Chinese remainder theorem over  $\mathbf{Z}$ .** The routines in this section have quadratic time complexity with respect to the input size; see the routines in the next two sections for quasi-linear time variants.

`GEN Z_chinese(GEN a, GEN b, GEN A, GEN B)` returns the integer in  $[0, \text{lcm}(A, B)[$  congruent to  $a \bmod A$  and  $b \bmod B$ , assuming it exists; in other words, that  $a$  and  $b$  are congruent mod  $\text{gcd}(A, B)$ .

`GEN Z_chinese_all(GEN a, GEN b, GEN A, GEN B, GEN *pC)` as `Z_chinese`, setting `*pC` to the lcm of  $A$  and  $B$ .

`GEN Z_chinese_coprime(GEN a, GEN b, GEN A, GEN B, GEN C)`, as `Z_chinese`, assuming that  $\text{gcd}(A, B) = 1$  and that  $C = \text{lcm}(A, B) = AB$ .

`ulong u_chinese_coprime(ulong a, ulong b, ulong A, ulong B, ulong C)`, as `Z_chinese_coprime` for `ulong` inputs and output.

`void Z_chinese_pre(GEN A, GEN B, GEN *pC, GEN *pU, GEN *pd)` initializes chinese remainder computations modulo  $A$  and  $B$ . Sets `*pC` to  $\text{lcm}(A, B)$ , `*pd` to  $\text{gcd}(A, B)$ , `*pU` to an integer

congruent to 0 mod  $(A/d)$  and 1 mod  $(B/d)$ . It is allowed to set `pd = NULL`, in which case,  $d$  is still computed, but not saved.

`GEN Z_chinese_post(GEN a, GEN b, GEN C, GEN U, GEN d)` returns the solution to the chinese remainder problem  $x$  congruent to  $a$  mod  $A$  and  $b$  mod  $B$ , where  $C, U, d$  were set in `Z_chinese_pre`. If  $d$  is `NULL`, assume the problem has a solution. Otherwise, return `NULL` if it has no solution.

The following pair of functions is used in homomorphic imaging schemes, when reconstructing an integer from its images modulo pairwise coprime integers. The idea is as follows: we want to discover an integer  $H$  which satisfies  $|H| < B$  for some known bound  $B$ ; we are given pairs  $(H_p, p)$  with  $H$  congruent to  $H_p$  mod  $p$  and all  $p$  pairwise coprime.

Given  $H$  congruent to  $H_p$  modulo a number of  $p$ , whose product is  $q$ , and a new pair  $(H_p, p)$ ,  $p$  coprime to  $q$ , the following incremental functions use the chinese remainder theorem (CRT) to find a new  $H$ , congruent to the preceding one modulo  $q$ , but also to  $H_p$  modulo  $p$ . It is defined uniquely modulo  $qp$ , and we choose the centered representative. When  $P$  is larger than  $2B$ , we have  $H = H$ , but of course, the value of  $H$  may stabilize sooner. In many applications it is possible to directly check that such a partial result is correct.

`GEN Z_init_CRT(ulong Hp, ulong p)` given a `Fl Hp` in  $[0, p-1]$ , returns the centered representative  $H$  congruent to  $H_p$  modulo  $p$ .

`int Z_incremental_CRT(GEN *H, ulong Hp, GEN *q, ulong p)` given a `t_INT *H`, centered modulo  $*q$ , a new pair  $(H_p, p)$  with  $p$  coprime to  $q$ , this function updates  $*H$  so that it also becomes congruent to  $(H_p, p)$ , and  $*q$  to the product  $qp = p \cdot *q$ . It returns 1 if the new value is equal to the old one, and 0 otherwise.

`GEN chinese1_coprime_Z(GEN v)` an alternative divide-and-conquer implementation:  $v$  is a vector of `t_INTMOD` with pairwise coprime moduli. Return the `t_INTMOD` solving the corresponding chinese remainder problem. This is a streamlined version of

`GEN chinese1(GEN v)`, which solves a general chinese remainder problem (not necessarily over  $\mathbf{Z}$ , moduli not assumed coprime).

As above, for  $H$  a `ZM`: we assume that  $H$  and all  $H_p$  have dimension  $> 0$ . The original  $*H$  is destroyed.

`GEN ZM_init_CRT(GEN Hp, ulong p)`

`int ZM_incremental_CRT(GEN *H, GEN Hp, GEN *q, ulong p)`

As above for  $H$  a `ZX`: note that the degree may increase or decrease. The original  $*H$  is destroyed.

`GEN ZX_init_CRT(GEN Hp, ulong p, long v)`

`int ZX_incremental_CRT(GEN *H, GEN Hp, GEN *q, ulong p)`

As above, for  $H$  a matrix whose coefficient are `ZX`. The original  $*H$  is destroyed. The entries of  $H$  are not normalized, use `ZX_renormalize` for this.

`GEN ZXM_init_CRT(GEN Hp, long deg, ulong p)` where `deg` is the maximal degree of all the  $H_p$

`int ZXM_incremental_CRT(GEN *H, GEN Hp, GEN *q, ulong p)`

### 7.3.32 Fast remainders.

The routines in these section are asymptotically fast (quasi-linear time in the input size).

**GEN Z\_ZV\_mod**(GEN A, GEN P) given a **t\_INT**  $A$  and a vector  $P$  of positive pairwise coprime integers of length  $n \geq 1$ , return a vector  $B$  of the same length such that  $B[i] = A \pmod{P[i]}$  and  $0 \leq B[i] < P[i]$  for all  $1 \leq i \leq n$ . The vector  $P$  may be a **t\_VEC** or a **t\_VECSMALL** (treated as **ulongs**) and  $B$  has the same type as  $P$ .

**GEN Z\_nv\_mod**(GEN A, GEN P) given a **t\_INT**  $A$  and a **t\_VECSMALL**  $P$  of positive pairwise coprime integers of length  $n \geq 1$ , return a **t\_VECSMALL**  $B$  of the same length such that  $B[i] = A \pmod{P[i]}$  and  $0 \leq B[i] < P[i]$  for all  $1 \leq i \leq n$ . The entries of  $P$  and  $B$  are treated as **ulongs**.

The following low level functions allow precomputations:

**GEN ZV\_producttree**(GEN P) where  $P$  is a vector of integers (or **t\_VECSMALL**) of length  $n \geq 1$ , return the vector of **t\_VECS**  $[f(P), f^2(P), \dots, f^k(P)]$  where  $f$  is the transformation  $[p_1, p_2, \dots, p_m] \mapsto [p_1 p_2, p_3 p_4, \dots, p_{m-1} p_m]$  if  $m$  is even and  $[p_1 p_2, p_3 p_4, \dots, p_{m-2} p_{m-1}, p_m]$  if  $m$  is odd, and  $k = O(\log m)$  is minimal so that  $f^k(P)$  has length 1; in other words,  $f^k(P) = [p_1 p_2 \dots p_m]$ .

**GEN Z\_ZV\_mod\_tree**(GEN A, GEN P, GEN T) as **Z\_ZV\_mod** where  $T$  is the tree **ZV\_producttree**(P).

**GEN ZV\_nv\_mod\_tree**(GEN A, GEN P, GEN T)  $A$  being a **ZV** and  $P$  a **t\_VECSMALL** of length  $n \geq 1$ , the elements of  $P$  being pairwise coprime, return the vector of **Flv**  $[A \pmod{P[1]}, \dots, A \pmod{P[n]}]$ , where  $T$  is the tree **ZV\_producttree**(P).

**GEN ZM\_nv\_mod\_tree**(GEN A, GEN P, GEN T)  $A$  being a **ZM** and  $P$  a **t\_VECSMALL** of length  $n \geq 1$ , the elements of  $P$  being pairwise coprime, return the vector of **Flm**  $[A \pmod{P[1]}, \dots, A \pmod{P[n]}]$ , where  $T$  is the tree **ZV\_producttree**(P).

**GEN ZX\_nv\_mod\_tree**(GEN A, GEN P, GEN T)  $A$  being a **ZX** and  $P$  a **t\_VECSMALL** of length  $n \geq 1$ , the elements of  $P$  being pairwise coprime, return the vector of **Flx** polynomials  $[A \pmod{P[1]}, \dots, A \pmod{P[n]}]$ , where  $T$  is the tree **ZV\_producttree**(P).

**GEN ZXC\_nv\_mod\_tree**(GEN A, GEN P, GEN T)  $A$  being a **ZXC** and  $P$  a **t\_VECSMALL** of length  $n \geq 1$ , the elements of  $P$  being pairwise coprime, return the vector of **FlxC**  $[A \pmod{P[1]}, \dots, A \pmod{P[n]}]$ , where  $T$  is the tree **ZV\_producttree**(P).

**GEN ZXM\_nv\_mod\_tree**(GEN A, GEN P, GEN T)  $A$  being a **ZXM** and  $P$  a **t\_VECSMALL** of length  $n \geq 1$ , the elements of  $P$  being pairwise coprime, return the vector of **FlxM**  $[A \pmod{P[1]}, \dots, A \pmod{P[n]}]$ , where  $T$  is the tree **ZV\_producttree**(P).

**GEN ZXX\_nv\_mod\_tree**(GEN A, GEN P, GEN T, long v)  $A$  being a **ZXX**, and  $P$  a **t\_VECSMALL** of length  $n \geq 1$ , the elements of  $P$  being pairwise coprime, return the vector of **FlxX**  $[A \pmod{P[1]}, \dots, A \pmod{P[n]}]$ , where  $T$  is assumed to be the tree created by **ZV\_producttree**(P).

**7.3.33 Fast Chinese remainder theorem over  $\mathbf{Z}$ .** The routines in these section are asymptotically fast (quasi-linear time in the input size) and should be used whenever the moduli are known from the start.

The simplest function is

`GEN ZV_chinese(GEN A, GEN P, GEN *pM)` let  $P$  be a vector of positive pairwise coprime integers, let  $A$  be a vector of integers of the same length  $n \geq 1$  such that  $0 \leq A[i] < P[i]$  for all  $i$ , and let  $M$  be the product of the elements of  $P$ . Returns the integer in  $[0, M[$  congruent to  $A[i] \bmod P[i]$  for all  $1 \leq i \leq n$ . If  $pM$  is not NULL, set  $*pM$  to  $M$ . We also allow `t_VECSMALLs` for  $A$  and  $P$  (seen as vectors of unsigned integers).

`GEN ZV_chinese_center(GEN A, GEN P, GEN *pM)` As `ZV_chinese` but return integers in  $[-M/2, M/2[$  instead.

The following functions allow to solve many Chinese remainder problems simultaneously, for a given set of moduli:

`GEN nxV_chinese_center(GEN A, GEN P, GEN *pt_mod)` where  $A$  is a vector of `nx` and  $P$  a `t_VECSMALL` of the same length  $n \geq 1$ , the elements of  $P$  being pairwise coprime, and  $M$  being the product of the elements of  $P$ , returns the `t_POL` whose entries are integers in  $[-M/2, M/2[$  congruent to  $A[i] \bmod P[i]$  for all  $1 \leq i \leq n$ . If  $pt\_mod$  is not NULL, set  $*pt\_mod$  to  $M$ .

`GEN ncV_chinese_center(GEN A, GEN P, GEN *pM)` where  $A$  is a vector of `VECSMALLs` (seen as vectors of unsigned integers) and  $P$  a `t_VECSMALL` of the same length  $n \geq 1$ , the elements of  $P$  being pairwise coprime, and  $M$  being the product of the elements of  $P$ , returns the `t_COL` whose entries are integers in  $[-M/2, M/2[$  congruent to  $A[i] \bmod P[i]$  for all  $1 \leq i \leq n$ . If  $pM$  is not NULL, set  $*pt\_mod$  to  $M$ .

`GEN nmV_chinese_center(GEN A, GEN P, GEN *pM)` where  $A$  is a vector of `MATSMALLs` (seen as matrices of unsigned integers) and  $P$  a `t_VECSMALL` of the same length  $n \geq 1$ , the elements of  $P$  being pairwise coprime, and  $M$  being the product of the elements of  $P$ , returns the matrix whose entries are integers in  $[-M/2, M/2[$  congruent to  $A[i] \bmod P[i]$  for all  $1 \leq i \leq n$ . If  $pM$  is not NULL, set  $*pM$  to  $M$ . N.B.: this function uses the parallel GP interface.

`GEN nxCV_chinese_center(GEN A, GEN P, GEN *pM)` where  $A$  is a vector of `nxCs` and  $P$  a `t_VECSMALL` of the same length  $n \geq 1$ , the elements of  $P$  being pairwise coprime, and  $M$  being the product of the elements of  $P$ , returns the `t_COL` whose entries are integers in  $[-M/2, M/2[$  congruent to  $A[i] \bmod P[i]$  for all  $1 \leq i \leq n$ . If  $pM$  is not NULL, set  $*pt\_mod$  to  $M$ .

`GEN nxMV_chinese_center(GEN A, GEN P, GEN *pM)` where  $A$  is a vector of `nxMs` and  $P$  a `t_VECSMALL` of the same length  $n \geq 1$ , the elements of  $P$  being pairwise coprime, and  $M$  being the product of the elements of  $P$ , returns the matrix whose entries are integers in  $[-M/2, M/2[$  congruent to  $A[i] \bmod P[i]$  for all  $1 \leq i \leq n$ . If  $pM$  is not NULL, set  $*pM$  to  $M$ . N.B.: this function uses the parallel GP interface.

The other routines allow for various precomputations :

`GEN ZV_chinesetree(GEN P, GEN T)` given  $P$  a vector of integers (or `t_VECSMALL`) and a product tree  $T$  from `ZV_producttree(P)` for the same  $P$ , return a “chinese remainder tree”  $R$ , preconditioning the solution of Chinese remainder problems modulo the  $P[i]$ .

`GEN ZV_chinese_tree(GEN A, GEN P, GEN T, GEN R)` return `ZV_chinese(A, P, NULL)`, where  $T$  is created by `ZV_producttree(P)` and  $R$  by `ZV_chinesetree(P, T)`.

GEN nmV\_chinese\_center\_tree(GEN A, GEN P, GEN T, GEN R) as nmV\_chinese\_center where  $T$  is assumed to be the tree created by ZV\_producttree(P) and  $R$  by ZV\_chinesetree(P, T).

GEN nxV\_chinese\_center\_tree(GEN A, GEN P, GEN T, GEN R) as nxV\_chinese\_center where  $T$  is assumed to be the tree created by ZV\_producttree(P) and  $R$  by ZV\_chinesetree(P, T).

### 7.3.34 Rational reconstruction.

int Fp\_ratlift(GEN x, GEN m, GEN amax, GEN bmax, GEN \*a, GEN \*b). Assuming that  $0 \leq x < m$ ,  $\text{amax} \geq 0$ , and  $\text{bmax} > 0$  are t\_INTs, and that  $2\text{amaxbmax} < m$ , attempts to recognize  $x$  as a rational  $a/b$ , i.e. to find t\_INTs  $a$  and  $b$  such that

- $a \equiv bx \text{ modulo } m$ ,
- $|a| \leq \text{amax}$ ,  $0 < b \leq \text{bmax}$ ,
- $\gcd(m, b) = \gcd(a, b)$ .

If unsuccessful, the routine returns 0 and leaves  $a, b$  unchanged; otherwise it returns 1 and sets  $a$  and  $b$ .

In almost all applications, we actually know that a solution exists, as well as a non-zero multiple  $B$  of  $b$ , and  $m = p^\ell$  is a prime power, for a prime  $p$  chosen coprime to  $B$  hence to  $b$ . Under the single assumption  $\gcd(m, b) = 1$ , if a solution  $a, b$  exists satisfying the three conditions above, then it is unique.

GEN FpM\_ratlift(GEN M, GEN m, GEN amax, GEN bmax, GEN denom) given an FpM modulo  $m$  with reduced or Fp\_center-ed entries, reconstructs a matrix with rational coefficients by applying Fp\_ratlift to all entries. Assume that all preconditions for Fp\_ratlift are satisfied, as well  $\gcd(m, b) = 1$  (so that the solution is unique if it exists). Return NULL if the reconstruction fails, and the rational matrix otherwise. If  $\text{denom}$  is not NULL check further that all denominators divide  $\text{denom}$ .

The function is not stack clean if one of the coefficients of  $M$  is negative (centered residues), but still suitable for gerepileupto.

GEN FpX\_ratlift(GEN P, GEN m, GEN amax, GEN bmax, GEN denom) as FpM\_ratlift, where  $P$  is an FpX.

GEN FpC\_ratlift(GEN P, GEN m, GEN amax, GEN bmax, GEN denom) as FpM\_ratlift, where  $P$  is an FpC.

### 7.3.35 Zp.

GEN Zp\_sqrt(GEN b, GEN p, long e)  $b$  and  $p$  being t\_INTs, with  $p$  a prime (possibly 2), returns a t\_INT  $a$  such that  $a^2 \equiv b \pmod{p^e}$ .

GEN Z2\_sqrt(GEN b, long e)  $b$  being a t\_INTs returns a t\_INT  $a$  such that  $a^2 \equiv b \pmod{2^e}$ .

GEN Zp\_sqrtlift(GEN b, GEN a, GEN p, long e) let  $a, b, p$  be t\_INTs, with  $p > 1$  odd, such that  $a^2 \equiv b \pmod{p}$ . Returns a t\_INT  $A$  such that  $A^2 \equiv b \pmod{p^e}$ . Special case of Zp\_sqrtnlift.

GEN Zp\_sqrtnlift(GEN b, GEN n, GEN a, GEN p, long e) let  $a, b, n, p$  be t\_INTs, with  $n, p > 1$ , and  $p$  coprime to  $n$ , such that  $a^n \equiv b \pmod{p}$ . Returns a t\_INT  $A$  such that  $A^n \equiv b \pmod{p^e}$ . Special case of ZpX\_liftroot.

GEN Zp\_teachmuller(GEN x, GEN p, long e, GEN pe) for  $p$  an odd prime,  $x$  a  $\mathbf{t\_INT}$  coprime to  $p$ , and  $pe = p^e$ , returns the  $(p-1)$ -th root of 1 congruent to  $x$  modulo  $p$ , modulo  $p^e$ . For convenience,  $p=2$  is also allowed and we return 1 ( $x$  is 1 mod 4) or  $2^e - 1$  ( $x$  is 3 mod 4).

GEN teachmullerinit(long p, long n) returns the values of Zp\_teachmuller at all  $x = 1, \dots, p-1$ .

### 7.3.36 ZpX.

GEN ZpX\_roots(GEN f, GEN p, long e)  $f$  a ZX with leading term prime to  $p$ , and without multiple roots mod  $p$ . Return a vector of  $\mathbf{t\_INT}$ s which are the roots of  $f$  mod  $p^e$ .

GEN ZpX\_liftroot(GEN f, GEN a, GEN p, long e)  $f$  a ZX with leading term prime to  $p$ , and  $a$  a root mod  $p$  such that  $v_p(f'(a)) = 0$ . Return a  $\mathbf{t\_INT}$  which is the root of  $f$  mod  $p^e$  congruent to  $a$  mod  $p$ .

GEN ZX\_Zp\_root(GEN f, GEN a, GEN p, long e) same as ZpX\_liftroot without the assumption  $v_p(f'(a)) = 0$ . Return a  $\mathbf{t\_VEC}$  of  $\mathbf{t\_INT}$ s, which are the  $p$ -adic roots of  $f$  congruent to  $a$  mod  $p$  (given modulo  $p^e$ ).

GEN ZpX\_liftroots(GEN f, GEN S, GEN p, long e)  $f$  a ZX with leading term prime to  $p$ , and  $S$  a vector of simple roots mod  $p$ . Return a vector of  $\mathbf{t\_INT}$ s which are the root of  $f$  mod  $p^e$  congruent to the  $S[i]$  mod  $p$ .

GEN ZpX\_liftfact(GEN A, GEN B, GEN pe, GEN p, long e) is the routine underlying pol\_hensellift. Here,  $p$  is prime defines a finite field  $\mathbf{F}_p$ .  $A$  is a polynomial in  $\mathbf{Z}[X]$ , whose leading coefficient is non-zero in  $\mathbf{F}_p$ .  $B$  is a vector of monic  $\mathbf{F}_p[X]$ , pairwise coprime in  $\mathbf{F}_p[X]$ , whose product is congruent to  $A/\text{lc}(A)$  in  $\mathbf{F}_p[X]$ . Lifts the elements of  $B$  mod  $pe = p^e$ .

GEN ZpX\_Frobenius(GEN T, GEN p, ulong e) returns the  $p$ -adic lift of the Frobenius automorphism of  $\mathbf{F}_p[X]/(T)$  to precision  $e$ .

long ZpX\_disc\_val(GEN f, GEN p) returns the valuation at  $p$  of the discriminant of  $f$ . Assume that  $f$  is a monic *separable* ZX and that  $p$  is a prime number. Proceeds by dynamically increasing the  $p$ -adic accuracy; infinite loop if the discriminant of  $f$  is 0.

long ZpX\_resultant\_val(GEN f, GEN g, GEN p, long M) returns the valuation at  $p$  of  $\text{Res}(f, g)$ . Assume  $f, g$  are both ZX, and that  $p$  is a prime number coprime to the leading coefficient of  $f$ . Proceeds by dynamically increasing the  $p$ -adic accuracy. To avoid an infinite loop when the resultant is 0, we return  $M$  if the Sylvester matrix mod  $p^M$  still does not have maximal rank.

GEN ZpX\_gcd(GEN f, GEN g, GEN p, GEN pm)  $f$  a monic ZX,  $g$  a ZX,  $pm = p^m$  a prime power. There is a unique integer  $r \geq 0$  and a monic  $h \in \mathbf{Q}_p[X]$  such that

$$p^r h \mathbf{Z}_p[X] + p^m \mathbf{Z}_p[X] = f \mathbf{Z}_p[X] + g \mathbf{Z}_p[X] + p^m \mathbf{Z}_p[X].$$

Return the 0 polynomial if  $r \geq m$  and a monic  $h \in \mathbf{Z}[1/p][X]$  otherwise (whose valuation at  $p$  is  $> -m$ ).

GEN ZpX\_reduced\_resultant(GEN f, GEN g, GEN p, GEN pm)  $f$  a monic ZX,  $g$  a ZX,  $pm = p^m$  a prime power. The  $p$ -adic *reduced resultant* of  $f$  and  $g$  is 0 if  $f, g$  not coprime in  $\mathbf{Z}_p[X]$ , and otherwise the generator of the form  $p^d$  of

$$(f \mathbf{Z}_p[X] + g \mathbf{Z}_p[X]) \cap \mathbf{Z}_p.$$

Return the reduced resultant modulo  $p^m$ .

**GEN ZpX\_reduced\_resultant\_fast**(GEN f, GEN g, GEN p, long M)  $f$  a monic ZX,  $g$  a ZX,  $p$  a prime. Returns the  $p$ -adic reduced resultant of  $f$  and  $g$  modulo  $p^M$ . This function computes resultants for a sequence of increasing  $p$ -adic accuracies (up to  $M$   $p$ -adic digits), returning as soon as it obtains a non-zero result. It is very inefficient when the resultant is 0, but otherwise usually more efficient than computations using a priori bounds.

**GEN ZpX\_monice\_factor**(GEN f, GEN p, long M)  $f$  a monic ZX,  $p$  a primer, return the  $p$ -adic factorization of  $f$ , modulo  $p^M$ . This is the underlying low-level recursive function behind **factorpadic** (using a combination of Round 4 factorization and Hensel lifting); the factors are not sorted and the function is not gerepile-clean.

### 7.3.37 ZpXQ.

**GEN ZpXQ\_invlift**(GEN b, GEN a, GEN T, GEN p, long e) let  $p$  be a prime t\_INT,  $a$  be a FpXQ (modulo  $(p, T)$ ) and  $b$  a ZpXQ such that  $ab \equiv 1 \pmod{(p, T)}$ . Returns an ZpXQ  $A$  such that  $A \equiv a \pmod{p}$  and  $Ab \equiv 1 \pmod{(p^e, T)}$ .

**GEN ZpXQ\_inv**(GEN b, GEN T, GEN p, long e) let  $p$  be a prime t\_INT and  $b$  be a FpXQ (modulo  $T, p^e$ ). Returns an FpXQ  $A$  such that  $Ab \equiv 1 \pmod{(p^e, T)}$ .

**GEN ZpXQ\_div**(GEN a, GEN b, GEN T, GEN q, GEN p, long e) let  $p$  be a prime t\_INT and  $a$  and  $b$  be a FpXQ (modulo  $T, p^e$ ). Returns an FpXQ  $c$  such that  $cb \equiv a \pmod{(p^e, T)}$ . The parameter  $q$  must be equal to  $p^e$ .

**GEN ZpXQ\_sqrtnlift**(GEN b, GEN n, GEN a, GEN T, GEN p, long e) let  $n, p$  be t\_INTs, with  $n, p > 1$  and  $p$  coprime to  $n$ , and  $a, b$  be FpXQs (modulo  $T$ ) such that  $a^n \equiv b \pmod{(p, T)}$ . Returns an Fq  $A$  such that  $A^n \equiv b \pmod{(p^e, T)}$ .

**GEN ZpXQ\_sqrt**(GEN b, GEN T, GEN p, long e) let  $p$  being a odd prime and  $b$  be a FpXQ (modulo  $T, p^e$ ), returns  $a$  such that  $a^2 \equiv b \pmod{(p^e, T)}$ .

**GEN ZpX\_ZpXQ\_liftroot**(GEN f, GEN a, GEN T, GEN p, long e) as ZpXQX\_liftroot, but  $f$  is a polynomial in  $\mathbf{Z}[X]$ .

**GEN ZpX\_ZpXQ\_liftroot\_ea**(GEN f, GEN a, GEN T, GEN p, long e, void \*E, int early(void \*E, GEN x, GEN q)) as ZpX\_ZpXQ\_liftroot with early abort: the function **early**(E,x,q) will be called with  $x$  is a root of  $f$  modulo  $q = p^n$  for some  $n$ . If **early** returns a non-zero value, the function returns  $x$  immediately.

**GEN ZpXQ\_log**(GEN a, GEN T, GEN p, long e)  $T$  being a ZpX irreducible modulo  $p$ , return the logarithm of  $a$  in  $\mathbf{Z}_p[X]/(T)$  to precision  $e$ , assuming that  $a \equiv 1 \pmod{p\mathbf{Z}_p[X]}$  if  $p$  odd or  $a \equiv 1 \pmod{4\mathbf{Z}_2[X]}$  if  $p = 2$ .

### 7.3.38 Zq.

**GEN Zq\_sqrtnlift**(GEN b, GEN n, GEN a, GEN T, GEN p, long e)

### 7.3.39 ZpXQM.

**GEN ZpXQM\_prodFrobenius**(GEN M, GEN T, GEN p, long e) returns the product of matrices  $M\sigma(M)\sigma^2(M)\dots\sigma^{n-1}(M)$  to precision  $e$  where  $\sigma$  is the lift of the Frobenius automorphism over  $\mathbf{Z}_p[X]/(T)$  and  $n$  is the degree of  $T$ .

### 7.3.40 ZpXQX.

GEN ZpXQX\_liftfact(GEN A, GEN B, GEN T, GEN pe, GEN p, long e) is the routine underlying polhensellift. Here,  $p$  is prime,  $T(Y)$  defines a finite field  $\mathbf{F}_q$ .  $A$  is a polynomial in  $\mathbf{Z}[X, Y]$ , whose leading coefficient is non-zero in  $\mathbf{F}_q$ .  $B$  is a vector of monic or  $\mathbf{F}_q[X]$ , pairwise coprime in  $\mathbf{F}_q[X]$ , whose product is congruent to  $A/\text{lc}(A)$  in  $\mathbf{F}_q[X]$ . Lifts the elements of  $B \bmod pe = p^e$ , such that the congruence now holds  $\bmod (T, p^e)$ .

GEN ZpXQX\_liftroot(GEN f, GEN a, GEN T, GEN p, long e) as ZpX\_liftroot, but  $f$  is now a polynomial in  $\mathbf{Z}[X, Y]$  and lift the root  $a$  in the unramified extension of  $\mathbf{Q}_p$  with residue field  $\mathbf{F}_p[Y]/(T)$ , assuming  $v_p(f(a)) > 0$  and  $v_p(f'(a)) = 0$ .

GEN ZpXQX\_liftroot\_vald(GEN f, GEN a, long v, GEN T, GEN p, long e) returns the foofs of  $f$  as ZpXQX\_liftroot, where  $v$  is the valuation of the content of  $f'$  and it is required that  $v_p(f(a)) > v$  and  $v_p(f'(a)) = v$ .

GEN ZpXQX\_roots(GEN F, GEN T, GEN p, long e)

GEN ZpXQX\_divrem(GEN x, GEN Sp, GEN T, GEN q, GEN p, long e, GEN \*pr) as FpXQX\_divrem. The parameter  $q$  must be equal to  $p^e$ .

GEN ZpXQX\_digits(GEN x, GEN B, GEN T, GEN q, GEN p, long e) As FpXQX\_digits. The parameter  $q$  must be equal to  $p^e$ .

### 7.3.41 ZqX.

GEN ZqX\_roots(GEN F, GEN T, GEN p, long e)

GEN ZqX\_liftfact(GEN A, GEN B, GEN T, GEN pe, GEN p, long e)

GEN ZqX\_liftroot(GEN f, GEN a, GEN T, GEN p, long e)

### 7.3.42 Other $p$ -adic functions.

GEN ZpM\_echelon(GEN M, long early\_abort, GEN p, GEN pm) given a ZM  $M$ , a prime  $p$  and  $pm = p^m$ , returns an echelon form  $E$  for  $M \bmod p^m$ . I.e. there exist a square integral matrix  $U$  with  $\det U$  coprime to  $p$  such that  $E = MU$  modulo  $p^m$ . If  $\text{early\_abort}$  is non-zero, return NULL as soon as one pivot in the echelon form is divisible by  $p^m$ . The echelon form is an upper triangular HNF, we do not waste time to reduce it to Gauss-Jordan form.

GEN zlm\_echelon(GEN M, long early\_abort, ulong p, ulong pm) variant of ZpM\_echelon, for a Zlm  $M$ .

GEN Zlm\_gauss(GEN a, GEN b, ulong p, long e, GEN C) as gauss with the following peculiarities:  $a$  and  $b$  are ZM, such that  $a$  is invertible modulo  $p$ . Optional  $C$  is an Flm that is an inverse of  $a \bmod p$  or NULL. Return the matrix  $x$  such that  $ax = b \bmod p^e$  and all elements of  $x$  are in  $[0, p^e - 1]$ . For efficiency, it is better to reduce  $a$  and  $b \bmod p^e$  first.

GEN padic\_to\_Q(GEN x) truncate the t\_PADIC to a t\_INT or t\_FRAC.

GEN padic\_to\_Q\_shallow(GEN x) shallow version of padic\_to\_Q

GEN QpV\_to\_QV(GEN v) apply padic\_to\_Q\_shallow

long padicprec(GEN x, GEN p) returns the absolute  $p$ -adic precision of the object  $x$ , by definition the minimum precision of the components of  $x$ . For a non-zero t\_PADIC, this returns  $\text{valp}(x) + \text{prec}(x)$ .



`long padicprec_relative(GEN x)` returns the relative  $p$ -adic precision of the `t_INT`, `t_FRAC`, or `t_PADIC`  $x$  (minimum precision of the components of  $x$  for `t_POL` or vector/matrices). For a `t_PADIC`, this returns `prec(x)` if  $x \neq 0$ , and 0 for  $x = 0$ .

### 7.3.42.1 low-level.

The following technical function returns an optimal sequence of  $p$ -adic accuracies, for a given target accuracy:

`ulong quadratic_prec_mask(long n)` we want to reach accuracy  $n \geq 1$ , starting from accuracy 1, using a quadratically convergent, self-correcting, algorithm; in other words, from inputs correct to accuracy  $l$  one iteration outputs a result correct to accuracy  $2l$ . For instance, to reach  $n = 9$ , we want to use accuracies  $[1, 2, 3, 5, 9]$  instead of  $[1, 2, 4, 8, 9]$ . The idea is to essentially double the accuracy at each step, and not overshoot in the end.

Let  $a_0 = 1, a_1 = 2, \dots, a_k = n$ , be the desired sequence of accuracies. To obtain it, we work backwards and set

$$a_k = n, \quad a_{i-1} = (a_i + 1) \setminus 2.$$

This is in essence what the function returns. But we do not want to store the  $a_i$  explicitly, even as a `t_VECSMALL`, since this would leave an object on the stack. Instead, we store  $a_i$  implicitly in a bitmask `MASK`: let  $a_0 = 1$ , if the  $i$ -th bit of the mask is set, set  $a_{i+1} = 2a_i - 1$ , and  $2a_i$  otherwise; in short the bits indicate the places where we do something special and do not quite double the accuracy (which would be the straightforward thing to do).

In fact, to avoid returning separately the mask and the sequence length  $k + 1$ , the function returns `MASK + 2k+1`, so the highest bit of the mask indicates the length of the sequence, and the following ones give an algorithm to obtain the accuracies. This is much simpler than it sounds, here is what it looks like in practice:

```
ulong mask = quadratic_prec_mask(n);
long l = 1;
while (mask > 1) {
    /* here, the result is known to accuracy l */
    l = 2*l; if (mask & 1) l--; /* new accuracy l for the iteration */
    mask >>= 1; /* pop low order bit */
    /* ... lift to the new accuracy ... */
}
/* we are done. At this point l = n */
```

We just pop the bits in `mask` starting from the low order bits, stop when `mask` is 1 (that last bit corresponds to the  $2^{k+1}$  that we added to the mask proper). Note that there is nothing specific to Hensel lifts in that function: it would work equally well for an Archimedean Newton iteration.

Note that in practice, we rather use an infinite loop, and insert an

```
if (mask == 1) break;
```

in the middle of the loop: the loop body usually includes preparations for the next iterations (e.g. lifting Bezout coefficients in a quadratic Hensel lift), which are costly and useless in the *last* iteration.

### 7.3.43 Conversions involving single precision objects.

#### 7.3.43.1 To single precision.

`ulong Rg_to_Fl(GEN z, ulong p)`,  $z$  which can be mapped to  $\mathbf{Z}/p\mathbf{Z}$ : a `t_INT`, a `t_INTMOD` whose modulus is divisible by  $p$ , a `t_FRAC` whose denominator is coprime to  $p$ , or a `t_PADIC` with underlying prime  $\ell$  satisfying  $p = \ell^n$  for some  $n$  (less than the accuracy of the input). Returns `lift(z * Mod(1,p))`, normalized, as an `Fl`.

`ulong Rg_to_F2(GEN z)`, as `Rg_to_Fl` for  $p = 2$ .

`ulong padic_to_Fl(GEN x, ulong p)` special case of `Rg_to_Fl`, for a  $x$  a `t_PADIC`.

`GEN RgX_to_F2x(GEN x)`,  $x$  a `t_POL`, returns the `F2x` obtained by applying `Rg_to_Fl` coefficientwise.

`GEN RgX_to_Flx(GEN x, ulong p)`,  $x$  a `t_POL`, returns the `Flx` obtained by applying `Rg_to_Fl` coefficientwise.

`GEN Rg_to_F2xq(GEN z, GEN T)`,  $z$  a `GEN` which can be mapped to  $\mathbf{F}_2[X]/(T)$ : anything `Rg_to_Fl` can be applied to, a `t_POL` to which `RgX_to_F2x` can be applied to, a `t_POLMOD` whose modulus is divisible by  $T$  (once mapped to a `F2x`), a suitable `t_RFRAC`. Returns  $z$  as an `F2xq`, normalized.

`GEN Rg_to_Flxq(GEN z, GEN T, ulong p)`,  $z$  a `GEN` which can be mapped to  $\mathbf{F}_p[X]/(T)$ : anything `Rg_to_Fl` can be applied to, a `t_POL` to which `RgX_to_Flx` can be applied to, a `t_POLMOD` whose modulus is divisible by  $T$  (once mapped to a `Flx`), a suitable `t_RFRAC`. Returns  $z$  as an `Flxq`, normalized.

`GEN RgX_to_FlxqX(GEN z, GEN T, ulong p)`,  $z$  a `GEN` which can be mapped to  $\mathbf{F}_p[x]/(T)[X]$ : anything `Rg_to_Flxq` can be applied to, a `t_POL` to which `RgX_to_Flx` can be applied to, a `t_POLMOD` whose modulus is divisible by  $T$  (once mapped to a `Flx`), a suitable `t_RFRAC`. Returns  $z$  as an `FlxqX`, normalized.

`GEN ZX_to_Flx(GEN x, ulong p)` reduce  $ZX\ x$  modulo  $p$  (yielding an `Flx`). Faster than `RgX_to_Flx`.

`GEN ZV_to_Flv(GEN x, ulong p)` reduce  $ZV\ x$  modulo  $p$  (yielding an `Flv`).

`GEN ZXV_to_FlxV(GEN v, ulong p)`, as `ZX_to_Flx`, repeatedly called on the vector's coefficients.

`GEN ZXT_to_FlxT(GEN v, ulong p)`, as `ZX_to_Flx`, repeatedly called on the tree leaves.

`GEN ZXX_to_FlxX(GEN B, ulong p, long v)`, as `ZX_to_Flx`, repeatedly called on the polynomial's coefficients.

`GEN zxX_to_FlxX(GEN z, ulong p)` as `zx_to_Flx`, repeatedly called on the polynomial's coefficients.

`GEN ZXXV_to_FlxXV(GEN V, ulong p, long v)`, as `ZXX_to_FlxX`, repeatedly called on the vector's coefficients.

`GEN ZXXT_to_FlxXT(GEN V, ulong p, long v)`, as `ZXX_to_FlxX`, repeatedly called on the tree leaves.

`GEN RgV_to_Flv(GEN x, ulong p)` reduce the `t_VEC/t_COL`  $x$  modulo  $p$ , yielding a `t_VECSMALL`.

`GEN RgM_to_Flm(GEN x, ulong p)` reduce the `t_MAT`  $x$  modulo  $p$ .

`GEN ZM_to_Flm(GEN x, ulong p)` reduce  $ZM\ x$  modulo  $p$  (yielding an `Flm`).

`GEN ZV_to_zv(GEN z)`, converts coefficients using `itos`

GEN ZV\_to\_nv(GEN z), converts coefficients using itou

GEN ZM\_to\_zm(GEN z), converts coefficients using itos

GEN FqC\_to\_FlxC(GEN x, GEN T, GEN p), converts coefficients in Fq to coefficient in Flx, result being a column vector.

GEN FqV\_to\_FlxV(GEN x, GEN T, GEN p), converts coefficients in Fq to coefficient in Flx, result being a line vector.

GEN FqM\_to\_FlxM(GEN x, GEN T, GEN p), converts coefficients in Fq to coefficient in Flx.

**7.3.43.2 From single precision.**

GEN Flx\_to\_ZX(GEN z), converts to ZX (t\_POL of non-negative t\_INTs in this case)

GEN Flx\_to\_FlxX(GEN z), converts to FlxX (t\_POL of constant Flx in this case).

GEN Flx\_to\_ZX\_inplace(GEN z), same as Flx\_to\_ZX, in place (z is destroyed).

GEN FlxX\_to\_ZXX(GEN B), converts an FlxX to a polynomial with ZX or t\_INT coefficients (repeated calls to Flx\_to\_ZX).

GEN FlxXC\_to\_ZXXC(GEN B), converts an FlxXC to a t\_COL with ZXX coefficients (repeated calls to FlxX\_to\_ZXX).

GEN FlxXM\_to\_ZXXM(GEN B), converts an FlxXM to a t\_MAT with ZXX coefficients (repeated calls to FlxX\_to\_ZXX).

GEN FlxC\_to\_ZXC(GEN x), converts a vector of Flx to a column vector of polynomials with t\_INT coefficients (repeated calls to  
    Flx\_to\_ZX).

GEN FlxV\_to\_ZXV(GEN x), as above but return a t\_VEC.

void F2xV\_to\_FlxV\_inplace(GEN v) v is destroyed.

void F2xV\_to\_ZXV\_inplace(GEN v) v is destroyed.

void FlxV\_to\_ZXV\_inplace(GEN v) v is destroyed.

GEN FlxM\_to\_ZXM(GEN z), converts a matrix of Flx to a matrix of polynomials with t\_INT coefficients (repeated calls to Flx\_to\_ZX).

GEN zx\_to\_ZX(GEN z), as Flx\_to\_ZX, without assuming the coefficients to be non-negative.

GEN zx\_to\_Flx(GEN z, ulong p) as Flx\_red without assuming the coefficients to be non-negative.

GEN Flc\_to\_ZC(GEN z), converts to ZC (t\_COL of non-negative t\_INTs in this case)

GEN Flc\_to\_ZC\_inplace(GEN z), same as Flc\_to\_ZC, in place (z is destroyed).

GEN Flv\_to\_ZV(GEN z), converts to ZV (t\_VEC of non-negative t\_INTs in this case)

GEN Flm\_to\_ZM(GEN z), converts to ZM (t\_MAT with non-negative t\_INTs coefficients in this case)

GEN Flm\_to\_ZM\_inplace(GEN z), same as Flm\_to\_ZM, in place (z is destroyed).

GEN zc\_to\_ZC(GEN z) as Flc\_to\_ZC, without assuming coefficients are non-negative.

GEN zv\_to\_ZV(GEN z) as Flv\_to\_ZV, without assuming coefficients are non-negative.

GEN zm\_to\_ZM(GEN z) as Flm\_to\_ZM, without assuming coefficients are non-negative.

GEN zv\_to\_Flv(GEN z, ulong p)

GEN zm\_to\_Flm(GEN z, ulong p)

**7.3.43.3 Mixed precision linear algebra.** Assumes dimensions are compatible. Multiply a multiprecision object by a single-precision one.

GEN RgM\_zc\_mul(GEN x, GEN y)

GEN RgMrow\_zc\_mul(GEN x, GEN y, long i)

GEN RgM\_zm\_mul(GEN x, GEN y)

GEN RgV\_zc\_mul(GEN x, GEN y)

GEN RgV\_zm\_mul(GEN x, GEN y)

GEN ZM\_zc\_mul(GEN x, GEN y)

GEN zv\_ZM\_mul(GEN x, GEN y)

GEN ZV\_zc\_mul(GEN x, GEN y)

GEN ZM\_zm\_mul(GEN x, GEN y)

GEN ZC\_z\_mul(GEN x, long y)

GEN ZM\_nm\_mul(GEN x, GEN y) the entries of  $y$  are ulongs.

GEN nm\_Z\_mul(GEN y, GEN c) the entries of  $y$  are ulongs.

#### 7.3.43.4 Miscellaneous involving Fl.

GEN Fl\_to\_Flx(ulong x, long evx) converts a unsigned long to a scalar Flx. Assume that  $evx = evalvarn(vx)$  for some variable number  $vx$ .

GEN Z\_to\_Flx(GEN x, ulong p, long sv) converts a  $t\_INT$  to a scalar Flx polynomial. Assume that  $sv = evalvarn(v)$  for some variable number  $v$ .

GEN Flx\_to\_Flv(GEN x, long n) converts from Flx to Flv with  $n$  components (assumed larger than the number of coefficients of  $x$ ).

GEN zx\_to\_zv(GEN x, long n) as Flx\_to\_Flv.

GEN Flv\_to\_Flx(GEN x, long sv) converts from vector (coefficient array) to (normalized) polynomial in variable  $v$ .

GEN zv\_to\_zx(GEN x, long n) as Flv\_to\_Flx.

GEN Flm\_to\_FlxV(GEN x, long sv) converts the columns of Flm  $x$  to an array of Flx in the variable  $v$  (repeated calls to Flv\_to\_Flx).

GEN zm\_to\_zxV(GEN x, long n) as Flm\_to\_FlxV.

GEN Flm\_to\_FlxX(GEN x, long sw, long sv) same as Flm\_to\_FlxV( $x, sv$ ) but returns the result as a (normalized) polynomial in variable  $w$ .

GEN FlxV\_to\_Flm(GEN v, long n) reverse Flm\_to\_FlxV, to obtain an Flm with  $n$  rows (repeated calls to Flx\_to\_Flv).

GEN FlxX\_to\_Flx(GEN P) Let  $P(x, X)$  be a FlxX, return  $P(0, X)$  as a Flx.

GEN FlxX\_to\_Flm(GEN v, long n) reverse Flm\_to\_FlxX, to obtain an Flm with n rows (repeated calls to Flx\_to\_Flv).

GEN FlxX\_to\_FlxC(GEN B, long n, long sv) see RgX\_to\_RgV. The coefficients of B are assumed to be in the variable v.

GEN FlxXV\_to\_FlxM(GEN V, long n, long sv) see RgXV\_to\_RgM. The coefficients of V[i] are assumed to be in the variable v.

GEN Fly\_to\_FlxY(GEN a, long sv) convert coefficients of a to constant Flx in variable v.

### 7.3.43.5 Miscellaneous involving F2x.

GEN F2x\_to\_F2v(GEN x, long n) converts from F2x to F2v with n components (assumed larger than the number of coefficients of x).

GEN F2xC\_to\_ZXC(GEN x), converts a vector of F2x to a column vector of polynomials with t\_INT coefficients (repeated calls to F2x\_to\_ZX).

GEN F2xC\_to\_FlxC(GEN x)

GEN FlxC\_to\_F2xC(GEN x)

GEN F2xV\_to\_F2m(GEN v, long n) F2x\_to\_F2v to each polynomial to get an F2m with n rows.

## 7.4 Higher arithmetic over Z: primes, factorization.

### 7.4.1 Pure powers.

long Z\_issquare(GEN n) returns 1 if the t\_INT n is a square, and 0 otherwise. This is tested first modulo small prime powers, then sqtremi is called.

long Z\_issquareall(GEN n, GEN \*sqrtn) as Z\_issquare. If n is indeed a square, set sqrtn to its integer square root. Uses a fast congruence test mod  $64 \times 63 \times 65 \times 11$  before computing an integer square root.

long Z\_ispow2(GEN x) returns 1 if the t\_INT x is a power of 2, and 0 otherwise.

long uissquare(ulong n) as Z\_issquare, for an ulong operand n.

long uissquareall(ulong n, ulong \*sqrtn) as Z\_issquareall, for an ulong operand n.

ulong usqrt(ulong a) returns the floor of the square root of a.

ulong usqrtn(ulong a, ulong n) returns the floor of the n-th root of a.

long Z\_ispower(GEN x, ulong k) returns 1 if the t\_INT n is a k-th power, and 0 otherwise; assume that  $k > 1$ .

long Z\_ispowerall(GEN x, ulong k, GEN \*pt) as Z\_ispower. If n is indeed a k-th power, set \*pt to its integer k-th root.

long Z\_isanypower(GEN x, GEN \*ptn) returns the maximal  $k \geq 2$  such that the t\_INT  $x = n^k$  is a perfect power, or 0 if no such k exist; in particular ispower(1), ispower(0), ispower(-1) all return 0. If the return value k is not 0 (so that  $x = n^k$ ) and ptn is not NULL, set \*ptn to n.

The following low-level functions are called by Z\_isanypower but can be directly useful:

`int is_357_power(GEN x, GEN *ptn, ulong *pmask)` tests whether the integer  $x > 0$  is a 3-rd, 5-th or 7-th power. The bits of `*mask` initially indicate which test is to be performed; bit 0: 3-rd, bit 1: 5-th, bit 2: 7-th (e.g. `*pmask = 7` performs all tests). They are updated during the call: if the “ $i$ -th power” bit is set to 0 then  $x$  is not a  $k$ -th power. The function returns 0 (not a 3-rd, 5-th or 7-th power), 3 (3-rd power, not a 5-th or 7-th power), 5 (5-th power, not a 7-th power), or 7 (7-th power); if an  $i$ -th power bit is initially set to 0, we take it at face value and assume  $x$  is not an  $i$ -th power without performing any test. If the return value  $k$  is non-zero, set `*ptn` to  $n$  such that  $x = n^k$ .

`int is_pth_power(GEN x, GEN *ptn, forprime_t *T, ulong cutoff)` let  $x > 0$  be an integer, `cutoff`  $> 0$  and  $T$  be an iterator over primes  $\geq 11$ , we look for the smallest prime  $p$  such that  $x = n^p$  (advancing  $T$  as we go along). The 11 is due to the fact that `is_357_power` and `issquare` are faster than the generic version for  $p < 11$ .

Fail and return 0 when the existence of  $p$  would imply  $2^{\text{cutoff}} > x^{1/p}$ , meaning that a possible  $n$  is so small that it should have been found by trial division; for maximal speed, you should start by a round of trial division, but the cut-off may also be set to 1 for a rigorous result without any trial division.

Otherwise returns the smallest suitable prime power  $p^i$  and set `*ptn` to the  $p^i$ -th root of  $x$  (which is now not a  $p$ -th power). We may immediately recall the function with the same parameters after setting  $x = \text{*ptn}$ : it will start at the next prime.

#### 7.4.2 Factorization.

`GEN Z_factor(GEN n)` factors the `t_INT`  $n$ . The “primes” in the factorization are actually strong pseudoprimes.

`GEN absZ_factor(GEN n)` returns `Z_factor(absi(n))`.

`long Z_issmooth(GEN n, ulong lim)` returns 1 if all the prime factors of the `t_INT`  $n$  are less or equal to  $lim$ .

`GEN Z_issmooth_fact(GEN n, ulong lim)` returns NULL if a prime factor of the `t_INT`  $n$  is  $> lim$ , and returns the factorization of  $n$  otherwise, as a `t_MAT` with `t_VECSMALL` columns (word-size primes and exponents). Neither memory-clean nor suitable for `gerepileupto`.

`GEN Z_factor_until(GEN n, GEN lim)` as `Z_factor`, but stop the factorization process as soon as the unfactored part is smaller than  $lim$ . The resulting factorization matrix only contains the factors found. No other assumptions can be made on the remaining factors.

`GEN Z_factor_limit(GEN n, ulong lim)` trial divide  $n$  by all primes  $p < lim$  in the precomputed list of prime numbers and return the corresponding factorization matrix. In this case, the last “prime” divisor in the first column of the factorization matrix may well be a proven composite.

If  $lim = 0$ , the effect is the same as setting  $lim = \text{maxprime}() + 1$ : use all precomputed primes.

`GEN absZ_factor_limit(GEN n, ulong all)` returns `Z_factor_limit(absi(n))`.

`GEN boundfact(GEN x, ulong lim)` as `Z_factor_limit`, applying to `t_INT` or `t_FRAC` inputs.

`GEN Z_smoothen(GEN n, GEN L, GEN *pP, GEN *pE)` given a `t_VECSMALL`  $L$  containing a list of small primes and a `t_INT`  $n$ , trial divide  $n$  by the elements of  $L$  and return the cofactor. Return NULL if the cofactor is  $\pm 1$ . `*P` and `*E` contain the list of prime divisors found and their exponents, as `t_VECSMALL`s. Neither memory-clean, nor suitable for `gerepileupto`.

**GEN Z\_factor\_listP**(GEN *N*, GEN *L*) given a **t\_INT** *N*, a vector or primes *L* containing all prime divisors of *N* (and possibly others). Return **factor**(*N*). Neither memory-clean, nor suitable for **gerepileupto**.

**GEN factor\_pn\_1**(GEN *p*, ulong *n*) returns the factorization of  $p^n - 1$ , where *p* is prime and *n* is a positive integer.

**GEN factor\_pn\_1\_limit**(GEN *p*, ulong *n*, ulong *B*) returns a partial factorization of  $p^n - 1$ , where *p* is prime and *n* is a positive integer. Don't actively search for prime divisors  $p > B$ , but we may find still find some due to Aurifeuillian factorizations. Any entry  $> B^2$  in the output factorization matrix is *a priori* not a prime (but may well be).

**GEN factor\_Aurifeuille\_prime**(GEN *p*, long *n*) an Aurifeuillian factor of  $\phi_n(p)$ , assuming *p* prime and an Aurifeuillian factor exists ( $p\zeta_n$  is a square in  $\mathbf{Q}(\zeta_n)$ ).

**GEN factor\_Aurifeuille**(GEN *a*, long *d*) an Aurifeuillian factor of  $\phi_n(a)$ , assuming *a* is a non-zero integer and  $n > 2$ . Returns 1 if no Aurifeuillian factor exists.

**GEN odd\_prime\_divisors**(GEN *a*) **t\_VEC** of all prime divisors of the **t\_INT** *a*.

**GEN factoru**(ulong *n*), returns the factorization of *n*. The result is a 2-component vector [*P*, *E*], where *P* and *E* are **t\_VECSMALL** containing the prime divisors of *n*, and the  $v_p(n)$ .

**GEN factoru\_pow**(ulong *n*), returns the factorization of *n*. The result is a 3-component vector [*P*, *E*, *C*], where *P*, *E* and *C* are **t\_VECSMALL** containing the prime divisors of *n*, the  $v_p(n)$  and the  $p^{v_p(n)}$ .

**GEN vecfactoru**(ulong *a*, ulong *b*), returns a **t\_VEC** *v* containing the factorizations (**factoru** format) of  $a, \dots, b$ ; assume that  $b \geq a > 0$ . Uses a sieve with primes up to  $\sqrt{b}$ . For all *c*,  $a \leq c \leq b$ , the factorization of *c* is given in  $v[c - a + 1]$ .

**GEN vecfactoroddu**(ulong *a*, ulong *b*), returns a **t\_VEC** *v* containing the factorizations (**factoru** format) of odd integers in  $a, \dots, b$ ; assume that  $b \geq a > 0$  are odd. Uses a sieve with primes up to  $\sqrt{b}$ . For all odd *c*,  $a \leq c \leq b$ , the factorization of *c* is given in  $v[(c - a)/2 + 1]$ .

**GEN vecfactoru\_i**(ulong *a*, ulong *b*), private version of **vecfactoru**, not memory clean.

**GEN vecfactoroddu\_i**(ulong *a*, ulong *b*), private version of **vecfactoroddu**, not memory clean.

**GEN vecfactorsquarefreeu**(ulong *a*, ulong *b*) return a **t\_VEC** *v* containing the prime divisors of squarefree integers in  $a, \dots, b$ ; assume that  $a \leq b$ . Uses a sieve with primes up to  $\sqrt{b}$ . For all squarefree *c*,  $a \leq c \leq b$ , the prime divisors of *c* (as a **t\_VECSMALL**) are given in  $v[c - a + 1]$ , and the other entries are NULL. Note that because of these NULL markers, *v* is not a valid **GEN**, it is not memory clean and cannot be used in garbage collection routines.

**GEN vecsquarefreeu**(ulong *a*, ulong *b*) return a **t\_VECSMALL** *v* containing the squarefree integers in  $a, \dots, b$ . Assume that  $a \leq b$ . Uses a sieve with primes up to  $\sqrt{b}$ .

ulong **tridiv\_bound**(GEN *n*) returns the trial division bound used by **Z\_factor**(*n*).

**GEN Z\_pollardbrent**(GEN *N*, long *n*, long *seed*) try to factor **t\_INT** *N* using  $n \geq 1$  rounds of Pollard iterations; *seed* is an integer whose value (mod 8) selects the quadratic polynomial use to generate Pollard's (pseudo)random walk. Returns NULL on failure, else a vector of 2 (possibly 3) integers whose product is *N*.

**GEN Z\_ECM**(GEN *N*, long *n*, long *seed*, ulong *B1*) try to factor **t\_INT** *N* using  $n \geq 1$  rounds of ECM iterations (on 8 to 64 curves simultaneously, depending on the size of *N*); *seed* is an integer

whose value selects the curves to be used: increase it by  $64n$  to make sure that a subsequent call with a factor of  $N$  uses a disjoint set of curves. Finally  $B_1 > 7$  determines the computations performed on the curves: we compute  $[k]P$  for some point in  $E(\mathbf{Z}/N\mathbf{Z})$  and  $k = q \prod p^{e_p}$  where  $p^{e_p} \leq B_1$  and  $q \leq B_2 := 110B_1$ ; a higher value of  $B_1$  means higher chances of hitting a factor and more time spent. The computation is deterministic for a given set of parameters. Returns NULL on failure, else a non trivial factor or N.

GEN `Q_factor`(GEN `x`) as `Z_factor`, where `x` is a `t_INT` or a `t_FRAC`.

GEN `Q_factor_limit`(GEN `x`, ulong `lim`) as `Z_factor_limit`, where `x` is a `t_INT` or a `t_FRAC`.

### 7.4.3 Coprime factorization.

Given  $a$  and  $b$  two non-zero integers, let **ppi**( $a, b$ ), **ppo**( $a, b$ ), **ppg**( $a, b$ ), **ppl**( $a, b$ ) (powers in  $a$  of primes inside  $b$ , outside  $b$ , greater than those in  $b$ , less than or equal to those in  $b$ ) be the integers defined by

- $v_p(\text{ppi}) = v_p(a)[v_p(b) > 0]$ ,
- $v_p(\text{ppo}) = v_p(a)[v_p(b) = 0]$ ,
- $v_p(\text{ppg}) = v_p(a)[v_p(a) > v_p(b)]$ ,
- $v_p(\text{ppl}) = v_p(a)[v_p(a) \leq v_p(b)]$ .

GEN `Z_ppo`(GEN `a`, GEN `b`) returns `ppo(a, b)`; shallow function.

ulong `u_ppo`(ulong `a`, ulong `b`) returns `ppo(a, b)`.

GEN `Z_ppgle`(GEN `a`, GEN `b`) returns `[ppg(a, b), ppl(a, b)]`; shallow function.

GEN `Z_ppio`(GEN `a`, GEN `b`) returns `[gcd(a, b), ppi(a, b), ppo(a, b)]`; shallow function.

GEN `Z_cba`(GEN `a`, GEN `b`) fast natural coprime base algorithm. Returns a vector of coprime divisors of  $a$  and  $b$  such that both  $a$  and  $b$  can be multiplicatively generated from this set. Perfect powers are not removed, is `Z_isanypower` if needed; shallow function.

GEN `ZV_cba_extend`(GEN `P`, GEN `b`) extend a coprime basis  $P$  by the integer  $b$ , the result being a coprime basis for  $P \cup \{b\}$ . Perfect powers are not removed; shallow function.

GEN `ZV_cba`(GEN `v`) given a vector of non-zero integers  $v$ , return a coprime basis for  $v$ . Perfect powers are not removed; shallow function.

### 7.4.4 Checks attached to arithmetic functions.

Arithmetic functions accept arguments of the following kind: a plain positive integer  $N$  (`t_INT`), the factorization  $fa$  of a positive integer (a `t_MAT` with two columns containing respectively primes and exponents), or a vector  $[N, fa]$ . A few functions accept non-zero integers (e.g. `omega`), and some others arbitrary integers (e.g. `factorint`, ...).

int `is_Z_factorpos`(GEN `f`) returns 1 if  $f$  looks like the factorization of a positive integer, and 0 otherwise. Useful for sanity checks but not 100% foolproof. Specifically, this routine checks that  $f$  is a two-column matrix all of whose entries are positive integers. It does *not* check that entries in the first column ("primes") are prime, or even pairwise coprime, nor that they are strictly increasing.

int `is_Z_factornon0`(GEN `f`) returns 1 if  $f$  looks like the factorization of a non-zero integer, and 0 otherwise. Useful for sanity checks but not 100% foolproof, analogous to `is_Z_factorpos`. (Entries in the first column need only be non-zero integers.)



`int is_Z_factor(GEN f)` returns 1 if  $f$  looks like the factorization of an integer, and 0 otherwise. Useful for sanity checks but not 100% foolproof. Specifically, this routine checks that  $f$  is a two-column matrix all of whose entries are integers. Entries in the second column (“exponents”) are all positive. Either it encodes the “factorization”  $0^e$ ,  $e > 0$ , or entries in the first column (“primes”) are all non-zero.

`GEN clean_Z_factor(GEN f)` assuming  $f$  is the factorization of an integer  $n$ , return the factorization of  $|n|$ , i.e. remove  $-1$  from the factorization. Shallow function.

`GEN fuse_Z_factor(GEN f, GEN B)` assuming  $f$  is the factorization of an integer  $n$ , return `boundfact(n, B)`, i.e. return a factorization where all primary factors for  $|p| \leq B$  are preserved, and all others are “fused” into a single composite integer; if that remainder is trivial, i.e. equal to 1, it is of course not included. Shallow function.

In the following three routines,  $f$  is the name of an arithmetic function, and  $n$  a supplied argument. They all raise exceptions if  $n$  does not correspond to an integer or an integer factorization of the expected shape.

`GEN check_arith_pos(GEN n, const char *f)` check whether  $n$  is attached to the factorization of a positive integer, and return NULL (plain `t_INT`) or a factorization extracted from  $n$  otherwise. May raise an `e_DOMAIN` ( $n \leq 0$ ) or an `e_TYPE` exception (other failures).

`GEN check_arith_non0(GEN n, const char *f)` check whether  $n$  is attached to the factorization of a non-0 integer, and return NULL (plain `t_INT`) or a factorization extracted from  $n$  otherwise. May raise an `e_TYPE` exception.

`GEN check_arith_all(GEN n, const char *f)` is attached to the factorization of an integer, and return NULL (plain `t_INT`) or a factorization extracted from  $n$  otherwise.

#### 7.4.5 Incremental integer factorization.

Routines attached to the dynamic factorization of an integer  $n$ , iterating over successive prime divisors. This is useful to implement high-level routines allowed to take shortcuts given enough partial information: e.g. `moebius(n)` can be trivially computed if we hit  $p$  such that  $p^2 \mid n$ . For efficiency, trial division by small primes should have already taken place. In any case, the functions below assume that no prime  $< 2^{14}$  divides  $n$ .

`GEN ifac_start(GEN n, int moebius)` schedules a new factorization attempt for the integer  $n$ . If `moebius` is non-zero, the factorization will be aborted as soon as a repeated factor is detected (Moebius mode). The function assumes that  $n > 1$  is a *composite* `t_INT` whose prime divisors satisfy  $p > 2^{14}$  and that one can write to  $n$  in place.

This function stores data on the stack, no `gerepile` call should delete this data until the factorization is complete. Returns `partial`, a data structure recording the partial factorization state.

`int ifac_next(GEN *partial, GEN *p, long *e)` deletes a primary factor  $p^e$  from `partial` and sets `p` (prime) and `e` (exponent), and normally returns 1. Whatever remains in the `partial` structure is now coprime to  $p$ .

Returns 0 if all primary factors have been used already, so we are done with the factorization. In this case `p` is set to NULL. If we ran in Moebius mode and the factorization was in fact aborted, we have  $e = 1$ , otherwise  $e = 0$ .

`int ifac_read(GEN part, GEN *k, long *e)` peeks at the next integer to be factored in the list  $k^e$ , where  $k$  is not necessarily prime and can be a perfect power as well, but will be factored by the next call to `ifac_next`. You can remove this factorization from the schedule by calling:

`void ifac_skip(GEN part)` removes the next scheduled factorization.

`int ifac_isprime(GEN n)` given  $n$  whose prime divisors are  $> 2^{14}$ , returns the decision the factoring engine would take about the compositeness of  $n$ : 0 if  $n$  is a proven composite, and 1 if we believe it to be prime; more precisely,  $n$  is a proven prime if `factor_proven` is set, and only a BPSW-pseudoprime otherwise.

#### 7.4.6 Integer core, squarefree factorization.

`long Z_issquarefree(GEN n)` returns 1 if the `t_INT`  $n$  is square-free, and 0 otherwise.

`long Z_isfundamental(GEN x)` returns 1 if the `t_INT`  $x$  is a fundamental discriminant, and 0 otherwise.

`GEN core(GEN n)` unique squarefree integer  $d$  dividing  $n$  such that  $n/d$  is a square. The core of 0 is defined to be 0.

`GEN core2(GEN n)` return  $[d, f]$  with  $d$  squarefree and  $n = df^2$ .

`GEN corepartial(GEN n, long lim)` as `core`, using `boundfact(n, lim)` to partially factor  $n$ . The result is not necessarily squarefree, but  $p^2 \mid n$  implies  $p > \text{lim}$ .

`GEN core2partial(GEN n, long lim)` as `core2`, using `boundfact(n, lim)` to partially factor  $n$ . The resulting  $d$  is not necessarily squarefree, but  $p^2 \mid n$  implies  $p > \text{lim}$ .

#### 7.4.7 Primes, primality and compositeness tests.

##### 7.4.7.1 Chebyshev's $\pi$ function, bounds.

`ulong uprimepi(ulong n)`, returns the number of primes  $p \leq n$  (Chebyshev's  $\pi$  function).

`double primepi_upper_bound(double x)` return a quick upper bound for  $\pi(x)$ , using Dusart bounds.

`GEN gprimepi_upper_bound(GEN x)` as `primepi_upper_bound`, returns a `t_REAL`.

`double primepi_lower_bound(double x)` return a quick lower bound for  $\pi(x)$ , using Dusart bounds.

`GEN gprimepi_lower_bound(GEN x)` as `primepi_lower_bound`, returns a `t_REAL` or `gen_0`.

##### 7.4.7.2 Primes, primes in intervals.

`ulong unextprime(ulong n)`, returns the smallest prime  $\geq n$ . Return 0 if it cannot be represented as an `ulong` ( $n$  bigger than  $2^{64} - 59$  or  $2^{32} - 5$  depending on the word size).

`ulong uprecprime(ulong n)`, returns the largest prime  $\leq n$ . Return 0 if  $n \leq 1$ .

`ulong uprime(long n)` returns the  $n$ -th prime, assuming it fits in an `ulong` (overflow error otherwise).

`GEN prime(long n)` same as `utoi(uprime(n))`.

`GEN primes_zv(long m)` returns the first  $m$  primes, in a `t_VECSMALL`.

`GEN primes(long m)` return the first  $m$  primes, as a `t_VEC` of `t_INTs`.

`GEN primes_interval(GEN a, GEN b)` return the primes in the interval  $[a, b]$ , as a `t_VEC` of `t_INT`s.

`GEN primes_interval_zv(ulong a, ulong b)` return the primes in the interval  $[a, b]$ , as a `t_VEC-SMALL` of `ulong`s.

`GEN primes_upto_zv(ulong b)` return the primes in the interval  $[2, b]$ , as a `t_VECSMALL` of `ulong`s.

#### 7.4.7.3 Tests.

`int uisprime(ulong p)`, returns 1 if  $p$  is a prime number and 0 otherwise.

`int uisprime_101(ulong p)`, assuming that  $p$  has no divisor  $\leq 101$ , returns 1 if  $p$  is a prime number and 0 otherwise.

`int uisprime_661(ulong p)`, assuming that  $p$  has no divisor  $\leq 661$ , returns 1 if  $p$  is a prime number and 0 otherwise.

`int isprime(GEN n)`, returns 1 if the `t_INT`  $n$  is a (fully proven) prime number and 0 otherwise.

`long isprimeAPRCL(GEN n)`, returns 1 if the `t_INT`  $n$  is a prime number and 0 otherwise, using only the APRCL test — not even trial division or compositeness tests. The workhorse `isprime` should be faster on average, especially if non-primes are included!

`long isprimeECPP(GEN n)`, returns 1 if the `t_INT`  $n$  is a prime number and 0 otherwise, using only the ECPP test. The workhorse `isprime` should be faster on average.

`long BPSW_psp(GEN n)`, returns 1 if the `t_INT`  $n$  is a Baillie-Pomerance-Selfridge-Wagstaff pseudoprime, and 0 otherwise (proven composite).

`int BPSW_isprime(GEN x)` assuming  $x$  is a BPSW-pseudoprime, rigorously prove its primality. The function `isprime` is currently implemented as

`BPSW_psp(x) && BPSW_isprime(x)`

`long millerrabin(GEN n, long k)` performs  $k$  strong Rabin-Miller compositeness tests on the `t_INT`  $n$ , using  $k$  random bases. This function also caches square roots of  $-1$  that are encountered during the successive tests and stops as soon as three distinct square roots have been produced; we have in principle factored  $n$  at this point, but unfortunately, there is currently no way for the factoring machinery to become aware of it. (It is highly implausible that hard to find factors would be exhibited in this way, though.) This should be slower than `BPSW_psp` for  $k \geq 4$  and we would expect it to be less reliable.

`GEN ecpp(GEN N)` returns an ECPP certificate for `t_INT`  $N$ ; underlies `primecert`.

`GEN ecppexport(GEN cert, long flag)` export a PARI ECPP certificate to MAGMA or Primo format; underlies `primecertexport`.

`long ecppisvalid(GEN cert)` checks whether a PARI ECPP certificate is valid; underlies `prime-certisvalid`.

### 7.4.8 Iterators over primes.

`int forprime_init(forprime_t *T, GEN a, GEN b)` initialize an iterator  $T$  over primes in  $[a, b]$ ; over primes  $\geq a$  if  $b = \text{NULL}$ . Return 0 if the range is known to be empty from the start (as if  $b < a$  or  $b < 0$ ), and return 1 otherwise. Use `forprime_next` to iterate over the prime collection.

`int forprimestep_init(forprime_t *T, GEN a, GEN b, GEN q)` initialize an iterator  $T$  over primes in an arithmetic progression in  $[a, b]$ ; over primes  $\geq a$  if  $b = \text{NULL}$ . The argument  $q$  is either a `t_INT` ( $p \equiv a \pmod{q}$ ) or a `t_INTMOD` `Mod(c, N)` and we restrict to that congruence class. Return 0 if the range is known to be empty from the start (as if  $b < a$  or  $b < 0$ ), and return 1 otherwise. Use `forprime_next` to iterate over the prime collection.

`GEN forprime_next(forprime_t *T)` returns the next prime in the range, assuming that  $T$  was initialized by `forprime_init`.

`int u_forprime_init(forprime_t *T, ulong a, ulong b)`

`ulong u_forprime_next(forprime_t *T)`

`void u_forprime_restrict(forprime_t *T, ulong c)` let  $T$  an iterator over primes initialized via `u_forprime_init(&T, a, b)`, possibly followed by a number of calls to `u_forprime_next`, and  $a \leq c \leq b$ . Restrict the range of primes considered to  $[a, c]$ .

`int u_forprime_arith_init(forprime_t *T, ulong a, ulong b, ulong c, ulong q)` initialize an iterator over primes in  $[a, b]$ , congruent to  $c$  modulo  $q$ . Subsequent calls to `u_forprime_next` will only return primes congruent to  $c$  modulo  $q$ . Note that unless  $(c, q) = 1$  there will be at most one such prime.

## 7.5 Integral, rational and generic linear algebra.

**7.5.1 ZC / ZV, ZM.** A ZV (resp. a ZM, resp. a ZX) is a `t_VEC` or `t_COL` (resp. `t_MAT`, resp. `t_POL`) with `t_INT` coefficients.

### 7.5.1.1 ZC / ZV.

`void RgV_check_ZV(GEN x, const char *s)` Assuming  $x$  is a `t_VEC` or `t_COL` raise an error if it is not a ZV ( $s$  should point to the name of the caller).

`int RgV_is_ZV(GEN x)` Assuming  $x$  is a `t_VEC` or `t_COL` return 1 if it is a ZV, and 0 otherwise.

`int RgV_is_ZVpos(GEN x)` Assuming  $x$  is a `t_VEC` or `t_COL` return 1 if it is a ZV with positive entries, and 0 otherwise.

`int RgV_is_ZVnon0(GEN x)` Assuming  $x$  is a `t_VEC` or `t_COL` return 1 if it is a ZV with non-zero entries, and 0 otherwise.

`int RgV_is_QV(GEN P)` return 1 if the RgV  $P$  has only `t_INT` and `t_FRAC` coefficients, and 0 otherwise.

`int ZV_equal0(GEN x)` returns 1 if all entries of the ZV  $x$  are zero, and 0 otherwise.

`int ZV_cmp(GEN x, GEN y)` compare two ZV, which we assume have the same length (lexicographic order, comparing absolute values).

`int ZV_abscmp(GEN x, GEN y)` compare two ZV, which we assume have the same length (lexicographic order).

`int ZV_equal(GEN x, GEN y)` returns 1 if the two ZV are equal and 0 otherwise. A `t_COL` and a `t_VEC` with the same entries are declared equal.

`GEN ZC_add(GEN x, GEN y)` adds  $x$  and  $y$ .

`GEN ZC_sub(GEN x, GEN y)` subtracts  $x$  and  $y$ .

`GEN ZC_Z_add(GEN x, GEN y)` adds  $y$  to  $x[1]$ .

`GEN ZC_Z_sub(GEN x, GEN y)` subtracts  $y$  to  $x[1]$ .

`GEN Z_ZC_sub(GEN a, GEN x)` returns the vector  $[a - x_1, -x_2, \dots, -x_n]$ .

`GEN ZC_copy(GEN x)` returns a (`t_COL`) copy of  $x$ .

`GEN ZC_neg(GEN x)` returns  $-x$  as a `t_COL`.

`void ZV_neg_inplace(GEN x)` negates the ZV  $x$  in place, by replacing each component by its opposite (the type of  $x$  remains the same, `t_COL` or `t_ROW`). If you want to save even more memory by avoiding the implicit component copies, use `ZV_togglesign`.

`void ZV_togglesign(GEN x)` negates  $x$  in place, by toggling the sign of its integer components. Universal constants `gen_1`, `gen_m1`, `gen_2` and `gen_m2` are handled specially and will not be corrupted. (We use `togglesign_safe`.)

`GEN ZC_Z_mul(GEN x, GEN y)` multiplies the ZC or ZV  $x$  (which can be a column or row vector) by the `t_INT`  $y$ , returning a ZC.

`GEN ZC_Z_divexact(GEN x, GEN y)` returns  $x/y$  assuming all divisions are exact.

`GEN ZC_Z_div(GEN x, GEN y)` returns  $x/y$ , where the resulting vector has rational entries.

`GEN ZV_dotproduct(GEN x, GEN y)` as `RgV_dotproduct` assuming  $x$  and  $y$  have `t_INT` entries.

`GEN ZV_dotsquare(GEN x)` as `RgV_dotsquare` assuming  $x$  has `t_INT` entries.

`GEN ZC_lincomb(GEN u, GEN v, GEN x, GEN y)` returns  $ux + vy$ , where  $u, v$  are `t_INT` and  $x, y$  are ZC or ZV. Return a ZC

`void ZC_lincomb1_inplace(GEN X, GEN Y, GEN v)` sets  $X \leftarrow X + vY$ , where  $v$  is a `t_INT` and  $X, Y$  are ZC or ZV. (The result has the type of  $X$ .) Memory efficient (e.g. no-op if  $v = 0$ ), but not gerepile-safe.

`void ZC_lincomb1_inplace_i(GEN X, GEN Y, GEN v, long n)` variant of `ZC_lincomb1_inplace`: only update  $X[1], \dots, X[n]$ , assuming that  $n < \lg(X)$ .

`GEN ZC_ZV_mul(GEN x, GEN y, GEN p)` multiplies the ZC  $x$  (seen as a column vector) by the ZV  $y$  (seen as a row vector, assumed to have compatible dimensions).

`GEN ZV_content(GEN x)` returns the GCD of all the components of  $x$ .

`GEN ZV_extgcd(GEN A)` given a vector of  $n$  integers  $A$ , returns  $[d, U]$ , where  $d$  is the content of  $A$  and  $U$  is a matrix in  $\text{GL}_n(\mathbf{Z})$  such that  $AU = [D, 0, \dots, 0]$ .

`GEN ZV_prod(GEN x)` returns the product of all the components of  $x$  (1 for the empty vector).

`GEN ZV_sum(GEN x)` returns the sum of all the components of  $x$  (0 for the empty vector).

`long ZV_max_lg(GEN x)` returns the effective length of the longest entry in  $x$ .

`int ZV_dvd(GEN x, GEN y)` assuming  $x, y$  are two ZVs of the same length, return 1 if  $y[i]$  divides  $x[i]$  for all  $i$  and 0 otherwise. Error if one of the  $y[i]$  is 0.

`GEN ZV_sort(GEN L)` sort the ZV  $L$ . Returns a vector with the same type as  $L$ .

`void ZV_sort_inplace(GEN L)` sort the ZV  $L$ , in place.

`GEN ZV_sort_uniq(GEN L)` sort the ZV  $L$ , removing duplicate entries. Returns a vector with the same type as  $L$ .

`long ZV_search(GEN L, GEN y)` look for the  $t\_INT$   $y$  in the sorted ZV  $L$ . Return an index  $i$  such that  $L[i] = y$ , and 0 otherwise.

`GEN ZV_indexsort(GEN L)` returns the permutation which, applied to the ZV  $L$ , would sort the vector. The result is a  $t\_VECSMALL$ .

`GEN ZV_union_shallow(GEN x, GEN y)` given two *sorted* ZV (as per `ZV_sort`, returns the union of  $x$  and  $y$ . Shallow function. In case two entries are equal in  $x$  and  $y$ , include the one from  $x$ .

`GEN ZC_union_shallow(GEN x, GEN y)` as `ZV_union_shallow` but return a  $t\_COL$ .

### 7.5.1.2 ZM.

`void RgM_check_ZM(GEN A, const char *s)` Assuming  $x$  is a  $t\_MAT$  raise an error if it is not a ZM ( $s$  should point to the name of the caller).

`GEN RgM_rescale_to_int(GEN x)` given a matrix  $x$  with real entries ( $t\_INT$ ,  $t\_FRAC$  or  $t\_REAL$ ), return a ZM which is very close to  $Dx$  for some well-chosen integer  $D$ . More precisely, if the input is exact,  $D$  is the denominator of  $x$ ; else it is a power of 2 chosen so that all inexact entries are correctly rounded to 1 ulp.

`GEN ZM_copy(GEN x)` returns a copy of  $x$ .

`int ZM_equal(GEN A, GEN B)` returns 1 if the two ZM are equal and 0 otherwise.

`int ZM_equal0(GEN A)` returns 1 if the ZM  $A$  is identically equal to 0.

`GEN ZM_add(GEN x, GEN y)` returns  $x + y$  (assumed to have compatible dimensions).

`GEN ZM_sub(GEN x, GEN y)` returns  $x - y$  (assumed to have compatible dimensions).

`GEN ZM_neg(GEN x)` returns  $-x$ .

`void ZM_togglesign(GEN x)` negates  $x$  in place, by toggling the sign of its integer components. Universal constants `gen_1`, `gen_m1`, `gen_2` and `gen_m2` are handled specially and will not be corrupted. (We use `togglesign_safe`.)

`GEN ZM_mul(GEN x, GEN y)` multiplies  $x$  and  $y$  (assumed to have compatible dimensions).

`GEN ZM_sqr(GEN x)` returns  $x^2$ , where  $x$  is a square ZM.

`GEN ZM_Z_mul(GEN x, GEN y)` multiplies the ZM  $x$  by the  $t\_INT$   $y$ .

`GEN ZM_ZC_mul(GEN x, GEN y)` multiplies the ZM  $x$  by the ZC  $y$  (seen as a column vector, assumed to have compatible dimensions).

`GEN ZM_ZX_mul(GEN x, GEN T)` returns  $x \times y$ , where  $y$  is `RgX_to_RgC(T, lg(x) - 1)`.

`GEN ZM_diag_mul(GEN d, GEN m)` given a vector  $d$  with integer entries and a ZM  $m$  of compatible dimensions, return `diagonal(d) * m`.

GEN ZM\_mul\_diag(GEN m, GEN d) given a vector  $d$  with integer entries and a ZM  $m$  of compatible dimensions, return  $m * \text{diagonal}(d)$ .

GEN ZM\_multosym(GEN x, GEN y)

GEN ZM\_transmultosym(GEN x, GEN y)

GEN ZM\_transmul(GEN x, GEN y)

GEN ZMrow\_ZC\_mul(GEN x, GEN y, long i) multiplies the  $i$ -th row of ZM  $x$  by the ZC  $y$  (seen as a column vector, assumed to have compatible dimensions). Assumes that  $x$  is non-empty and  $0 < i < \text{lg}(x[1])$ .

GEN ZV\_ZM\_mul(GEN x, GEN y) multiplies the ZV  $x$  by the ZM  $y$ . Returns a t\_VEC.

GEN ZM\_Z\_divexact(GEN x, GEN y) returns  $x/y$  assuming all divisions are exact.

GEN ZM\_Z\_div(GEN x, GEN y) returns  $x/y$ , where the resulting matrix has rational entries.

GEN ZC\_Q\_mul(GEN x, GEN y) returns  $x*y$ , where  $y$  is a rational number and the resulting t\_COL has rational entries.

GEN ZM\_Q\_mul(GEN x, GEN y) returns  $x*y$ , where  $y$  is a rational number and the resulting matrix has rational entries.

GEN ZM\_pow(GEN x, GEN n) returns  $x^n$ , assuming  $x$  is a square ZM and  $n \geq 0$ .

GEN ZM\_powu(GEN x, ulong n) returns  $x^n$ , assuming  $x$  is a square ZM and  $n \geq 0$ .

GEN ZM\_det(GEN M) if  $M$  is a ZM, returns the determinant of  $M$ . This is the function underlying `matdet` whenever  $M$  is a ZM.

GEN ZM\_permanent(GEN M) if  $M$  is a ZM, returns its permanent. This is the function underlying `mat-permanent` whenever  $M$  is a ZM. It assumes that the matrix is square of dimension  $< \text{BITS\_IN\_LONG}$ .

GEN ZM\_detmult(GEN M) if  $M$  is a ZM, returns a multiple of the determinant of the lattice generated by its columns. This is the function underlying `detint`.

GEN ZM\_supnorm(GEN x) return the sup norm of the ZM  $x$ .

GEN ZM\_charpoly(GEN M) returns the characteristic polynomial (in variable 0) of the ZM  $M$ .

GEN ZM\_imagecompl(GEN x) returns `matimagecompl(x)`.

long ZM\_rank(GEN x) returns `matrank(x)`.

GEN ZM\_ker(GEN x) returns `matker(x)`

GEN ZM\_indexrank(GEN x) returns `matindexrank(x)`.

GEN ZM\_indeximage(GEN x) returns `gel(ZM_indexrank(x), 2)`.

long ZM\_max\_lg(GEN x) returns the effective length of the longest entry in  $x$ .

GEN ZM\_inv(GEN M, GEN \*pd) if  $M$  is a ZM, return a primitive matrix  $H$  such that  $MH$  is  $d$  times the identity and set `*pd` to  $d$ . Uses a multimodular algorithm up to Hadamard's bound. If you suspect that the denominator is much smaller than  $\det M$ , you may use `ZM_inv_ratlift`.

GEN ZM\_inv\_ratlift(GEN M, GEN \*pd) if  $M$  is a ZM, return a primitive matrix  $H$  such that  $MH$  is  $d$  times the identity and set `*pd` to  $d$ . Uses a multimodular algorithm, attempting rational

reconstruction along the way. To be used when you expect that the denominator of  $M^{-1}$  is much smaller than  $\det M$  else use `ZM_inv`.

`GEN ZM_pseudoinv(GEN M, GEN *pv, GEN *pd)` if  $M$  is a non-empty ZM, let  $v = [y, z]$  returned by `indexrank` and let  $M_1$  be the corresponding square invertible matrix. Return a primitive left-inverse  $H$  such that  $HM_1$  is  $d$  times the identity and set `*pd` to  $d$ . If `pv` is not NULL, set `*pv` to  $v$ . Not gerepile-safe.

`GEN ZM_gauss(GEN a, GEN b)` as `gauss`, where  $a$  and  $b$  coefficients are `t_INTs`.

`GEN ZM_det_triangular(GEN x)` returns the product of the diagonal entries of  $x$  (its determinant if it is indeed triangular).

`int ZM_isidentity(GEN x)` return 1 if the ZM  $x$  is the identity matrix, and 0 otherwise.

`int ZM_isdiagonal(GEN x)` return 1 if the ZM  $x$  is diagonal, and 0 otherwise.

`int ZM_isscalar(GEN x, GEN s)` given a ZM  $x$  and a `t_INT`  $s$ , return 1 if  $x$  is equal to  $s$  times the identity, and 0 otherwise. If  $s$  is NULL, test whether  $x$  is an arbitrary scalar matrix.

`long ZC_is_ei(GEN x)` return  $i$  if the ZC  $x$  has 0 entries, but for a 1 at position  $i$ .

`int ZM_ishnf(GEN x)` return 1 if  $x$  is in HNF form, i.e. is upper triangular with positive diagonal coefficients, and for  $j > i$ ,  $x_{i,i} > x_{i,j} \geq 0$ .

### 7.5.2 QM.

`GEN QM_charpoly_ZX(GEN M)` returns the characteristic polynomial (in variable 0) of the QM  $M$ , assuming that the result has integer coefficients.

`GEN QM_charpoly_ZX_bound(GEN M, long b)` as `QM_charpoly_ZX` assuming that the sup norm of the (integral) result is  $\leq 2^b$ .

`GEN QM_gauss(GEN a, GEN b)` as `gauss`, where  $a$  and  $b$  coefficients are `t_FRACs`.

`GEN QM_indexrank(GEN x)` returns `matindexrank(x)`.

`GEN QM_inv(GEN M)` return the inverse of the QM  $M$ .

`long QM_rank(GEN x)` returns `matrank(x)`.

### 7.5.3 Qevproj.

`GEN Qevproj_init(GEN M)` let  $M$  be a  $n \times d$  ZM of maximal rank  $d \leq n$ , representing the basis of a  $\mathbf{Q}$ -subspace  $V$  of  $\mathbf{Q}^n$ . Return a projector on  $V$ , to be used by `Qevproj_apply`. The interface details may change in the future, but this function currently returns  $[M, B, D, p]$ , where  $p$  is a `t_VECSMALL` with  $d$  entries such that the submatrix  $A = \text{rowpermute}(M, p)$  is invertible,  $B$  is a ZM and  $d$  a `t_INT` such that  $AB = D\text{Id}_d$ .

`GEN Qevproj_apply(GEN T, GEN pro)` let  $T$  be an  $n \times n$  QM, stabilizing a  $\mathbf{Q}$ -subspace  $V \subset \mathbf{Q}^n$  of dimension  $d$ , and let `pro` be a projector on that subspace initialized by `Qevproj_init(M)`. Return the  $d \times d$  matrix representing  $T|_V$  on the basis given by the columns of  $M$ .

`GEN Qevproj_apply_vecei(GEN T, GEN pro, long k)` as `Qevproj_apply`, return only the image of the  $k$ -th basis vector  $M[k]$  (still on the basis given by the columns of  $M$ ).

`GEN Qevproj_down(GEN T, GEN pro)` given a ZC (resp. a ZM)  $T$  representing an element (resp. a vector of elements) in the subspace  $V$  return a QC (resp. a QM)  $U$  such that  $T = MU$ .



#### 7.5.4 zv, zm.

GEN `zv_neg`(GEN `x`) return  $-x$ . No check for overflow is done, which occurs in the fringe case where an entry is equal to  $2^{\text{BITS\_IN\_LONG}-1}$ .

GEN `zv_neg_inplace`(GEN `x`) negates  $x$  in place and return it. No check for overflow is done, which occurs in the fringe case where an entry is equal to  $2^{\text{BITS\_IN\_LONG}-1}$ .

GEN `zm_zc_mul`(GEN `x`, GEN `y`)

GEN `zm_mul`(GEN `x`, GEN `y`)

GEN `zv_z_mul`(GEN `x`, long `n`) return  $nx$ . No check for overflow is done.

long `zv_content`(GEN `x`) returns the gcd of the entries of  $x$ .

long `zv_dotproduct`(GEN `x`, GEN `y`)

long `zv_prod`(GEN `x`) returns the product of all the components of  $x$  (assumes no overflow occurs).

GEN `zv_prod_Z`(GEN `x`) returns the product of all the components of  $x$ ; consider all  $x[i]$  as `ulong`s.

long `zv_sum`(GEN `x`) returns the sum of all the components of  $x$  (assumes no overflow occurs).

int `zv_cmp0`(GEN `x`) returns 1 if all entries of the `zv`  $x$  are 0, and 0 otherwise.

int `zv_equal`(GEN `x`, GEN `y`) returns 1 if the two `zv` are equal and 0 otherwise.

int `zv_equal0`(GEN `x`) returns 1 if all entries are 0, and return 0 otherwise.

long `zv_search`(GEN `L`, long `y`) look for  $y$  in the sorted `zv`  $L$ . Return an index  $i$  such that  $L[i] = y$ , and 0 otherwise.

GEN `zv_copy`(GEN `x`) as `Flv_copy`.

GEN `zm_transpose`(GEN `x`) as `Flm_transpose`.

GEN `zm_copy`(GEN `x`) as `Flm_copy`.

GEN `zero_zm`(long `m`, long `n`) as `zero_Flm`.

GEN `zero_zv`(long `n`) as `zero_Flv`.

GEN `zm_row`(GEN `A`, long `x0`) as `Flm_row`.

GEN `zm_permanent`(GEN `M`) return the permanent of  $M$ . The function assumes that the matrix is square of dimension  $< \text{BITS\_IN\_LONG}$ .

int `zvV_equal`(GEN `x`, GEN `y`) returns 1 if the two `zvV` (vectors of `zv`) are equal and 0 otherwise.

#### 7.5.5 ZMV / zmV (vectors of ZM/zm).

int `RgV_is_ZMV`(GEN `x`) Assuming  $x$  is a `t_VEC` or `t_COL` return 1 if its components are ZM, and 0 otherwise.

GEN `ZMV_to_zmV`(GEN `z`)

GEN `zmV_to_ZMV`(GEN `z`)

GEN `ZMV_to_FlmV`(GEN `z`, ulong `m`)

### 7.5.6 QC / QV, QM.

GEN QM\_mul(GEN x, GEN y) multiplies  $x$  and  $y$  (assumed to have compatible dimensions).

GEN QM\_QC\_mul(GEN x, GEN y) multiplies  $x$  and  $y$  (assumed to have compatible dimensions).

GEN QM\_det(GEN M) returns the determinant of  $M$ .

GEN QM\_ker(GEN x) returns `matker(x)`.

### 7.5.7 RgC / RgV, RgM.

RgC and RgV routines assume the inputs are VEC or COL of the same dimension. RgM assume the inputs are MAT of compatible dimensions.

#### 7.5.7.1 Matrix arithmetic.

void RgM\_dimensions(GEN x, long \*m, long \*n) sets  $m$ , resp.  $n$ , to the number of rows, resp. columns of the `t_MAT`  $x$ .

GEN RgC\_add(GEN x, GEN y) returns  $x + y$  as a `t_COL`.

GEN RgC\_neg(GEN x) returns  $-x$  as a `t_COL`.

GEN RgC\_sub(GEN x, GEN y) returns  $x - y$  as a `t_COL`.

GEN RgV\_add(GEN x, GEN y) returns  $x + y$  as a `t_VEC`.

GEN RgV\_neg(GEN x) returns  $-x$  as a `t_VEC`.

GEN RgV\_sub(GEN x, GEN y) returns  $x - y$  as a `t_VEC`.

GEN RgM\_add(GEN x, GEN y) return  $x + y$ .

GEN RgM\_neg(GEN x) returns  $-x$ .

GEN RgM\_sub(GEN x, GEN y) returns  $x - y$ .

GEN RgM\_Rg\_add(GEN x, GEN y) assuming  $x$  is a square matrix and  $y$  a scalar, returns the square matrix  $x + y * \text{Id}$ .

GEN RgM\_Rg\_add\_shallow(GEN x, GEN y) as RgM\_Rg\_add with much fewer copies. Not suitable for gerepileupto.

GEN RgM\_Rg\_sub(GEN x, GEN y) assuming  $x$  is a square matrix and  $y$  a scalar, returns the square matrix  $x - y * \text{Id}$ .

GEN RgM\_Rg\_sub\_shallow(GEN x, GEN y) as RgM\_Rg\_sub with much fewer copies. Not suitable for gerepileupto.

GEN RgC\_Rg\_add(GEN x, GEN y) assuming  $x$  is a non-empty column vector and  $y$  a scalar, returns the vector  $[x_1 + y, x_2, \dots, x_n]$ .

GEN RgC\_Rg\_sub(GEN x, GEN y) assuming  $x$  is a non-empty column vector and  $y$  a scalar, returns the vector  $[x_1 - y, x_2, \dots, x_n]$ .

GEN Rg\_RgC\_sub(GEN a, GEN x) assuming  $x$  is a non-empty column vector and  $a$  a scalar, returns the vector  $[a - x_1, -x_2, \dots, -x_n]$ .

GEN RgC\_Rg\_div(GEN x, GEN y)

GEN RgM\_Rg\_div(GEN x, GEN y) returns  $x/y$  ( $y$  treated as a scalar).

`GEN RgC_Rg_mul(GEN x, GEN y)`  
`GEN RgV_Rg_mul(GEN x, GEN y)`  
`GEN RgM_Rg_mul(GEN x, GEN y)` returns  $x \times y$  ( $y$  treated as a scalar).  
`GEN RgV_RgC_mul(GEN x, GEN y)` returns  $x \times y$ .  
`GEN RgV_RgM_mul(GEN x, GEN y)` returns  $x \times y$ .  
`GEN RgM_RgC_mul(GEN x, GEN y)` returns  $x \times y$ .  
`GEN RgM_RgX_mul(GEN x, GEN T)` returns  $x \times y$ , where  $y$  is `RgX_to_RgC(T, lg(x) - 1)`.  
`GEN RgM_mul(GEN x, GEN y)` returns  $x \times y$ .  
`GEN RgM_transmul(GEN x, GEN y)` returns  $x^\sim \times y$ .  
`GEN RgM_multosym(GEN x, GEN y)` returns  $x \times y$ , assuming the result is a symmetric matrix (about twice faster than a generic matrix multiplication).  
`GEN RgM_transmultosym(GEN x, GEN y)` returns  $x^\sim \times y$ , assuming the result is a symmetric matrix (about twice faster than a generic matrix multiplication).  
`GEN RgMrow_RgC_mul(GEN x, GEN y, long i)` multiplies the  $i$ -th row of `RgM x` by the `RgC y` (seen as a column vector, assumed to have compatible dimensions). Assumes that  $x$  is non-empty and  $0 < i < \text{lg}(x[1])$ .  
`GEN RgM_mulreal(GEN x, GEN y)` returns the real part of  $x \times y$  (whose entries are `t_INT`, `t_FRAC`, `t_REAL` or `t_COMPLEX`).  
`GEN RgM_sqr(GEN x)` returns  $x^2$ .  
`GEN RgC_RgV_mul(GEN x, GEN y)` returns  $x \times y$  (the matrix  $(x_i y_j)$ ).  
The following two functions are not well defined in general and only provided for convenience in specific cases:  
`GEN RgC_RgM_mul(GEN x, GEN y)` returns  $x \times y[1, ]$  if  $y$  is a row matrix  $1 \times n$ , error otherwise.  
`GEN RgM_RgV_mul(GEN x, GEN y)` returns  $x \times y[, 1]$  if  $y$  is a column matrix  $n \times 1$ , error otherwise.  
`GEN RgM_powers(GEN x, long n)` returns  $[x^0, \dots, x^n]$  as a `t_VEC` of `RgMs`.  
`GEN RgV_sum(GEN v)` sum of the entries of  $v$   
`GEN RgV_prod(GEN v)` product of the entries of  $v$ , using a divide and conquer strategy  
`GEN RgV_sumpart(GEN v, long n)` returns the sum  $v[1] + \dots + v[n]$  (assumes that  $\text{lg}(v) > n$ ).  
`GEN RgV_sumpart2(GEN v, long m, long n)` returns the sum  $v[m] + \dots + v[n]$  (assumes that  $\text{lg}(v) > n$  and  $m > 0$ ). Returns `gen_0` when  $m > n$ .  
`GEN RgM_sumcol(GEN v)` returns a `t_COL`, sum of the columns of the `t_MAT v`.  
`GEN RgV_dotproduct(GEN x, GEN y)` returns the scalar product of  $x$  and  $y$   
`GEN RgV_dotsquare(GEN x)` returns the scalar product of  $x$  with itself.  
`GEN RgV_kill0(GEN v)` returns a shallow copy of  $v$  where entries matched by `gequal0` are replaced by `NULL`. The return value is not a valid `GEN` and must be handled specially. The idea is to pre-treat a vector of coefficients to speed up later linear combinations or scalar products.

GEN `gram_matrix`(GEN `v`) returns the Gram matrix  $(v_i \cdot v_j)$  attached to the entries of `v` (matrix, or vector of vectors).

GEN `RgV_polint`(GEN `X`, GEN `Y`, long `v`) `X` and `Y` being two vectors of the same length, returns the polynomial  $T$  in variable  $v$  such that  $T(X[i]) = Y[i]$  for all  $i$ . The special case  $X = \text{NULL}$  corresponds to  $X = [1, 2, \dots, n]$ , where  $n$  is the length of `Y`.

### 7.5.7.2 Special shapes.

The following routines check whether matrices or vectors have a special shape, using `gequal1` and `gequal0` to test components. (This makes a difference when components are inexact.)

int `RgV_isscalar`(GEN `x`) return 1 if all the entries of `x` are 0 (as per `gequal0`), except possibly the first one. The name comes from vectors expressing polynomials on the standard basis  $1, T, \dots, T^{n-1}$ , or on `nf.zk` (whose first element is 1).

int `QV_isscalar`(GEN `x`) as `RgV_isscalar`, assuming `x` is a QV (`t_INT` and `t_FRAC` entries only).

int `ZV_isscalar`(GEN `x`) as `RgV_isscalar`, assuming `x` is a ZV (`t_INT` entries only).

int `RgM_isscalar`(GEN `x`, GEN `s`) return 1 if `x` is the scalar matrix equal to `s` times the identity, and 0 otherwise. If `s` is NULL, test whether `x` is an arbitrary scalar matrix.

int `RgM_isidentity`(GEN `x`) return 1 if the `t_MAT` `x` is the identity matrix, and 0 otherwise.

int `RgM_isdiagonal`(GEN `x`) return 1 if the `t_MAT` `x` is a diagonal matrix, and 0 otherwise.

long `RgC_is_ei`(GEN `x`) return  $i$  if the `t_COL` `x` has 0 entries, but for a 1 at position  $i$ .

int `RgM_is_ZM`(GEN `x`) return 1 if the `t_MAT` `x` has only `t_INT` coefficients, and 0 otherwise.

int `RgM_is_QM`(GEN `x`) return 1 if the `t_MAT` `x` has only `t_INT` or `t_FRAC` coefficients, and 0 otherwise.

long `RgV_isin`(GEN `v`, GEN `x`) return the first index  $i$  such that  $v[i] = x$  if it exists, and 0 otherwise. Naive search in linear time, does not assume that `v` is sorted.

GEN `RgM_diagonal`(GEN `m`) returns the diagonal of `m` as a `t_VEC`.

GEN `RgM_diagonal_shallow`(GEN `m`) shallow version of `RgM_diagonal`

### 7.5.7.3 Conversion to floating point entries.

GEN `RgC_gtofp`(GEN `x`, GEN `prec`) returns the `t_COL` obtained by applying `gtofp(gel(x,i), prec)` to all coefficients of `x`.

GEN `RgV_gtofp`(GEN `x`, GEN `prec`) returns the `t_VEC` obtained by applying `gtofp(gel(x,i), prec)` to all coefficients of `x`.

GEN `RgC_gtomp`(GEN `x`, long `prec`) returns the `t_COL` obtained by applying `gtomp(gel(x,i), prec)` to all coefficients of `x`.

GEN `RgC_fpnorml2`(GEN `x`, long `prec`) returns (a stack-clean variant of)

`gnorml2( RgC_gtofp(x, prec) )`

GEN `RgM_gtofp`(GEN `x`, GEN `prec`) returns the `t_MAT` obtained by applying `gtofp(gel(x,i), prec)` to all coefficients of `x`.

GEN `RgM_gtomp`(GEN `x`, long `prec`) returns the `t_MAT` obtained by applying `gtomp(gel(x,i), prec)` to all coefficients of `x`.

GEN RgM\_fpnorml2(GEN x, long prec) returns (a stack-clean variant of)

```
gnorml2( RgM_gtofp(x, prec) )
```

#### 7.5.7.4 Linear algebra, linear systems.

GEN RgM\_inv(GEN a) returns a left inverse of  $a$  (which needs not be square), or NULL if this turns out to be impossible. The latter happens when the matrix does not have maximal rank (or when rounding errors make it appear so).

GEN RgM\_inv\_upper(GEN a) as RgM\_inv, assuming that  $a$  is a non-empty invertible upper triangular matrix, hence a little faster.

GEN RgM\_RgC\_invmage(GEN A, GEN B) returns a  $\mathbf{t\_COL}$   $X$  such that  $AX = B$  if one such exists, and NULL otherwise.

GEN RgM\_invmage(GEN A, GEN B) returns a  $\mathbf{t\_MAT}$   $X$  such that  $AX = B$  if one such exists, and NULL otherwise.

GEN RgM\_Hadamard(GEN a) returns a upper bound for the absolute value of  $\det(a)$ . The bound is a  $\mathbf{t\_INT}$ .

GEN RgM\_solve(GEN a, GEN b) returns  $a^{-1}b$  where  $a$  is a square  $\mathbf{t\_MAT}$  and  $b$  is a  $\mathbf{t\_COL}$  or  $\mathbf{t\_MAT}$ . Returns NULL if  $a^{-1}$  cannot be computed, see RgM\_inv.

If  $b = \text{NULL}$ , the matrix  $a$  need no longer be square, and we strive to return a left inverse for  $a$  (NULL if it does not exist).

GEN RgM\_solve\_realimag(GEN M, GEN b)  $M$  being a  $\mathbf{t\_MAT}$  with  $r_1 + r_2$  rows and  $r_1 + 2r_2$  columns,  $y$  a  $\mathbf{t\_COL}$  or  $\mathbf{t\_MAT}$  such that the equation  $Mx = y$  makes sense, returns  $x$  under the following simplifying assumptions: the first  $r_1$  rows of  $M$  and  $y$  are real (the  $r_2$  others are complex), and  $x$  is real. This is stabler and faster than calling RgM\_solve( $M, b$ ) over  $\mathbf{C}$ . In most applications,  $M$  approximates the complex embeddings of an integer basis in a number field, and  $x$  is actually rational.

GEN split\_realimag(GEN x, long r1, long r2)  $x$  is a  $\mathbf{t\_COL}$  or  $\mathbf{t\_MAT}$  with  $r_1 + r_2$  rows, whose first  $r_1$  rows have real entries (the  $r_2$  others are complex). Return an object of the same type as  $x$  and  $r_1 + 2r_2$  rows, such that the first  $r_1 + r_2$  rows contain the real part of  $x$ , and the  $r_2$  following ones contain the imaginary part of the last  $r_2$  rows of  $x$ . Called by RgM\_solve\_realimag.

GEN RgM\_det\_triangular(GEN x) returns the product of the diagonal entries of  $x$  (its determinant if it is indeed triangular).

GEN Frobeniusform(GEN V, long n) given the vector  $V$  of elementary divisors for  $M - x\text{Id}$ , where  $M$  is an  $n \times n$  square matrix. Returns the Frobenius form of  $M$ .

int RgM\_QR\_init(GEN x, GEN \*pB, GEN \*pQ, GEN \*pL, long prec) QR-decomposition of a square invertible  $\mathbf{t\_MAT}$   $x$  with real coefficients. Sets \*pB to the vector of squared lengths of the  $x[i]$ , \*pL to the Gram-Schmidt coefficients and \*pQ to a vector of successive Householder transforms. If  $R$  denotes the transpose of  $L$  and  $Q$  is the result of applying \*pQ to the identity matrix, then  $x = QR$  is the QR decomposition of  $x$ . Returns 0 if  $x$  is not invertible or we hit a precision problem, and 1 otherwise.

int QR\_init(GEN x, GEN \*pB, GEN \*pQ, GEN \*pL, long prec) as RgM\_QR\_init, assuming further that  $x$  has  $\mathbf{t\_INT}$  or  $\mathbf{t\_REAL}$  coefficients.

GEN `R_from_QR`(GEN `x`, long `prec`) assuming that  $x$  is a square invertible `t_MAT` with `t_INT` or `t_REAL` coefficients, return the upper triangular  $R$  from the  $QR$  decomposition of  $x$ . Not memory clean. If the matrix is not known to have `t_INT` or `t_REAL` coefficients, apply `RgM_gtomp` first.

GEN `gaussred_from_QR`(GEN `x`, long `prec`) assuming that  $x$  is a square invertible `t_MAT` with `t_INT` or `t_REAL` coefficients, returns `qfgaussred(x~* x)`; this is essentially the upper triangular  $R$  matrix from the  $QR$  decomposition of  $x$ , renormalized to accomodate `qfgaussred` conventions. Not memory clean.

GEN `RgM_gram_schmidt`(GEN `e`, GEN `*ptB`) naive (unstable) Gram-Schmidt orthogonalization of the basis  $(e_i)$  given by the columns of `t_MAT`  $e$ . Return the  $e_i^*$  (as columns of a `t_MAT`) and set `*ptB` to the vector of squared lengths  $|e_i^*|^2$ .

GEN `RgM_Babai`(GEN `M`, GEN `y`) given a `t_MAT`  $M$  of maximal rank  $n$  and a `t_COL`  $y$  of the same dimension, apply Babai's nearest plane algorithm to return an *integral*  $x$  such that  $y - Mx$  has small  $L_2$  norm. This yields an approximate solution to the closest vector problem: if  $M$  is LLL-reduced, then

$$\|y - Mx\|_2 \leq 2(2/\sqrt{3})^n \|y - MX\|_2$$

for all  $X \in \mathbf{Z}^n$ .

### 7.5.8 ZG.

Let  $G$  be a multiplicative group with neutral element  $1_G$  whose multiplication is supported by `gmul` and where equality test is performed using `gidentical`, e.g. a matrix group. The following routines implement basic computations in the group algebra  $\mathbf{Z}[G]$ . All of them are shallow for efficiency reasons. A `ZG` is either

- a `t_INT`  $n$ , representing  $n[1_G]$
- or a “factorization matrix” with two columns  $[g, e]$ : the first one contains group elements, sorted according to `cmp_universal`, and the second one contains integer “exponents”, representing  $\sum e_i [g_i]$ .

Note that `to_famat` and `to_famat_shallow(g, e)` allow to build the `ZG`  $e[g]$  from  $e \in \mathbf{Z}$  and  $g \in G$ .

GEN `ZG_normalize`(GEN `x`) given a `t_INT`  $x$  or a factorization matrix *without* assuming that the first column is properly sorted. Return a valid (sorted) `ZG`. Shallow function.

GEN `ZG_add`(GEN `x`, GEN `y`) return  $x + y$ ; shallow function.

GEN `ZG_neg`(GEN `x`) return  $-x$ ; shallow function.

GEN `ZG_sub`(GEN `x`, GEN `y`) return  $x - y$ ; shallow function.

GEN `ZG_mul`(GEN `x`, GEN `y`) return  $xy$ ; shallow function.

GEN `ZG_G_mul`(GEN `x`, GEN `y`) given a `ZG`  $x$  and  $y \in G$ , return  $xy$ ; shallow function.

GEN `G_ZG_mul`(GEN `x`, GEN `y`) given a `ZG`  $y$  and  $x \in G$ , return  $xy$ ; shallow function.

GEN `ZG_Z_mul`(GEN `x`, GEN `n`) given a `ZG`  $x$  and  $y \in \mathbf{Z}$ , return  $xy$ ; shallow function.

GEN `ZGC_G_mul`(GEN `v`, GEN `x`) given  $v$  a vector of `ZG` and  $x \in G$  return the vector (with the same type as  $v$  with entries  $v[i] \cdot x$ . Shallow function.

void `ZGC_G_mul_inplace`(GEN `v`, GEN `x`) as `ZGC_G_mul`, modifying  $v$  in place.

GEN ZGC\_Z\_mul(GEN v, GEN n) given  $v$  a vector of ZG and  $n \in Z$  return the vector (with the same type as  $v$  with entries  $n \cdot v[i]$ ). Shallow function.

GEN G\_ZGC\_mul(GEN x, GEN v) given  $v$  a vector of ZG and  $x \in G$  return the vector of  $x \cdot v[i]$ . Shallow function.

GEN ZGCs\_add(GEN x, GEN y) add two sparse vectors of ZG elements (see Blackbox linear algebra below).

### 7.5.9 Blackbox linear algebra.

A sparse column zCs  $v$  is a t\_COL with two components  $C$  and  $E$  which are t\_VECSMALL of the same length, representing  $\sum_i E[i] * e_{C[i]}$ , where  $(e_j)$  is the canonical basis. A sparse matrix (zMs) is a t\_VEC of zCs.

FpCs and FpMs are identical to the above, but  $E[i]$  is now interpreted as a *signed* C long integer representing an element of  $\mathbf{F}_p$ . This is important since  $p$  can be so large that  $p + E[i]$  would not fit in a C long.

RgCs and RgMs are similar, except that the type of the components of  $E$  is now unspecified. Functions handling those later objects must not depend on the type of those components.

It is not possible to derive the space dimension (number of rows) from the above data. Thus most functions take an argument nbrow which is the number of rows of the corresponding column/matrix in dense representation.

GEN zCs\_to\_ZC(GEN C, long nbrow) convert the sparse vector  $C$  to a dense ZC of dimension nbrow.

GEN zMs\_to\_ZM(GEN M, long nbrow) convert the sparse matrix  $M$  to a dense ZM whose columns have dimension nbrow.

GEN FpMs\_FpC\_mul(GEN M, GEN B, GEN p) multiply the sparse matrix  $M$  (over  $\mathbf{F}_p$ ) by the FpC  $B$ . The result is an FpC, i.e. a dense vector.

GEN zMs\_ZC\_mul(GEN M, GEN B, GEN p) multiply the sparse matrix  $M$  by the ZC  $B$  (over  $\mathbf{Z}$ ). The result is an ZC, i.e. a dense vector.

GEN FpV\_FpMs\_mul(GEN B, GEN M, GEN p) multiply the FpV  $B$  by the sparse matrix  $M$  (over  $\mathbf{F}_p$ ). The result is an FpV, i.e. a dense vector.

GEN ZV\_zMs\_mul(GEN B, GEN M, GEN p) multiply the FpV  $B$  (over  $\mathbf{Z}$ ) by the sparse matrix  $M$ . The result is an ZV, i.e. a dense vector.

void RgMs\_structelim(GEN M, long nbrow, GEN A, GEN \*p\_col, GEN \*p\_row)  $M$  being a RgMs with nbrow rows,  $A$  being a list of row indices, Perform structured elimination on  $M$  by removing some rows and columns until the number of effectively present rows is equal to the number of columns. the result is stored in two t\_VECSMALLs, \*p\_col and \*p\_row: \*p\_col is a map from the new columns indices to the old one. \*p\_row is a map from the old rows indices to the new one (0 if removed).

GEN FpMs\_leftkernel\_elt(GEN M, long nbrow, GEN p)  $M$  being a sparse matrix over  $\mathbf{F}_p$ , return a non-zero kbdFpV  $X$  such that  $XM$  components are almost all 0.

GEN FpMs\_FpCs\_solve(GEN M, GEN B, long nbrow, GEN p) solve the equation  $MX = B$ , where  $M$  is a sparse matrix and  $B$  is a sparse vector, both over  $\mathbf{F}_p$ . Return either a solution as a t\_COL

(dense vector), the index of a column which is linearly dependent from the others as a `t_VECSMALL` with a single component, or `NULL` (can happen if  $B$  is not in the image of  $M$ ).

`GEN FpMs_FpCs_solve_safe(GEN M, GEN B, long nbrow, GEN p)` as above, but in the event that  $p$  is not a prime and an impossible division occurs, return `NULL`.

`GEN ZpMs_ZpCs_solve(GEN M, GEN B, long nbrow, GEN p, long e)` solve the equation  $MX = B$ , where  $M$  is a sparse matrix and  $B$  is a sparse vector, both over  $\mathbf{Z}/p^e\mathbf{Z}$ . Return either a solution as a `t_COL` (dense vector), or the index of a column which is linearly dependent from the others as a `t_VECSMALL` with a single component.

`GEN gen_FpM_Wiedemann(void *E, GEN (*f)(void*, GEN), GEN B, GEN p)` solve the equation  $f(X) = B$  over  $\mathbf{F}_p$ , where  $B$  is a `FpV`, and  $f$  is a blackbox endomorphism, where  $f(E, X)$  computes the value of  $f$  at the (dense) column vector  $X$ . Returns either a solution `t_COL`, or a kernel vector as a `t_VEC`.

`GEN gen_ZpM_Dixon(void *E, GEN (*f)(void*, GEN), GEN B, GEN p, long e)` solve equation  $f(X) = B$  over  $\mathbf{Z}/p^e\mathbf{Z}$ , where  $B$  is a `ZV`, and  $f$  is a blackbox endomorphism, where  $f(E, X)$  computes the value of  $f$  at the (dense) column vector  $X$ . Returns either a solution `t_COL`, or a kernel vector as a `t_VEC`.

### 7.5.10 Obsolete functions.

The functions in this section are kept for backward compatibility only and will eventually disappear.

`GEN image2(GEN x)` compute the image of  $x$  using a very slow algorithm. Use `image` instead.

## 7.6 Integral, rational and generic polynomial arithmetic.

### 7.6.1 ZX.

`void RgX_check_ZX(GEN x, const char *s)` Assuming  $x$  is a `t_POL` raise an error if it is not a `ZX` ( $s$  should point to the name of the caller).

`GEN ZX_copy(GEN x, GEN p)` returns a copy of  $x$ .

`long ZX_max_lg(GEN x)` returns the effective length of the longest component in  $x$ .

`GEN scalar_ZX(GEN x, long v)` returns the constant `ZX` in variable  $v$  equal to the `t_INT`  $x$ .

`GEN scalar_ZX_shallow(GEN x, long v)` returns the constant `ZX` in variable  $v$  equal to the `t_INT`  $x$ . Shallow function not suitable for `gerepile` and friends.

`GEN ZX_renormalize(GEN x, long l)`, as `normalizepol`, where  $l = \lg(x)$ , in place.

`int ZX_equal(GEN x, GEN y)` returns 1 if the two `ZX` have the same `degpol` and their coefficients are equal. Variable numbers are not checked.

`int ZX_equal1(GEN x)` returns 1 if the `ZX`  $x$  is equal to 1 and 0 otherwise.

`int ZX_is_monic(GEN x)` returns 1 if the `ZX`  $x$  is monic and 0 otherwise. The zero polynomial considered not monic.

`GEN ZX_add(GEN x, GEN y)` adds  $x$  and  $y$ .

`GEN ZX_sub(GEN x, GEN y)` subtracts  $x$  and  $y$ .



GEN ZX\_neg(GEN x) returns  $-x$ .

GEN ZX\_Z\_add(GEN x, GEN y) adds the integer  $y$  to the ZX  $x$ .

GEN ZX\_Z\_add\_shallow(GEN x, GEN y) shallow version of ZX\_Z\_add.

GEN ZX\_Z\_sub(GEN x, GEN y) subtracts the integer  $y$  to the ZX  $x$ .

GEN Z\_ZX\_sub(GEN x, GEN y) subtracts the ZX  $y$  to the integer  $x$ .

GEN ZX\_Z\_mul(GEN x, GEN y) multiplies the ZX  $x$  by the integer  $y$ .

GEN ZX\_mulu(GEN x, ulong y) multiplies  $x$  by the integer  $y$ .

GEN ZX\_shifti(GEN x, long n) shifts all coefficients of  $x$  by  $n$  bits, which can be negative.

GEN ZX\_Z\_divexact(GEN x, GEN y) returns  $x/y$  assuming all divisions are exact.

GEN ZX\_remi2n(GEN x, long n) reduces all coefficients of  $x$  to  $n$  bits, using remi2n.

GEN ZX\_mul(GEN x, GEN y) multiplies  $x$  and  $y$ .

GEN ZX\_sqr(GEN x, GEN p) returns  $x^2$ .

GEN ZX\_mulspec(GEN a, GEN b, long na, long nb). Internal routine:  $a$  and  $b$  are arrays of coefficients representing polynomials  $\sum_{i=0}^{na-1} a[i]X^i$  and  $\sum_{i=0}^{nb-1} b[i]X^i$ . Returns their product (as a true GEN) in variable 0.

GEN ZX\_sqrspec(GEN a, long na). Internal routine:  $a$  is an array of coefficients representing polynomial  $\sum_{i=0}^{na-1} a[i]X^i$ . Return its square (as a true GEN) in variable 0.

GEN ZX\_rem(GEN x, GEN y) returns the remainder of the Euclidean division of  $x$  mod  $y$ . Assume that  $x, y$  are two ZX and that  $y$  is monic.

GEN ZX\_mod\_Xnm1(GEN T, ulong n) return  $T$  modulo  $X^n - 1$ . Shallow function.

GEN ZX\_div\_by\_X\_1(GEN T, GEN \*r) return the quotient of  $T$  by  $X - 1$ . If  $r$  is not NULL set it to  $T(1)$ .

GEN ZX\_gcd(GEN x, GEN y) returns a gcd of the ZX  $x$  and  $y$ . Not memory-clean, but suitable for gerepileupto.

GEN ZX\_gcd\_all(GEN x, GEN y, GEN \*pX) returns a gcd  $d$  of  $x$  and  $y$ . If  $pX$  is not NULL, set  $*pX$  to a (non-zero) integer multiple of  $x/d$ . If  $x$  and  $y$  are both monic, then  $d$  is monic and  $*pX$  is exactly  $x/d$ . Not memory clean if the gcd is 1 (in that case  $*pX$  is set to  $x$ ).

GEN ZX\_radical(GEN x) returns the largest squarefree divisor of the ZX  $x$ . Not memory clean.

GEN ZX\_content(GEN x) returns the content of the ZX  $x$ .

long ZX\_val(GEN P) as RgX\_val, but assumes P has t\_INT coefficients.

long ZX\_valrem(GEN P, GEN \*z) as RgX\_valrem, but assumes P has t\_INT coefficients.

GEN ZX\_to\_monic(GEN q, GEN \*L) given  $q$  a non-zero ZX, returns a monic integral polynomial  $Q$  such that  $Q(x) = Cq(x/L)$ , for some rational  $C$  and positive integer  $L > 0$ . If  $L$  is not NULL, set  $*L$  to  $L$ ; if  $L = 1$ ,  $*L$  is set to gen\_1. Not suitable for gerepileupto.

GEN ZX\_primitive\_to\_monic(GEN q, GEN \*L) as ZX\_to\_monic except  $q$  is assumed to have trivial content, which avoids recomputing it. The result is suboptimal if  $q$  is not primitive ( $L$  larger than necessary), but remains correct.

GEN ZX\_Z\_normalize(GEN q, GEN \*L) a restricted version of ZX\_primitive\_to\_monic, where  $q$  is a *monic* ZX of degree  $> 0$ . Finds the largest integer  $L > 0$  such that  $Q(X) := L^{-\deg q} q(Lx)$  is integral and return  $Q$ ; this is not well-defined if  $q$  is a monomial, in that case, set  $L = 1$  and  $Q = q$ . If  $L$  is not NULL, set  $*L$  to  $L$ .

GEN ZX\_Q\_normalize(GEN q, GEN \*L) a variant of ZX\_Z\_normalize where  $L > 0$  is allowed to be rational, the monic  $Q \in \mathbf{Z}[X]$  has possibly smaller coefficients.

GEN ZX\_Q\_mul(GEN x, GEN y) returns  $x * y$ , where  $y$  is a rational number and the resulting  $\mathbf{t\_POL}$  has rational entries.

long ZX\_deflate\_order(GEN P) given a non-constant ZX  $P$ , returns the largest exponent  $d$  such that  $P$  is of the form  $P(x^d)$ .

long ZX\_deflate\_max(GEN P, long \*d). Given a non-constant polynomial with integer coefficients  $P$ , sets  $d$  to ZX\_deflate\_order( $P$ ) and returns RgX\_deflate( $P, d$ ). Shallow function.

GEN ZX\_rescale(GEN P, GEN h) returns  $h^{\deg(P)} P(x/h)$ .  $P$  is a ZX and  $h$  is a non-zero integer. Neither memory-clean nor suitable for gerepileupto.

GEN ZX\_rescale2n(GEN P, long n) returns  $2^{n \deg(P)} P(x \gg n)$  where  $P$  is a ZX. Neither memory-clean nor suitable for gerepileupto.

GEN ZX\_rescale\_lt(GEN P) returns the monic integral polynomial  $h^{\deg(P)-1} P(x/h)$ , where  $P$  is a non-zero ZX and  $h$  is its leading coefficient. Neither memory-clean nor suitable for gerepileupto.

GEN ZX\_translate(GEN P, GEN c) assume  $P$  is a ZX and  $c$  an integer. Returns  $P(X+c)$  (optimized for  $c = \pm 1$ ).

GEN ZX\_unscale(GEN P, GEN h) given a ZX  $P$  and a  $\mathbf{t\_INT}$   $h$ , returns  $P(hx)$ . Not memory clean.

GEN ZX\_z\_unscale(GEN P, long h) given a ZX  $P$ , returns  $P(hx)$ . Not memory clean.

GEN ZX\_unscale2n(GEN P, long n) given a ZX  $P$ , returns  $P(x \ll n)$ . Not memory clean.

GEN ZX\_unscale\_div(GEN P, GEN h) given a ZX  $P$  and a  $\mathbf{t\_INT}$   $h$  such that  $h \mid P(0)$ , returns  $P(hx)/h$ . Not memory clean.

GEN ZX\_eval1(GEN P) returns the integer  $P(1)$ .

GEN ZX\_graeffe(GEN p) returns the Graeffe transform of  $p$ , i.e. the ZX  $q$  such that  $p(x)p(-x) = q(x^2)$ .

GEN ZX\_deriv(GEN x) returns the derivative of  $x$ .

GEN ZX\_resultant(GEN A, GEN B) returns the resultant of the ZX  $A$  and  $B$ .

GEN ZX\_disc(GEN T) returns the discriminant of the ZX  $T$ .

GEN ZX\_factor(GEN T) returns the factorization of the primitive part of  $T$  over  $\mathbf{Q}[X]$  (the content is lost).

int ZX\_is\_squarefree(GEN T) returns 1 if the ZX  $T$  is squarefree, 0 otherwise.

long ZX\_is\_irred(GEN T) returns 1 if  $T$  is irreducible, and 0 otherwise.

GEN ZX\_squff(GEN T, GEN \*E) write  $T$  as a product  $\prod T_i^{e_i}$  with the  $e_1 < e_2 < \dots$  all distinct and the  $T_i$  pairwise coprime. Return the vector of the  $T_i$ , and set  $*E$  to the vector of the  $e_i$ , as a  $\mathbf{t\_VECSMALL}$ .

**GEN ZX\_Uspensky**(GEN P, GEN ab, long flag, long bitprec) let P be a primitive ZX polynomial whose real roots are simple and bitprec is the relative precision in bits.

- If flag is 0 returns a list of intervals that isolate the real roots of P. The return value is a column of elements which are either vectors [a,b] meaning that there is a single root in the open interval (a,b) or elements x0 such that x0 is a root of P. There is no guarantee that all rational roots are found (at most those with denominator a power of 2 can be found and even those are not guaranteed). Beware that the limits of the open intervals can be roots of the polynomial.

- If flag is 1 returns an approximation of the real roots of P.

- If flag is 2 returns the number of roots.

The argument ab specify the interval in which the roots are searched. The default interval is  $(-\infty, \infty)$ . If ab is an integer or fraction a then the interval is  $[a, \infty)$ . If ab is a vector [a, b], where t\_INT, t\_FRAC or t\_INFINITY are allowed for a and b, the interval is [a, b].

**long ZX\_sturm**(GEN P) number of real roots of the non-constant squarefree ZX P. For efficiency, it is advised to make P primitive first.

**long ZX\_sturmpart**(GEN P, GEN ab) number of real roots of the non-constant squarefree ZX P in the interval specified by ab: either NULL (no restriction) or a t\_VEC [a, b] with two real components (of type t\_INT, t\_FRAC or t\_INFINITY). For efficiency, it is advised to make P primitive first.

## 7.6.2 Resultants.

**GEN ZX\_ZXY\_resultant**(GEN A, GEN B) under the assumption that A in  $\mathbf{Z}[Y]$ , B in  $\mathbf{Q}[Y][X]$ , and  $R = \text{Res}_Y(A, B) \in \mathbf{Z}[X]$ , returns the resultant R.

**GEN ZX\_compositum\_disjoint**(GEN A, GEN B) given two irreducible ZX defining linearly disjoint extensions, returns a ZX defining their compositum.

**GEN ZX\_ZXY\_rnfequation**(GEN A, GEN B, long \*lambda), assume A in  $\mathbf{Z}[Y]$ , B in  $\mathbf{Q}[Y][X]$ , and  $R = \text{Res}_Y(A, B) \in \mathbf{Z}[X]$ . If lambda = NULL, returns R as in ZX\_ZXY\_resultant. Otherwise, lambda must point to some integer, e.g. 0 which is used as a seed. The function then finds a small  $\lambda \in \mathbf{Z}$  (starting from \*lambda) such that  $R_\lambda(X) := \text{Res}_Y(A, B(X + \lambda Y))$  is squarefree, resets \*lambda to the chosen value and returns  $R_\lambda$ .

## 7.6.3 ZXV.

**GEN ZXV\_equal**(GEN x, GEN y) returns 1 if the two vectors of ZX are equal, as per ZX\_equal (variables are not checked to be equal) and 0 otherwise.

**GEN ZXV\_Z\_mul**(GEN x, GEN y) multiplies the vector of ZX x by the integer y.

**GEN ZXV\_remi2n**(GEN x, long n) applies ZX\_remi2n to all coefficients of x.

**GEN ZXV\_dotproduct**(GEN x, GEN y) as RgV\_dotproduct assuming x and y have ZX entries.

## 7.6.4 ZXT.

**GEN ZXT\_remi2n**(GEN x, long n) applies ZX\_remi2n to all leaves of the tree x.

### 7.6.5 ZXQ.

GEN ZXQ\_mul(GEN x, GEN y, GEN T) returns  $x * y \bmod T$ , assuming that all inputs are ZXs and that  $T$  is monic.

GEN ZXQ\_sqr(GEN x, GEN T) returns  $x^2 \bmod T$ , assuming that all inputs are ZXs and that  $T$  is monic.

GEN ZXQ\_charpoly(GEN A, GEN T, long v): let  $T$  and  $A$  be ZXs, returns the characteristic polynomial of  $\text{Mod}(A, T)$ . More generally,  $A$  is allowed to be a QX, hence possibly has rational coefficients, *assuming* the result is a ZX, i.e. the algebraic number  $\text{Mod}(A, T)$  is integral over  $\mathbb{Z}$ .

### 7.6.6 ZXn.

GEN ZXn\_mul(GEN x, GEN y, long n) return  $xy \pmod{X^n}$ .

GEN ZXn\_sqr(GEN x, long n) return  $x^2 \pmod{X^n}$ .

GEN eta\_ZXn(long r, long n) return  $\eta(X^r) = \prod_{i>0} (1 - X^{ri}) \pmod{X^n}$ ,  $r > 0$ .

GEN eta\_product\_ZXn(GEN DR, long n):  $DR = [D, R]$  being a vector with two t\_VECSMALL components, return  $\prod_i \eta(X^{d_i})^{r_i}$ . Shallow function.

### 7.6.7 ZXQM.

ZXQM are matrices of ZXQ. All entries must be integers or polynomials of degree strictly less than the degree of  $T$ .

GEN ZXQM\_mul(GEN x, GEN y, GEN T) returns  $x * y \bmod T$ , assuming that all inputs are ZXs and that  $T$  is monic.

GEN ZXQM\_sqr(GEN x, GEN T) returns  $x^2 \bmod T$ , assuming that all inputs are ZXs and that  $T$  is monic.

### 7.6.8 ZXQX.

GEN ZXQX\_mul(GEN x, GEN y, GEN T) returns  $x * y$ , assuming that all inputs are ZXQXs and that  $T$  is monic.

GEN ZXQX\_sqr(GEN x, GEN T) returns  $x^2$ , assuming that all inputs are ZXQXs and that  $T$  is monic.

### 7.6.9 ZXX.

void RgX\_check\_ZXX(GEN x, const char \*s) Assuming  $x$  is a t\_POL raise an error if it one of its coefficients is not an integer or a ZX ( $s$  should point to the name of the caller).

GEN ZXX\_renormalize(GEN x, long l), as `normalizepol`, where  $l = \text{lg}(x)$ , in place.

long ZXX\_max\_lg(GEN x) returns the effective length of the longest component in  $x$ ; assume all coefficients are t\_INT or ZXs.

GEN ZXX\_Z\_mul(GEN x, GEN y) returns  $xy$ .

GEN ZXX\_Z\_add\_shallow(GEN x, GEN y) returns  $x + y$ . Shallow function.

GEN ZXX\_Z\_divexact(GEN x, GEN y) returns  $x/y$  assuming all integer divisions are exact.

GEN ZXX\_to\_Kronecker(GEN P, long n) Assuming  $P(X, Y)$  is a polynomial of degree in  $X$  strictly less than  $n$ , returns  $P(X, X^{2*n-1})$ , the Kronecker form of  $P$ . Shallow function.

GEN ZXX\_to\_Kronecker\_spec(GEN Q, long lQ, long n) return ZXX\_to\_Kronecker( $P, n$ ), where  $P$  is the polynomial  $\sum_{i=0}^{lQ-1} Q[i]x^i$ . To be used when splitting the coefficients of genuine polynomials into blocks. Shallow function.

GEN Kronecker\_to\_ZXX(GEN z, long n, long v) recover  $P(X, Y)$  from its Kronecker form  $P(X, X^{2n-1})$ ,  $v$  is the variable number corresponding to  $Y$ . Shallow function.

GEN ZXX\_mul\_Kronecker(GEN P, GEN Q, long n) return ZX\_mul applied to the Kronecker forms  $P(X, X^{2n-1})$  and  $Q(X, X^{2n-1})$  of  $P$  and  $Q$ . Not memory clean.

GEN ZXX\_sqr\_Kronecker(GEN P, long n) return ZX\_sqr applied to the Kronecker forms  $P(X, X^{2n-1})$  of  $P$ . Not memory clean.

#### 7.6.10 QX.

void RgX\_check\_QX(GEN x, const char \*s) Assuming  $x$  is a t\_POL raise an error if it is not a QX ( $s$  should point to the name of the caller).

GEN QX\_mul(GEN x, GEN y)

GEN QX\_sqr(GEN x)

GEN QX\_ZX\_rem(GEN x, GEN y)  $y$  is assumed to be monic.

GEN QX\_gcd(GEN x, GEN y) returns a gcd of the QX  $x$  and  $y$ .

GEN QX\_disc(GEN T) returns the discriminant of the QX  $T$ .

GEN QX\_factor(GEN T) as ZX\_factor.

GEN QX\_resultant(GEN A, GEN B) returns the resultant of the QX  $A$  and  $B$ .

GEN QX\_complex\_roots(GEN p, long l) returns the complex roots of the QX  $p$  at accuracy  $l$ , where real roots are returned as t\_REALs. More efficient when  $p$  is irreducible and primitive. Special case of cleanroots.

#### 7.6.11 QXQ.

GEN QXQ\_norm(GEN A, GEN B)  $A$  being a QX and  $B$  being a ZX, returns the norm of the algebraic number  $A \bmod B$ , using a modular algorithm. To ensure that  $B$  is a ZX, one may replace it by Q\_primpart( $B$ ), which of course does not change the norm.

If  $A$  is not a ZX — it has a denominator —, but the result is nevertheless known to be an integer, it is much more efficient to call QXQ\_intnorm instead.

GEN QXQ\_intnorm(GEN A, GEN B)  $A$  being a QX and  $B$  being a ZX, returns the norm of the algebraic number  $A \bmod B$ , *assuming* that the result is an integer, which is for instance the case is  $A \bmod B$  is an algebraic integer, in particular if  $A$  is a ZX. To ensure that  $B$  is a ZX, one may replace it by Q\_primpart( $B$ ) (which of course does not change the norm).

If the result is not known to be an integer, you must use QXQ\_norm instead, which is slower.

GEN QXQ\_mul(GEN A, GEN B, GEN T) returns the product of  $A$  and  $B$  modulo  $T$  where both  $A$  and  $B$  are a QX and  $T$  is a monic ZX.

GEN QXQ\_sqr(GEN A, GEN T) returns the square of  $A$  modulo  $T$  where  $A$  is a QX and  $T$  is a monic ZX.

GEN QXQ\_inv(GEN A, GEN B) returns the inverse of  $A$  modulo  $B$  where  $A$  is a QX and  $B$  is a ZX. Should you need this for a QX  $B$ , just use

```
QXQ_inv(A, Q_primpart(B));
```

But in all cases where modular arithmetic modulo  $B$  is desired, it is much more efficient to replace  $B$  by  $Q\_primpart(B)$  once and for all.

GEN QXQ\_div\_ratlift(GEN C, GEN A, GEN B) returns  $C/A$  modulo  $B$  where  $A$  and  $C$  are QX and  $B$  is a ZX. Use this function when the result is known to be “small” compared to  $A^{-1} \bmod B$ , it will be faster than QXQ\_inv in this case.

GEN QXQ\_charpoly(GEN A, GEN T, long v) where  $A$  is a QX and  $T$  is a ZX, returns the characteristic polynomial of  $\text{Mod}(A, T)$ . If the result is known to be a ZX, then calling ZXQ\_charpoly will be faster.

GEN QXQ\_powers(GEN x, long n, GEN T) returns  $[x^0, \dots, x^n]$  as RgXQ\_powers would, but in a more efficient way when  $x$  has a huge integer denominator (we start by removing that denominator). Meant to be used to precompute powers of algebraic integers in  $\mathbf{Q}[t]/(T)$ . The current implementation does not require  $x$  to be a QX: any polynomial to which Q\_remove\_denom can be applied is fine.

GEN QXQ\_reverse(GEN f, GEN T) as RgXQ\_reverse, assuming  $f$  is a QX.

GEN QX\_ZXQV\_eval(GEN f, GEN nV, GEN dV) as RgX\_RgXQV\_eval, except that  $f$  is assumed to be a QX,  $V$  is given implicitly by a numerator  $nV$  (ZV) and denominator  $dV$  (a positive t\_INT or NULL for trivial denominator). Not memory clean, but suitable for gerepileupto.

GEN QXV\_QXQ\_eval(GEN v, GEN a, GEN T)  $v$  is a vector of QXs (possibly scalars, i.e. rational numbers, for convenience),  $a$  and  $T$  both QX. Return the vector of evaluations at  $a$  modulo  $T$ . Not memory clean, nor suitable for gerepileupto.

GEN QXX\_QXQ\_eval(GEN P, GEN a, GEN T)  $P(X, Y)$  is a t\_POL with QX coefficients (possibly scalars, i.e. rational numbers, for convenience),  $a$  and  $T$  both QX. Return the QX  $P(X, a \bmod T)$ . Not memory clean, nor suitable for gerepileupto.

GEN nfgcd(GEN P, GEN Q, GEN T, GEN den) given  $P$  and  $Q$  in  $\mathbf{Z}[X, Y]$ ,  $T$  monic irreducible in  $\mathbf{Z}[Y]$ , returns the primitive  $d$  in  $\mathbf{Z}[X, Y]$  which is a gcd of  $P, Q$  in  $K[X]$ , where  $K$  is the number field  $\mathbf{Q}[Y]/(T)$ . If not NULL,  $den$  is a multiple of the integral denominator of the (monic) gcd of  $P, Q$  in  $K[X]$ .

GEN nfgcd\_all(GEN P, GEN Q, GEN T, GEN den, GEN \*Pnew) as nfgcd. If  $Pnew$  is not NULL, set  $*Pnew$  to a non-zero integer multiple of  $P/d$ . If  $P$  and  $Q$  are both monic, then  $d$  is monic and  $*Pnew$  is exactly  $P/d$ . Not memory clean if the gcd is 1 (in that case  $*Pnew$  is set to  $P$ ).

#### 7.6.12 QXQM.

QXQM are matrices of QXQ. All entries must be t\_INT, t\_FRAC or polynomials of degree strictly less than the degree of  $T$ , which must be a monic ZX.

GEN QXQM\_mul(GEN x, GEN y, GEN T) returns  $x * y \bmod T$ .

GEN QXQM\_sqr(GEN x, GEN T) returns  $x^2 \bmod T$ .

### 7.6.13 zx.

GEN `zero_zx(long sv)` returns a zero `zx` in variable  $v$ .

GEN `polx_zx(long sv)` returns the variable  $v$  as degree 1 `Flx`.

GEN `zx_renormalize(GEN x, long l)`, as `Flx_renormalize`, where  $l = \lg(x)$ , in place.

GEN `zx_shift(GEN T, long n)` returns  $T$  multiplied by  $x^n$ , assuming  $n \geq 0$ .

### 7.6.14 RgX.

#### 7.6.14.1 Tests.

long `RgX_degree(GEN x, long v)`  $x$  being a `t_POL` and  $v \geq 0$ , returns the degree in  $v$  of  $x$ . Error if  $x$  is not a polynomial in  $v$ .

int `RgX_isscalar(GEN x)` return 1 if all the coefficients of  $x$  of degree  $> 0$  are 0 (as per `gequal0`).

int `RgX_is_rational(GEN P)` return 1 if the `RgX`  $P$  has only rational coefficients (`t_INT` and `t_FRAC`), and 0 otherwise.

int `RgX_is_QX(GEN P)` return 1 if the `RgX`  $P$  has only `t_INT` and `t_FRAC` coefficients, and 0 otherwise.

int `RgX_is_ZX(GEN P)` return 1 if the `RgX`  $P$  has only `t_INT` coefficients, and 0 otherwise.

int `RgX_is_monomial(GEN x)` returns 1 (true) if  $x$  is a non-zero monomial in its main variable, 0 otherwise.

long `RgX_equal(GEN x, GEN y)` returns 1 if the `t_POLs`  $x$  and  $y$  have the same `degpol` and their coefficients are equal (as per `gequal`). Variable numbers are not checked. Note that this is more stringent than `gequal(x,y)`, which only checks whether  $x - y$  satisfies `gequal0`; in particular, they may have different apparent degrees provided the extra leading terms are 0.

long `RgX_equal_var(GEN x, GEN y)` returns 1 if  $x$  and  $y$  have the same variable number and `RgX_equal(x,y)` is 1.

#### 7.6.14.2 Coefficients, blocks.

GEN `RgX_coeff(GEN P, long n)` return the coefficient of  $x^n$  in  $P$ , defined as `gen_0` if  $n < 0$  or  $n > \degpol(P)$ . Shallow function.

int `RgX_blocks(GEN P, long n, long m)` writes  $P(X) = a_0(X) + X^n * a_1(X) * X^n + \dots + X^{n*(m-1)} a_{m-1}(X)$ , where the  $a_i$  are polynomial of degree at most  $n - 1$  (except possibly for the last one) and returns  $[a_0(X), a_1(X), \dots, a_{m-1}(X)]$ . Shallow function.

void `RgX_even_odd(GEN p, GEN *pe, GEN *po)` write  $p(X) = E(X^2) + XO(X^2)$  and set `*pe = E`, `*po = O`. Shallow function.

GEN `RgX_splitting(GEN P, long k)` write  $P(X) = a_0(X^k) + X a_1(X^k) + \dots + X^{k-1} a_{k-1}(X^k)$  and return  $[a_0(X), a_1(X), \dots, a_{k-1}(X)]$ . Shallow function.

GEN `RgX_copy(GEN x)` returns (a deep copy of)  $x$ .

GEN `RgX_renormalize(GEN x)` remove leading terms in  $x$  which are equal to (necessarily inexact) zeros.

GEN `RgX_renormalize_lg(GEN x, long lx)` as `setlg(x, lx)` followed by `RgX_renormalize(x)`. Assumes that  $lx \leq \lg(x)$ .

GEN `RgX_recip`(GEN `P`) returns the reverse of the polynomial  $P$ , i.e.  $X^{\deg P} P(1/X)$ .

GEN `RgX_recip_shallow`(GEN `P`) shallow function of `RgX_recip`, where we further assume that  $P(0) \neq 0$ , so that the degree of the output is the degree of  $P$ .

GEN `RgX_deflate`(GEN `P`, long `d`) assuming  $P$  is a polynomial of the form  $Q(X^d)$ , return  $Q$ . Shallow function, not suitable for `gerepileupto`.

long `RgX_deflate_order`(GEN `P`) given a non-constant polynomial  $P$ , returns the largest exponent  $d$  such that  $P$  is of the form  $P(x^d)$  (use `gequal0` to check whether coefficients are 0).

long `RgX_deflate_max`(GEN `P`, long `*d`) given a non-constant polynomial  $P$ , sets `d` to `RgX_deflate_order(P)` and returns `RgX_deflate(P,d)`. Shallow function.

GEN `RgX_inflate`(GEN `P`, long `d`) return  $P(X^d)$ . Shallow function, not suitable for `gerepileupto`.

GEN `RgX_rescale_to_int`(GEN `x`) given a polynomial  $x$  with real entries (`t_INT`, `t_FRAC` or `t_REAL`), return a `ZX` which is very close to  $Dx$  for some well-chosen integer  $D$ . More precisely, if the input is exact,  $D$  is the denominator of  $x$ ; else it is a power of 2 chosen so that all inexact entries are correctly rounded to 1 ulp.

### 7.6.14.3 Shifts, valuations.

GEN `RgX_shift`(GEN `x`, long `n`) returns  $x * t^n$  if  $n \geq 0$ , and  $x \backslash t^{-n}$  otherwise.

GEN `RgX_shift_shallow`(GEN `x`, long `n`) as `RgX_shift`, but shallow (coefficients are not copied).

GEN `RgX_rotate_shallow`(GEN `P`, long `k`, long `p`) returns  $P * X^k \pmod{X^p - 1}$ , assuming the degree of  $P$  is strictly less than  $p$ , and  $k \geq 0$ .

void `RgX_shift_inplace_init`(long `v`)  $v \geq 0$ , prepare for a later call to `RgX_shift_inplace`. Reserves  $v$  words on the stack.

GEN `RgX_shift_inplace`(GEN `x`, long `v`)  $v \geq 0$ , assume that `RgX_shift_inplace_init(v)` has been called (reserving  $v$  words on the stack), immediately followed by a `t_POL`  $x$ . Return `RgX_shift(x,v)` by shifting  $x$  in place. To be used as follows

```
RgX_shift_inplace_init(v);
av = avma;
...
x = gerepileupto(av, ...); /* a t_POL */
return RgX_shift_inplace(x, v);
```

long `RgX_valrem`(GEN `P`, GEN `*pz`) returns the valuation  $v$  of the `t_POL`  $P$  with respect to its main variable  $X$ . Check whether coefficients are 0 using `isexactzero`. Set `*pz` to `RgX_shift_shallow(P, -v)`.

long `RgX_val`(GEN `P`) returns the valuation  $v$  of the `t_POL`  $P$  with respect to its main variable  $X$ . Check whether coefficients are 0 using `isexactzero`.

long `RgX_valrem_inexact`(GEN `P`, GEN `*z`) as `RgX_valrem`, using `gequal0` instead of `isexactzero`.



#### 7.6.14.4 Basic arithmetic.

GEN RgX\_add(GEN x, GEN y) adds x and y.

GEN RgX\_sub(GEN x, GEN y) subtracts x and y.

GEN RgX\_neg(GEN x) returns  $-x$ .

GEN RgX\_Rg\_add(GEN y, GEN x) returns  $x + y$ .

GEN RgX\_Rg\_add\_shallow(GEN y, GEN x) returns  $x + y$ ; shallow function.

GEN Rg\_RgX\_sub(GEN x, GEN y)

GEN RgX\_Rg\_sub(GEN y, GEN x) returns  $x - y$

GEN RgX\_Rg\_mul(GEN y, GEN x) multiplies the RgX y by the scalar x.

GEN RgX\_muls(GEN y, long s) multiplies the RgX y by the long s.

GEN RgX\_Rg\_div(GEN y, GEN x) divides the RgX y by the scalar x.

GEN RgX\_divs(GEN y, long s) divides the RgX y by the long s.

GEN RgX\_Rg\_divexact(GEN x, GEN y) exact division of the RgX y by the scalar x.

GEN RgX\_Rg\_eval\_bk(GEN f, GEN x) returns  $f(x)$  using Brent and Kung algorithm. (Use poleval for Horner algorithm.)

GEN RgX\_RgV\_eval(GEN f, GEN V) as RgX\_Rg\_eval\_bk(f, x), assuming V was output by gpowers(x, n) for some  $n \geq 1$ .

GEN RgXV\_RgV\_eval(GEN f, GEN V) apply RgX\_RgV\_eval\_bk(, V) to all the components of the vector f.

GEN RgX\_normalize(GEN x) divides x by its leading coefficient. If the latter is 1, x itself is returned, not a copy. Leading coefficients equal to 0 are stripped, e.g.

$$0.*t^3 + \text{Mod}(0,3)*t^2 + 2*t$$

is normalized to t.

GEN RgX\_mul(GEN x, GEN y) multiplies the two t\_POL (in the same variable) x and y. Detect the coefficient ring and use an appropriate algorithm.

GEN RgX\_mul\_i(GEN x, GEN y) multiplies the two t\_POL (in the same variable) x and y. Do not detect the coefficient ring. Use a generic Karatsuba algorithm.

GEN RgX\_mul\_normalized(GEN A, long a, GEN B, long b) returns  $(X^a + A)(X^b + B) - X^{(a+b)}$ , where we assume that  $\deg A < a$  and  $\deg B < b$  are polynomials in the same variable X.

GEN RgX\_sqr(GEN x) squares the t\_POL x. Detect the coefficient ring and use an appropriate algorithm.

GEN RgX\_sqr\_i(GEN x) squares the t\_POL x. Do not detect the coefficient ring. Use a generic Karatsuba algorithm.

GEN RgX\_divrem(GEN x, GEN y, GEN \*r) by default, returns the Euclidean quotient and store the remainder in r. Three special values of r change that behavior • NULL: do not store the remainder, used to implement RgX\_div,

- ONLY\_REM: return the remainder, used to implement RgX\_rem,

- **ONLY\_DIVIDES**: return the quotient if the division is exact, and **NULL** otherwise.

**GEN** RgX\_div(**GEN** x, **GEN** y)

**GEN** RgX\_div\_by\_X\_x(**GEN** A, **GEN** a, **GEN** \*r) returns the quotient of the **RgX** A by  $(X - a)$ , and sets r to the remainder  $A(a)$ .

**GEN** RgX\_rem(**GEN** x, **GEN** y)

**GEN** RgX\_pseudodivrem(**GEN** x, **GEN** y, **GEN** \*ptr) compute a pseudo-quotient  $q$  and pseudo-remainder  $r$  such that  $\text{lc}(y)^{\deg(x)-\deg(y)+1}x = qy + r$ . Return  $q$  and set \*ptr to  $r$ .

**GEN** RgX\_pseudorem(**GEN** x, **GEN** y) return the remainder in the pseudo-division of  $x$  by  $y$ .

**GEN** RgXQX\_pseudorem(**GEN** x, **GEN** y, **GEN** T) return the remainder in the pseudo-division of  $x$  by  $y$  over  $R[X]/(T)$ .

**int** ZXQX\_dvd(**GEN** x, **GEN** y, **GEN** T) let  $T$  be a monic irreducible **ZX**, let  $x, y$  be **t\_POL** whose coefficients are either **t\_INTs** or **ZX** in the same variable as  $T$ . Assume further that the leading coefficient of  $y$  is an integer. Return 1 if  $y|x$  in  $(\mathbf{Z}[Y]/(T))[X]$ , and 0 otherwise.

**GEN** RgXQX\_pseudodivrem(**GEN** x, **GEN** y, **GEN** T, **GEN** \*ptr) compute a pseudo-quotient  $q$  and pseudo-remainder  $r$  such that  $\text{lc}(y)^{\deg(x)-\deg(y)+1}x = qy + r$  in  $R[X]/(T)$ . Return  $q$  and set \*ptr to  $r$ .

**GEN** RgX\_mulXn(**GEN** a, **long** n) returns  $a * X^n$ . This may be a **t\_FRAC** if  $n < 0$  and the valuation of  $a$  is not large enough.

**GEN** RgX\_addmulXn(**GEN** a, **GEN** b, **long** n) returns  $a + b * X^n$ , assuming that  $n > 0$ .

**GEN** RgX\_addmulXn\_shallow(**GEN** a, **GEN** b, **long** n) shallow variant of **RgX\_addmulXn**.

**GEN** RgX\_digits(**GEN** x, **GEN** B) returns a vector of **RgX**  $[c_0, \dots, c_n]$  of degree less than the degree of  $B$  and such that  $x = \sum_{i=0}^n c_i B^i$ .

#### 7.6.14.5 Internal routines working on coefficient arrays.

These routines operate on coefficient blocks which are invalid **GENs**. A **GEN** argument  $a$  or  $b$  in routines below is actually a coefficient arrays representing the polynomials  $\sum_{i=0}^{na-1} a[i]X^i$  and  $\sum_{i=0}^{nb-1} b[i]X^i$ . Note that  $a[0]$  and  $b[0]$  contain coefficients and not the mandatory **GEN** codeword. This allows to implement divide-and-conquer methods directly, without needing to allocate wrappers around coefficient blocks.

**GEN** RgX\_mulspec(**GEN** a, **GEN** b, **long** na, **long** nb). Internal routine: given two coefficient arrays representing polynomials, return their product (as a true **GEN**) in variable 0.

**GEN** RgX\_sqrspec(**GEN** a, **long** na). Internal routine: given a coefficient array representing a polynomial r return its square (as a true **GEN**) in variable 0.

**GEN** RgX\_addspec(**GEN** x, **GEN** y, **long** nx, **long** ny) given two coefficient arrays representing polynomials, return their sum (as a true **GEN**) in variable 0.

**GEN** RgX\_addspec\_shallow(**GEN** x, **GEN** y, **long** nx, **long** ny) shallow variant of **RgX\_addspec**.

#### 7.6.14.6 GCD, Resultant.

GEN `RgX_gcd`(GEN `x`, GEN `y`) returns the GCD of `x` and `y`, assumed to be `t_POLs` in the same variable.

GEN `RgX_gcd_simple`(GEN `x`, GEN `y`) as `RgX_gcd` using a standard extended Euclidean algorithm. Usually slower than `RgX_gcd`.

GEN `RgX_extgcd`(GEN `x`, GEN `y`, GEN `*u`, GEN `*v`) returns  $d = \text{GCD}(x, y)$ , and sets `*u`, `*v` to the Bezout coefficients such that  $*ux + *vy = d$ . Uses a generic subresultant algorithm.

GEN `RgX_extgcd_simple`(GEN `x`, GEN `y`, GEN `*u`, GEN `*v`) as `RgX_extgcd` using a standard extended Euclidean algorithm. Usually slower than `RgX_extgcd`.

GEN `RgX_disc`(GEN `x`) returns the discriminant of the `t_POL` `x` with respect to its main variable.

GEN `RgX_resultant_all`(GEN `x`, GEN `y`, GEN `*sol`) returns `resultant(x, y)`. If `sol` is not NULL, sets it to the last non-constant remainder in the polynomial remainder sequence if it exists and to `gen_0` otherwise (e.g. one polynomial has degree 0).

#### 7.6.14.7 Other operations.

GEN `RgX_gtofp`(GEN `x`, GEN `prec`) returns the polynomial obtained by applying

`gtofp(gel(x, i), prec)`

to all coefficients of `x`.

GEN `RgX_fpnorml2`(GEN `x`, long `prec`) returns (a stack-clean variant of)

`gnorml2( RgX_gtofp(x, prec) )`

GEN `RgX_deriv`(GEN `x`) returns the derivative of `x` with respect to its main variable.

GEN `RgX_integ`(GEN `x`) returns the primitive of `x` vanishing at 0, with respect to its main variable.

GEN `RgX_rescale`(GEN `P`, GEN `h`) returns  $h^{\deg(P)}P(x/h)$ . `P` is an `RgX` and `h` is non-zero. (Leaves small objects on the stack. Suitable but inefficient for `gerepileupto`.)

GEN `RgX_unscale`(GEN `P`, GEN `h`) returns  $P(hx)$ . (Leaves small objects on the stack. Suitable but inefficient for `gerepileupto`.)

GEN `RgXV_unscale`(GEN `v`, GEN `h`) apply `RgX_unscale` to a vector of `RgX`.

GEN `RgX_translate`(GEN `P`, GEN `c`) assume `c` is a scalar or a polynomials whose main variable has lower priority than the main variable `X` of `P`. Returns  $P(X + c)$  (optimized for  $c = \pm 1$ ).

#### 7.6.14.8 Function related to modular forms.

GEN `RgX_act_GL2Q`(GEN `g`, long `k`) let  $R$  be a commutative ring and  $g = [a, b; c, d]$  be in  $\text{GL}_2(\mathbf{Q})$ ,  $g$  acts (on the left) on homogeneous polynomials of degree  $k - 2$  in  $V := R[X, Y]_{k-2}$  via

$$g \cdot P := P(dX - cY, -bX + aY) = (\det g)^{k-2} P((X, Y) \cdot g^{-1}).$$

This function returns the matrix in  $M_{k-1}(R)$  of  $P \mapsto g \cdot P$  in the basis  $(X^{k-2}, \dots, Y^{k-2})$  of  $V$ .

GEN `RgX_act_ZGL2Q`(GEN `z`, long `k`) let  $G := \text{GL}_2(\mathbf{Q})$ , acting on  $R[X, Y]_{k-2}$  and  $z \in \mathbf{Z}[G]$ . Return the matrix giving  $P \mapsto z \cdot P$  in the basis  $(X^{k-2}, \dots, Y^{k-2})$ .

### 7.6.15 RgXn.

GEN RgXn\_red\_shallow(GEN x, long n) return  $x \% t^n$ , where  $n \geq 0$ . Shallow function.

GEN RgXn\_recip\_shallow(GEN P) returns  $X^n P(1/X)$ . Shallow function.

GEN RgXn\_mul(GEN a, GEN b, long n) returns  $ab$  modulo  $X^n$ , where  $a, b$  are two  $\mathbf{t\_POL}$  in the same variable  $X$  and  $n \geq 0$ . Uses Karatsuba algorithm (Mulders, Hanrot-Zimmermann variant).

GEN RgXn\_sqr(GEN a, long n) returns  $a^2$  modulo  $X^n$ , where  $a$  is a  $\mathbf{t\_POL}$  in the variable  $X$  and  $n \geq 0$ . Uses Karatsuba algorithm (Mulders, Hanrot-Zimmermann variant).

GEN RgX\_mulhigh\_i(GEN f, GEN g, long n) return the Euclidean quotient of  $f(x) * g(x)$  by  $x^n$  (high product). Uses RgXn\_mul applied to the reciprocal polynomials of  $f$  and  $g$ . Not suitable for gerepile.

GEN RgX\_sqrhigh\_i(GEN f, long n) return the Euclidean quotient of  $f(x)^2$  by  $x^n$  (high product). Uses RgXn\_sqr applied to the reciprocal polynomial of  $f$ . Not suitable for gerepile.

GEN RgXn\_inv(GEN a, long n) returns  $a^{-1}$  modulo  $X^n$ , where  $a$  is a  $\mathbf{t\_POL}$  in the variable  $X$  and  $n \geq 0$ . Uses Newton-Raphson algorithm.

GEN RgXn\_inv\_i(GEN a, long n) as RgXn\_inv without final garbage collection (suitable for gerepileupto).

GEN RgXn\_powers(GEN x, long m, long n) returns  $[x^0, \dots, x^m]$  modulo  $X^n$  as a  $\mathbf{t\_VEC}$  of RgXns.

GEN RgXn\_powu(GEN x, ulong m, long n) returns  $x^m$  modulo  $X^n$ .

GEN RgXn\_powu\_i(GEN x, ulong m, long n) as RgXn\_powu, not memory clean.

GEN RgXn\_sqrt(GEN a, long n) returns  $a^{1/2}$  modulo  $X^n$ , where  $a$  is a  $\mathbf{t\_POL}$  in the variable  $X$  and  $n \geq 0$ . Assume that  $a = 1 \bmod X$ . Uses Newton algorithm.

GEN RgXn\_exp(GEN a, long n) returns  $\exp(a)$  modulo  $X^n$ , assuming  $a = 0 \bmod X$ . Uses Hanrot-Zimmermann algorithm.

GEN RgXn\_eval(GEN Q, GEN x, long n) special case of RgX\_RgXQ\_eval, when the modulus is a monomial: returns  $Q(x)$  modulo  $t^n$ , where  $x \in R[t]$ .

GEN RgX\_RgXn\_eval(GEN f, GEN x, long n) returns  $f(x)$  modulo  $X^n$ .

GEN RgX\_RgXnV\_eval(GEN f, GEN V, long n) as RgX\_RgXn\_eval(f, x, n), assuming  $V$  was output by RgXn\_powers(x, m, n) for some  $m \geq 1$ .

GEN RgXn\_reverse(GEN f, long n) assuming that  $f = ax \bmod x^2$  with  $a$  invertible, returns a  $\mathbf{t\_POL}$   $g$  of degree  $< n$  such that  $(g \circ f)(x) = x \bmod x^n$ .

### 7.6.16 RgXnV.

GEN RgXnV\_red\_shallow(GEN x, long n) apply RgXn\_red\_shallow to all the components of the vector  $x$ .

### 7.6.17 RgXQ.

GEN RgXQ\_mul(GEN y, GEN x, GEN T) computes  $xy \bmod T$

GEN RgXQ\_sqr(GEN x, GEN T) computes  $x^2 \bmod T$

GEN RgXQ\_inv(GEN x, GEN T) return the inverse of  $x \bmod T$ .

GEN RgXQ\_pow(GEN x, GEN n, GEN T) computes  $x^n \bmod T$

GEN RgXQ\_powu(GEN x, ulong n, GEN T) computes  $x^n \bmod T$ ,  $n$  being an ulong.

GEN RgXQ\_powers(GEN x, long n, GEN T) returns  $[x^0, \dots, x^n]$  as a t\_VEC of RgXQs.

GEN RgXQ\_matrix\_pow(GEN y, long n, long m, GEN P) returns RgXQ\_powers(y, m-1, P), as a matrix of dimension  $n \geq \deg P$ .

GEN RgXQ\_norm(GEN x, GEN T) returns the norm of  $\text{Mod}(x, T)$ .

GEN RgXQ\_charpoly(GEN x, GEN T, long v) returns the characteristic polynomial of  $\text{Mod}(x, T)$ , in variable  $v$ .

GEN RgX\_RgXQ\_eval(GEN f, GEN x, GEN T) returns  $f(x)$  modulo  $T$ .

GEN RgX\_RgXQV\_eval(GEN f, GEN V, GEN T) as RgX\_RgXQ\_eval(f, x, T), assuming  $V$  was output by RgXQ\_powers(x, n, T) for some  $n \geq 1$ .

int RgXQ\_ratlift(GEN x, GEN T, long amax, long bmax, GEN \*P, GEN \*Q) Assuming that  $\text{amax} + \text{bmax} < \deg T$ , attempts to recognize  $x$  as a rational function  $a/b$ , i.e. to find t\_POLs  $P$  and  $Q$  such that

- $P \equiv Qx \bmod T$ ,
- $\deg P \leq \text{amax}$ ,  $\deg Q \leq \text{bmax}$ ,
- $\gcd(T, P) = \gcd(P, Q)$ .

If unsuccessful, the routine returns 0 and leaves  $P, Q$  unchanged; otherwise it returns 1 and sets  $P$  and  $Q$ .

GEN RgXQ\_reverse(GEN f, GEN T) returns a t\_POL  $g$  of degree  $< n = \deg T$  such that  $T(x)$  divides  $(g \circ f)(x) - x$ , by solving a linear system. Low-level function underlying `modreverse`: it returns a lift of `[modreverse(f,T)]`; faster than the high-level function since it needs not compute the characteristic polynomial of  $f \bmod T$  (often already known in applications). In the trivial case where  $n \leq 1$ , returns a scalar, not a constant t\_POL.

### 7.6.18 RgXQV, RgXQC.

GEN RgXQC\_red(GEN z, GEN T)  $z$  a vector whose coefficients are RgXs (arbitrary GENs in fact), reduce them to RgXQs (applying `greduce` coefficientwise) in a t\_COL.

GEN RgXQV\_red(GEN z, GEN T)  $z$  a vector whose coefficients are RgXs (arbitrary GENs in fact), reduce them to RgXQs (applying `greduce` coefficientwise) in a t\_VEC.

GEN RgXQV\_RgXQ\_mul(GEN z, GEN x, GEN T)  $z$  multiplies the RgXQV  $z$  by the scalar (RgXQ)  $x$ .

### 7.6.19 RgXQM.

GEN RgXQM\_red(GEN z, GEN T) z a matrix whose coefficients are RgXs (arbitrary GENs in fact), reduce them to RgXQs (applying grem coefficientwise).

GEN RgXQM\_mul(GEN x, GEN y, GEN T)

### 7.6.20 RgXQX.

GEN RgXQX\_red(GEN z, GEN T) z a t\_POL whose coefficients are RgXs (arbitrary GENs in fact), reduce them to RgXQs (applying grem coefficientwise).

GEN RgXQX\_mul(GEN x, GEN y, GEN T)

GEN RgXQX\_RgXQ\_mul(GEN x, GEN y, GEN T) multiplies the RgXQX y by the scalar (RgXQ) x.

GEN RgXQX\_sqr(GEN x, GEN T)

GEN RgXQX\_powers(GEN x, long n, GEN T)

GEN RgXQX\_divrem(GEN x, GEN y, GEN T, GEN \*pr)

GEN RgXQX\_div(GEN x, GEN y, GEN T, GEN \*r)

GEN RgXQX\_rem(GEN x, GEN y, GEN T, GEN \*r)

GEN RgXQX\_translate(GEN P, GEN c, GEN T) assume the main variable  $X$  of  $P$  has higher priority than the main variable  $Y$  of  $T$  and  $c$ . Return a lift of  $P(X + \text{Mod}(c(Y), T(Y)))$ .

GEN Kronecker\_to\_mod(GEN z, GEN T)  $z \in R[X]$  represents an element  $P(X, Y)$  in  $R[X, Y] \bmod T(Y)$  in Kronecker form, i.e.  $z = P(X, X^{2^{*n-1}})$

Let  $R$  be some commutative ring,  $n = \deg T$  and let  $P(X, Y) \in R[X, Y]$  lift a polynomial in  $K[Y]$ , where  $K := R[X]/(T)$  and  $\deg_X P < 2n - 1$  — such as would result from multiplying minimal degree lifts of two polynomials in  $K[Y]$ . Let  $z = P(t, t^{2^{*n-1}})$  be a Kronecker form of  $P$ , this function returns the image of  $P(X, t)$  in  $K[t]$ , with t\_POLMOD coefficients. Not stack-clean. Note that  $t$  need not be the same variable as  $Y$ !

## Chapter 8:

### Black box algebraic structures

The generic routines like `gmul` or `gadd` allow handling objects belonging to a fixed list of basic types, with some natural polymorphism (you can mix rational numbers and polynomials, etc.), at the expense of efficiency and sometimes of clarity when the recursive structure becomes complicated, e.g. a few levels of `t_POLMODs` attached to different polynomials and variable numbers for quotient structures. This is the only possibility in GP.

On the other hand, the Level 2 Kernel allows dedicated routines to handle efficiently objects of a very specific type, e.g. polynomials with coefficients in the same finite field. This is more efficient, but involves a lot of code duplication since polymorphism is no longer possible.

A third and final option, still restricted to library programming, is to define an arbitrary algebraic structure (currently groups, fields, rings, algebras and  $\mathbf{Z}_p$ -modules) by providing suitable methods, then using generic algorithms. For instance naive Gaussian pivoting applies over all base fields and need only be implemented once. The difference with the first solution is that we no longer depend on the way functions like `gmul` or `gadd` will guess what the user is trying to do. We can then implement independently various groups / fields / algebras in a clean way.

#### 8.1 Black box groups.

A black box group is defined by a `bb_group` struct, describing methods available to handle group elements:

```
struct bb_group
{
    GEN (*mul)(void*, GEN, GEN);
    GEN (*pow)(void*, GEN, GEN);
    GEN (*rand)(void*);
    ulong (*hash)(GEN);
    int (*equal)(GEN, GEN);
    int (*equal1)(GEN);
    GEN (*easylog)(void *E, GEN, GEN, GEN);
};
```

`mul(E,x,y)` returns the product  $xy$ .

`pow(E,x,n)` returns  $x^n$  ( $n$  integer, possibly negative or zero).

`rand(E)` returns a random element in the group.

`hash(x)` returns a hash value for  $x$  (`hash.GEN` is suitable for this field).

`equal(x,y)` returns one if  $x = y$  and zero otherwise.

`equal1(x)` returns one if  $x$  is the neutral element in the group, and zero otherwise.

`easylog(E,a,g,o)` (optional) returns either NULL or the discrete logarithm  $n$  such that  $g^n = a$ , the element  $g$  being of order  $o$ . This provides a short-cut in situation where a better algorithm than the generic one is known.

A group is thus described by a `struct bb_group` as above and auxiliary data typecast to `void*`. The following functions operate on black box groups:

`GEN gen_Shanks_log(GEN x, GEN g, GEN N, void *E, const struct bb_group *grp)`  
 Generic baby-step/giant-step algorithm (Shanks's method). Assuming that  $g$  has order  $N$ , compute an integer  $k$  such that  $g^k = x$ . Return `cgetg(1, t_VEC)` if there are no solutions. This requires  $O(\sqrt{N})$  group operations and uses an auxiliary table containing  $O(\sqrt{N})$  group elements.

The above is useful for a one-shot computation. If many discrete logs are desired: `GEN gen_Shanks_init(GEN g, long n, void *E, const struct bb_group *grp)` return an auxiliary data structure  $T$  required to compute a discrete log in base  $g$ . Compute and store all powers  $g^i$ ,  $i < n$ .

`GEN gen_Shanks(GEN T, GEN x, ulong N, void *E, const struct bb_group *grp)` Let  $T$  be computed by `gen_Shanks_init(g, n, ...)`. Return  $k < nN$  such that  $g^k = x$  or `NULL` if no such index exist. It uses  $O(N)$  operation in the group and fast table lookups (in time  $O(\log n)$ ). The interface is such that the function may be used when the order of the base  $g$  is unknown, and hence compute it given only an upper bound  $B$  for it: e.g. choose  $n, N$  such that  $nN \geq B$  and compute the discrete log  $l$  of  $g^{-1}$  in base  $g$ , then use `gen_order` with multiple  $N = l + 1$ .

`GEN gen_Pollard_log(GEN x, GEN g, GEN N, void *E, const struct bb_group *grp)`  
 Generic Pollard rho algorithm. Assuming that  $g$  has order  $N$ , compute an integer  $k$  such that  $g^k = x$ . This requires  $O(\sqrt{N})$  group operations in average and  $O(1)$  storage. Will enter an infinite loop if there are no solutions.

`GEN gen_plog(GEN x, GEN g, GEN N, void *E, const struct bb_group *grp)` Assuming that  $g$  has prime order  $N$ , compute an integer  $k$  such that  $g^k = x$ , using either `gen_Shanks_log` or `gen_Pollard_log`. Return `cgetg(1, t_VEC)` if there are no solutions.

`GEN gen_Shanks_sqrtn(GEN a, GEN n, GEN N, GEN *zetan, void *E, const struct bb_group *grp)` returns one solution of  $x^n = a$  in a black box cyclic group of order  $N$ . Return `NULL` if no solution exists. If `zetan` is not `NULL` it is set to an element of exact order  $n$ . This function uses `gen_plog` for all prime divisors of  $\gcd(n, N)$ .

`GEN gen_PH_log(GEN a, GEN g, GEN N, void *E, const struct bb_group *grp)` returns an integer  $k$  such that  $g^k = x$ , assuming that  $g$  has order  $N$ , by Pohlig-Hellman algorithm. Return `cgetg(1, t_VEC)` if there are no solutions. This calls `gen_plog` repeatedly for all prime divisors  $p$  of  $N$ .

In the following functions the integer parameter `ord` can be given in all the formats recognized for the argument of arithmetic functions, i.e. either as a positive `t_INT`  $N$ , or as its factorization matrix  $faN$ , or (preferred) as a pair  $[N, faN]$ .

`GEN gen_order(GEN x, GEN ord, void *E, const struct bb_group *grp)` computes the order of  $x$ ; `ord` is a multiple of the order, for instance the group order.

`GEN gen_factored_order(GEN x, GEN ord, void *E, const struct bb_group *grp)` returns a pair  $[o, F]$ , where  $o$  is the order of  $x$  and  $F$  is the factorization of  $o$ ; `ord` is as in `gen_order`.

`GEN gen_gener(GEN ord, void *E, const struct bb_group *grp)` returns a random generator of the group, assuming it is of order exactly `ord`.

`GEN get_arith_Z(GEN ord)` given `ord` as above in one of the formats recognized for arithmetic functions, i.e. a positive `t_INT`  $N$ , its factorization  $faN$ , or the pair  $[N, faN]$ , return  $N$ .



GEN `get_arith_ZZM`(GEN `ord`) given `ord` as above, return the pair  $[N, faN]$ . This may require factoring  $N$ .

GEN `gen_select_order`(GEN `v`, void `*E`, const struct `bb_group *grp`) Let  $v$  be a vector of possible orders for the group; try to find the true order by checking orders of random points. This will not terminate if there is an ambiguity.

### 8.1.1 Black box groups with pairing.

These functions handle groups of rank at most 2 equipped with a family of bilinear pairings which behave like the Weil pairing on elliptic curves over finite field. In the descriptions below, the function `pairorder`(`E`, `P`, `Q`, `m`, `F`) must return the order of the  $m$ -pairing of  $P$  and  $Q$ , both of order dividing  $m$ , where  $F$  is the factorization matrix of a multiple of  $m$ .

GEN `gen_ellgroup`(GEN `o`, GEN `d`, GEN `*pt_m`, void `*E`, const struct `bb_group *grp`, GEN `pairorder`(void `*E`, GEN `P`, GEN `Q`, GEN `m`, GEN `F`)) returns the elementary divisors  $[d_1, d_2]$  of the group, assuming it is of order exactly  $o > 1$ , and that  $d_2$  divides  $d$ . If  $d_2 = 1$  then  $[o]$  is returned, otherwise `m=*pt_m` is set to the order of the pairing required to verify a generating set which is to be used with `gen_ellgens`. For the parameter  $o$ , all formats recognized by arithmetic functions are allowed, preferably a factorization matrix or a pair  $[n, \text{factor}(n)]$ .

GEN `gen_ellgens`(GEN `d1`, GEN `d2`, GEN `m`, void `*E`, const struct `bb_group *grp`, GEN `pairorder`(void `*E`, GEN `P`, GEN `Q`, GEN `m`, GEN `F`)) the parameters  $d_1, d_2, m$  being as returned by `gen_ellgroup`, returns a pair of generators  $[P, Q]$  such that  $P$  is of order  $d_1$  and the  $m$ -pairing of  $P$  and  $Q$  is of order  $m$ . (Note:  $Q$  needs not be of order  $d_2$ ). For the parameter  $d_1$ , all formats recognized by arithmetic functions are allowed, preferably a factorization matrix or a pair  $[n, \text{factor}(n)]$ .

### 8.1.2 Functions returning black box groups.

const struct `bb_group *` `get_Flxq_star`(void `**E`, GEN `T`, ulong `p`)

const struct `bb_group *` `get_FpXQ_star`(void `**E`, GEN `T`, GEN `p`) returns a pointer to the black box group  $(\mathbf{F}_p[x]/(T))^*$ .

const struct `bb_group *` `get_FpE_group`(void `**pE`, GEN `a4`, GEN `a6`, GEN `p`) returns a pointer to a black box group and set `*pE` to the necessary data for computing in the group  $E(\mathbf{F}_p)$  where  $E$  is the elliptic curve  $E: y^2 = x^3 + a_4x + a_6$ , with  $a_4$  and  $a_6$  in  $\mathbf{F}_p$ .

const struct `bb_group *` `get_FpXQE_group`(void `**pE`, GEN `a4`, GEN `a6`, GEN `T`, GEN `p`) returns a pointer to a black box group and set `*pE` to the necessary data for computing in the group  $E(\mathbf{F}_p[X]/(T))$  where  $E$  is the elliptic curve  $E: y^2 = x^3 + a_4x + a_6$ , with  $a_4$  and  $a_6$  in  $\mathbf{F}_p[X]/(T)$ .

const struct `bb_group *` `get_FlxqE_group`(void `**pE`, GEN `a4`, GEN `a6`, GEN `T`, ulong `p`) idem for small  $p$ .

const struct `bb_group *` `get_F2xqE_group`(void `**pE`, GEN `a2`, GEN `a6`, GEN `T`) idem for  $p = 2$ .

## 8.2 Black box fields.

A black box field is defined by a `bb_field` struct, describing methods available to handle field elements:

```
struct bb_field
{
    GEN (*red)(void *E ,GEN);
    GEN (*add)(void *E ,GEN, GEN);
    GEN (*mul)(void *E ,GEN, GEN);
    GEN (*neg)(void *E ,GEN);
    GEN (*inv)(void *E ,GEN);
    int (*equal0)(GEN);
    GEN (*s)(void *E, long);
};
```

In contrast of black box group, elements can have non canonical forms, and only `red` is required to return a canonical form. For instance a black box implementation of finite fields, all methods except `red` may return arbitrary representatives in  $\mathbf{Z}[X]$  of the correct congruence class modulo  $(p, T(X))$ .

`red(E,x)` returns the canonical form of  $x$ .

`add(E,x,y)` returns the sum  $x + y$ .

`mul(E,x,y)` returns the product  $xy$ .

`neg(E,x)` returns  $-x$ .

`inv(E,x)` returns the inverse of  $x$ .

`equal0(x)`  $x$  being in canonical form, returns one if  $x = 0$  and zero otherwise.

`s(n)`  $n$  being a small signed integer, returns  $n$  times the unit element.

A field is thus described by a `struct bb_field` as above and auxiliary data typecast to `void*`. The following functions operate on black box fields:

```
GEN gen_Gauss(GEN a, GEN b, void *E, const struct bb_field *ff)
GEN gen_Gauss_pivot(GEN x, long *rr, void *E, const struct bb_field *ff)
GEN gen_det(GEN a, void *E, const struct bb_field *ff)
GEN gen_ker(GEN x, long deplin, void *E, const struct bb_field *ff)
GEN gen_matcolinvimage(GEN a, GEN b, void *E, const struct bb_field *ff)
GEN gen_matcolmul(GEN a, GEN b, void *E, const struct bb_field *ff)
GEN gen_matid(long n, void *E, const struct bb_field *ff)
GEN gen_matinvimage(GEN a, GEN b, void *E, const struct bb_field *ff)
GEN gen_matmul(GEN a, GEN b, void *E, const struct bb_field *ff)
```

### 8.2.1 Functions returning black box fields.

```
const struct bb_field * get_Fp_field(void **pE, GEN p)
const struct bb_field * get_Fq_field(void **pE, GEN T, GEN p)
const struct bb_field * get_Flxq_field(void **pE, GEN T, ulong p)
const struct bb_field * get_F2xq_field(void **pE, GEN T)
const struct bb_field * get_nf_field(void **pE, GEN nf)
```

## 8.3 Black box algebra.

A black box algebra is defined by a `bb_algebra` struct, describing methods available to handle algebra elements:

```
struct bb_algebra
{
    GEN (*red)(void *E, GEN x);
    GEN (*add)(void *E, GEN x, GEN y);
    GEN (*sub)(void *E, GEN x, GEN y);
    GEN (*mul)(void *E, GEN x, GEN y);
    GEN (*sqr)(void *E, GEN x);
    GEN (*one)(void *E);
    GEN (*zero)(void *E);
};
```

In contrast with black box groups, elements can have non canonical forms, but only `add` is allowed to return a non canonical form.

`red(E,x)` returns the canonical form of  $x$ .

`add(E,x,y)` returns the sum  $x + y$ .

`sub(E,x,y)` returns the difference  $x - y$ .

`mul(E,x,y)` returns the product  $xy$ .

`sqr(E,x)` returns the square  $x^2$ .

`one(E)` returns the unit element.

`zero(E)` returns the zero element.

An algebra is thus described by a `struct bb_algebra` as above and auxiliary data typecast to `void*`. The following functions operate on black box algebra:

`GEN gen_bkeval(GEN P, long d, GEN x, int use_sqr, void *E, const struct bb_algebra *ff, GEN cmul(void *E, GEN P, long a, GEN x))`  $x$  being an element of the black box algebra, and  $P$  some black box polynomial of degree  $d$  over the base field, returns  $P(x)$ . The function `cmul(E,P,a,y)` must return the coefficient of degree  $a$  of  $P$  multiplied by  $y$ . `cmul` is allowed to return a non canonical form; it is also allowed to return `NULL` instead of an exact 0.

The flag `use_sqr` has the same meaning as for `gen_powers`. This implements an algorithm of Brent and Kung (1978).

GEN `gen_bkeval_powers`(GEN P, long d, GEN V, void \*E, const struct bb\_algebra \*ff, GEN `cmul`(void \*E, GEN P, long a, GEN x)) as `gen_RgX_bkeval` assuming  $V$  was output by `gen_powers`( $x, l, E, ff$ ) for some  $l \geq 1$ . For optimal performance,  $l$  should be computed by `brent_kung_optpow`.

long `brent_kung_optpow`(long d, long n, long m) returns the optimal parameter  $l$  for the evaluation of  $n/m$  polynomials of degree  $d$ . Fractional values can be used if the evaluations are done with different accuracies, and thus have different weights.

### 8.3.1 Functions returning black box algebras.

const struct bb\_algebra \* `get_FpX_algebra`(void \*\*E, GEN p, long v) return the algebra of polynomials over  $\mathbf{F}_p$  in variable  $v$ .

const struct bb\_algebra \* `get_FpXQ_algebra`(void \*\*E, GEN T, GEN p) return the algebra  $\mathbf{F}_p[X]/(T(X))$ .

const struct bb\_algebra \* `get_FpXQX_algebra`(void \*\*E, GEN T, GEN p, long v) return the algebra of polynomials over  $\mathbf{F}_p[X]/(T(X))$  in variable  $v$ .

const struct bb\_algebra \* `get_FlxqXQ_algebra`(void \*\*E, GEN S, GEN T, ulong p) return the algebra  $\mathbf{F}_p[X, Y]/(S(X, Y), T(X))$  (for ulong  $p$ ).

const struct bb\_algebra \* `get_FpXQXQ_algebra`(void \*\*E, GEN S, GEN T, GEN p) return the algebra  $\mathbf{F}_p[X, Y]/(S(X, Y), T(X))$ .

const struct bb\_algebra \* `get_Rg_algebra`(void) return the generic algebra.

## 8.4 Black box ring.

A black box ring is defined by a `bb_ring` struct, describing methods available to handle ring elements:

```
struct bb_ring
{
    GEN (*add)(void *E, GEN x, GEN y);
    GEN (*mul)(void *E, GEN x, GEN y);
    GEN (*sqr)(void *E, GEN x);
};
```

`add(E, x, y)` returns the sum  $x + y$ .

`mul(E, x, y)` returns the product  $xy$ .

`sqr(E, x)` returns the square  $x^2$ .

GEN `gen_fromdigits`(GEN v, GEN B, void \*E, struct bb\_ring \*r) where  $B$  is a ring element and  $v = [c_0, \dots, c_{n-1}]$  a vector of ring elements, return  $\sum_{i=0}^n c_i B^i$  using binary splitting.

GEN `gen_digits`(GEN x, GEN B, long n, void \*E, struct bb\_ring \*r, GEN (\*div)(void \*E, GEN x, GEN y, GEN \*r))

(Require the ring to be Euclidean)

`div(E, x, y, &r)` performs the Euclidean division of  $x$  by  $y$  in the ring  $R$ , returning the quotient  $q$  and setting  $r$  to the residue so that  $x = qy + r$  holds. The residue must belong to a fixed set of representatives of  $R/(y)$ .

The argument  $x$  being a ring element, `gen_digits` returns a vector of ring elements  $[c_0, \dots, c_{n-1}]$  such that  $x = \sum_{i=0}^n c_i B^i$ . Furthermore for all  $i \neq n-1$ , the elements  $c_i$  belonging to the fixed set of representatives of  $R/(B)$ .

## 8.5 Black box free $\mathbf{Z}_p$ -modules.

(Very experimental)

`GEN gen_ZpX_Dixon(GEN F, GEN V, GEN q, GEN p, long N, void *E, GEN lin(void *E, GEN F, GEN z, GEN q), GEN invl(void *E, GEN z))`

Let  $F$  be a `ZpXT` representing the coefficients of some abstract linear mapping  $f$  over  $\mathbf{Z}_p[X]$  seen as a free  $\mathbf{Z}_p$ -module, let  $V$  be an element of  $\mathbf{Z}_p[X]$  and let  $q = p^N$ . Return  $y \in \mathbf{Z}_p[X]$  such that  $f(y) = V \pmod{p^N}$  assuming the following holds for  $n \leq N$ :

- `lin(E, FpX_red(F, p^n), z, p^n)  $\equiv f(z) \pmod{p^n}$`
- `f(invl(E, z))  $\equiv z \pmod{p}$`

The rationale for the argument  $F$  being that it allows `gen_ZpX_Dixon` to reduce it to the required  $p$ -adic precision.

`GEN gen_ZpX_Newton(GEN x, GEN p, long n, void *E, GEN eval(void *E, GEN a, GEN q), GEN invd(void *E, GEN b, GEN v, GEN q, long N))`

Let  $x$  be an element of  $\mathbf{Z}_p[X]$  seen as a free  $\mathbf{Z}_p$ -module, and  $f$  some differentiable function over  $\mathbf{Z}_p[X]$  such that  $f(x) \equiv 0 \pmod{p}$ . Return  $y$  such that  $f(y) \equiv 0 \pmod{p^n}$ , assuming the following holds for all  $a, b \in \mathbf{Z}_p[X]$  and  $M \leq N$ :

- `v = eval(E, a, p^N)` is a vector of elements of  $\mathbf{Z}_p[X]$ ,
- `w = invd(E, b, v, p^M, M)` is an element in  $\mathbf{Z}_p[X]$ ,
- `v[1]  $\equiv f(a) \pmod{p^N \mathbf{Z}_p[X]}$` ,
- `df_a(w)  $\equiv b \pmod{p^M \mathbf{Z}_p[X]}$`

and  $df_a$  denotes the differential of  $f$  at  $a$ . Motivation: `eval` allows to evaluate  $f$  and `invd` allows to invert its differential. Frequently, data useful to compute the differential appear as a subproduct of computing the function. The vector  $v$  allows `eval` to provide these to `invd`. The implementation of `invd` will generally involves the use of the function `gen_ZpX_Dixon`.



## Chapter 9:

### Operations on general PARI objects

#### 9.1 Assignment.

It is in general easier to use a direct conversion, e.g. `y = stoi(s)`, than to allocate a target of correct type and sufficient size, then assign to it:

```
GEN y = cgeti(3); affsi(s, y);
```

These functions can still be moderately useful in complicated garbage collecting scenarios but you will be better off not using them.

`void gaffsg(long s, GEN x)` assigns the `long s` into the object `x`.

`void gaffect(GEN x, GEN y)` assigns the object `x` into the object `y`. Both `x` and `y` must be scalar types. Type conversions (e.g. from `t_INT` to `t_REAL` or `t_INTMOD`) occur if legitimate.

`int is_universal_constant(GEN x)` returns 1 if `x` is a global PARI constant you should never assign to (such as `gen_1`), and 0 otherwise.

#### 9.2 Conversions.

##### 9.2.1 Scalars.

`double rtodbl(GEN x)` applied to a `t_REAL x`, converts `x` into a `double` if possible.

`GEN dbltor(double x)` converts the `double x` into a `t_REAL`.

`long dblexpo(double x)` returns `expo(dbltor(x))`, but faster and without cluttering the stack.

`ulong dblmantissa(double x)` returns the most significant word in the mantissa of `dbltor(x)`.

`double gtodouble(GEN x)` if `x` is a real number (not necessarily a `t_REAL`), converts `x` into a `double` if possible.

`long gtos(GEN x)` converts the `t_INT x` to a small integer if possible, otherwise raise an exception. This function is similar to `itos`, slightly slower since it checks the type of `x`.

`double dbllog2r(GEN x)` assuming that `x` is a non-zero `t_REAL`, returns an approximation to `log2(|x|)`.

`double dblmodulus(GEN x)` return an approximation to `|x|`.

`long gtolong(GEN x)` if `x` is an integer (not necessarily a `t_INT`), converts `x` into a `long` if possible.

`GEN fractor(GEN x, long l)` applied to a `t_FRAC x`, converts `x` into a `t_REAL` of length `prec`.

`GEN quadtofp(GEN x, long l)` applied to a `t_QUAD x`, converts `x` into a `t_REAL` or `t_COMPLEX` depending on the sign of the discriminant of `x`, to precision `l BITS_IN_LONG`-bit words.

GEN `upper_to_cx`(GEN `x`, long `*prec`) valid for a `t_COMPLEX` or `t_QUAD` belonging to the upper half-plane. If a `t_QUAD`, convert it to `t_COMPLEX` using accuracy `*prec`. If `x` is inexact, sets `*prec` to the precision of `x`.

GEN `cxtofp`(GEN `x`, long `prec`) converts the `t_COMPLEX` `x` to a complex whose real and imaginary parts are `t_REAL` of length `prec` (special case of `gtofp`).

GEN `cxcompotor`(GEN `x`, long `prec`) converts the `t_INT`, `t_REAL` or `t_FRAC` `x` to a `t_REAL` of length `prec`. These are all the real types which may occur as components of a `t_COMPLEX`; special case of `gtofp` (introduced so that the latter is not recursive and can thus be inlined).

GEN `cxtoreal`(GEN `x`) converts the complex (`t_INT`, `t_REAL`, `t_FRAC` or `t_COMPLEX`) `x` to a real number if its imaginary part is 0. Shallow function.

converts the `t_COMPLEX` `x` to a complex whose real and imaginary parts are `t_REAL` of length `prec` (special case of `gtofp`).

GEN `gtofp`(GEN `x`, long `prec`) converts the complex number `x` (`t_INT`, `t_REAL`, `t_FRAC`, `t_QUAD` or `t_COMPLEX`) to either a `t_REAL` or `t_COMPLEX` whose components are `t_REAL` of precision `prec`; not necessarily of *length* `prec`: a real 0 may be given as `real_0(...)`. If the result is a `t_COMPLEX` extra care is taken so that its modulus really has accuracy `prec`: there is a problem if the real part of the input is an exact 0; indeed, converting it to `real_0(prec)` would be wrong if the imaginary part is tiny, since the modulus would then become equal to 0, as in  $1.E-100+0.E-28=0.E-28$ .

GEN `gtomp`(GEN `z`, long `prec`) converts the real number `x` (`t_INT`, `t_REAL`, `t_FRAC`, real `t_QUAD`) to either a `t_INT` or a `t_REAL` of precision `prec`. Not memory clean if `x` is a `t_INT`: we return `x` itself and not a copy.

GEN `gcvtop`(GEN `x`, GEN `p`, long `l`) converts `x` into a `t_PADIC` of precision `l`. Works componentwise on recursive objects, e.g. `t_POL` or `t_VEC`. Converting 0 yields  $O(p^l)$ ; converting a non-zero number yield a result well defined modulo  $p^{v_p(x)+l}$ .

GEN `cvtop`(GEN `x`, GEN `p`, long `l`) as `gcvtop`, assuming that `x` is a scalar.

GEN `cvtop2`(GEN `x`, GEN `y`) `y` being a  $p$ -adic, converts the scalar `x` to a  $p$ -adic of the same accuracy. Shallow function.

GEN `cvstop2`(long `s`, GEN `y`) `y` being a  $p$ -adic, converts the scalar `s` to a  $p$ -adic of the same accuracy. Shallow function.

GEN `gprec`(GEN `x`, long `l`) returns a copy of `x` whose precision is changed to `l` digits. The precision change is done recursively on all components of `x`. Digits means *decimal*,  $p$ -adic and  $X$ -adic digits for `t_REAL`, `t_SER`, `t_PADIC` components, respectively.

GEN `gprec_w`(GEN `x`, long `l`) returns a shallow copy of `x` whose `t_REAL` components have their precision changed to `l words`. This is often more useful than `gprec`.

GEN `gprec_wtrunc`(GEN `x`, long `l`) returns a shallow copy of `x` whose `t_REAL` components have their precision *truncated* to `l words`. Contrary to `gprec_w`, this function may never increase the precision of `x`.

GEN `gprec_wensure`(GEN `x`, long `l`) returns a shallow copy of `x` whose `t_REAL` components have their precision *increased* to at least `l words`. Contrary to `gprec_w`, this function may never decrease the precision of `x`.



### 9.2.2 Modular objects / lifts.

GEN `gmodulo`(GEN `x`, GEN `y`) creates the object **Mod**(`x`,`y`) on the PARI stack, where `x` and `y` are either both `t_INTs`, and the result is a `t_INTMOD`, or `x` is a scalar or a `t_POL` and `y` a `t_POL`, and the result is a `t_POLMOD`.

GEN `gmodulgs`(GEN `x`, long `y`) same as **gmodulo** except `y` is a long.

GEN `gmodulsg`(long `x`, GEN `y`) same as **gmodulo** except `x` is a long.

GEN `gmodulss`(long `x`, long `y`) same as **gmodulo** except both `x` and `y` are longs.

GEN `lift_shallow`(GEN `x`) shallow version of `lift`

GEN `liftall_shallow`(GEN `x`) shallow version of `liftall`

GEN `liftint_shallow`(GEN `x`) shallow version of `liftint`

GEN `liftpol_shallow`(GEN `x`) shallow version of `liftpol`

GEN `centerlift0`(GEN `x`, long `v`) DEPRECATED, kept for backward compatibility only: use either `lift0(x,v)` or `centerlift(x)`.

### 9.2.3 Between polynomials and coefficient arrays.

GEN `gtopoly`(GEN `x`, long `v`) converts or truncates the object `x` into a `t_POL` with main variable number `v`. A common application would be the conversion of coefficient vectors (coefficients are given by decreasing degree). E.g. `[2,3]` goes to  $2*v + 3$

GEN `gtopolyrev`(GEN `x`, long `v`) converts or truncates the object `x` into a `t_POL` with main variable number `v`, but vectors are converted in reverse order compared to `gtopoly` (coefficients are given by increasing degree). E.g. `[2,3]` goes to  $3*v + 2$ . In other words the vector represents a polynomial in the basis  $(1, v, v^2, v^3, \dots)$ .

GEN `normalizpol`(GEN `x`) applied to an unnormalized `t_POL` `x` (with all coefficients correctly set except that `leading_term(x)` might be zero), normalizes `x` correctly in place and returns `x`. For internal use. Normalizing means deleting all leading *exact* zeroes (as per `isexactzero`), except if the polynomial turns out to be 0, in which case we try to find a coefficient `c` which is a non-rational zero, and return the constant polynomial `c`. (We do this so that information about the base ring is not lost.)

GEN `normalizpol_lg`(GEN `x`, long `l`) applies `normalizpol` to `x`, pretending that `lg(x)` is `l`, which must be less than or equal to `lg(x)`. If equal, the function is equivalent to `normalizpol(x)`.

GEN `normalizpol_approx`(GEN `x`, long `lx`) as `normalizpol_lg`, with the difference that we just delete all leading zeroes (as per `gequal0`). This rougher normalization is used when we have no other choice, for instance before attempting a Euclidean division by `x`.

The following routines do *not* copy coefficients on the stack (they only move pointers around), hence are very fast but not suitable for `gerepile` calls. Recall that an `RgV` (resp. an `RgX`, resp. an `RgM`) is a `t_VEC` or `t_COL` (resp. a `t_POL`, resp. a `t_MAT`) with arbitrary components. Similarly, an `RgXV` is a `t_VEC` or `t_COL` with `RgX` components, etc.

GEN `RgV_to_RgX`(GEN `x`, long `v`) converts the `RgV` `x` to a (normalized) polynomial in variable `v` (as `gtopolyrev`, without copy).

GEN `RgV_to_RgX_reverse`(GEN `x`, long `v`) converts the `RgV` `x` to a (normalized) polynomial in variable `v` (as `gtopoly`, without copy).

GEN `RgX_to_RgC`(GEN `x`, long `N`) converts the `t_POL` `x` to a `t_COL` `v` with `N` components. Coefficients of `x` are listed by increasing degree, so that `y[i]` is the coefficient of the term of degree  $i - 1$  in `x`.

GEN `Rg_to_RgC`(GEN `x`, long `N`) as `RgX_to_RgV`, except that other types than `t_POL` are allowed for `x`, which is then considered as a constant polynomial.

GEN `RgM_to_RgXV`(GEN `x`, long `v`) converts the `RgM` `x` to a `t_VEC` of `RgX`, by repeated calls to `RgV_to_RgX`.

GEN `RgV_to_RgM`(GEN `v`, long `N`) converts the vector `v` to a `t_MAT` with `N` rows, by repeated calls to `Rg_to_RgV`.

GEN `RgXV_to_RgM`(GEN `v`, long `N`) converts the vector of `RgX` `v` to a `t_MAT` with `N` rows, by repeated calls to `RgX_to_RgV`.

GEN `RgM_to_RgXX`(GEN `x`, long `v`, long `w`) converts the `RgM` `x` into a `t_POL` in variable `v`, whose coefficients are `t_POL`s in variable `w`. This is a shortcut for

`RgV_to_RgX( RgM_to_RgXV(x, w), v );`

There are no consistency checks with respect to variable priorities: the above is an invalid object if `varncmp(v, w) ≥ 0`.

GEN `RgXX_to_RgM`(GEN `x`, long `N`) converts the `t_POL` `x` with `RgX` (or constant) coefficients to a matrix with `N` rows.

long `RgXY_degreeex`(GEN `P`) return the degree of `P` with respect to the secondary variable.

GEN `RgXY_swap`(GEN `P`, long `n`, long `w`) converts the bivariate polynomial  $P(u, v)$  (a `t_POL` with `t_POL` or scalar coefficients) to  $P(\text{pol.x}[w], u)$ , assuming `n` is an upper bound for  $\deg_v(P)$ .

GEN `RgXY_swapspec`(GEN `C`, long `n`, long `w`, long `lP`) as `RgXY_swap` where the coefficients of `P` are given by `gel(C, 0), ..., gel(C, lP-1)`.

GEN `RgX_to_ser`(GEN `x`, long `l`) convert the `t_POL` `x` to a *shallow* `t_SER` of length  $l \geq 2$ . Unless the polynomial is an exact zero, the coefficient of lowest degree  $T^d$  of the result is not an exact zero (as per `isexactzero`). The remainder is  $O(T^{d+l-2})$ .

GEN `RgX_to_ser_inexact`(GEN `x`, long `l`) convert the `t_POL` `x` to a *shallow* `t_SER` of length  $l \geq 2$ . Unless the polynomial is zero, the coefficient of lowest degree  $T^d$  of the result is not zero (as per `gequal0`). The remainder is  $O(T^{d+l-2})$ .

GEN `RgV_to_ser`(GEN `x`, long `v`, long `l`) convert the `t_VEC` `x`, to a *shallow* `t_SER` of length  $l \geq 2$ .

GEN `rfrac_to_ser`(GEN `F`, long `l`) applied to a `t_RFRAC` `F`, creates a `t_SER` of length  $l \geq 2$  congruent to `F`. Not memory-clean but suitable for `gerepileupto`.

GEN `rfracrecip_to_ser_absolute`(GEN `F`, long `d`) applied to a `t_RFRAC` `F`, creates the `t_SER`  $F(1/t) + O(t^d)$ . Note that we use absolute and not relative precision here.

GEN `gtoser`(GEN `s`, long `v`, long `d`). This function is deprecated, kept for backward compatibility: it follows the semantic of `Ser(s, v)`, with  $d = \text{seriesprecision}$  implied and is hard to use as a general conversion function. Use `gtoser_prec` instead.

It converts the object `s` into a `t_SER` with main variable number `v` and  $d > 0$  significant terms, but the argument `d` is sometimes ignored. More precisely

- if  $s$  is a scalar (with respect to variable  $v$ ), we return a constant power series with  $d$  significant terms;

- if  $s$  is a `t_POL` in variable  $v$ , it is truncated to  $d$  terms if needed;

- if  $s$  is a vector, the coefficients of the vector are understood to be the coefficients of the power series starting from the constant term (as in `Polrev`), and the precision  $d$  is *ignored*;

- if  $s$  is already a power series in  $v$ , we return a copy, and the precision  $d$  is again *ignored*.

`GEN gtoser_prec(GEN s, long v, long d)` this function is a variant of `gtoser` following the semantic of `Ser(s,v,d)`: the precision  $d$  is always taken into account.

`GEN gtocol(GEN x)` converts the object  $x$  into a `t_COL`

`GEN gtomat(GEN x)` converts the object  $x$  into a `t_MAT`.

`GEN gtovect(GEN x)` converts the object  $x$  into a `t_VEC`.

`GEN gtovectsmall(GEN x)` converts the object  $x$  into a `t_VECSMALL`.

`GEN normalize(GEN x)` applied to an unnormalized `t_SER`  $x$  (i.e. type `t_SER` with all coefficients correctly set except that  $x[2]$  might be zero), normalizes  $x$  correctly in place. Returns  $x$ . For internal use.

`GEN serchop0(GEN s)` given a `t_SER` of the form  $x^v s(x)$ , with  $s(0) \neq 0$ , return  $x^v(s-s(0))$ . Shallow function.

`GEN serchop_i(GEN x, long n)` returns a shallow copy of `t_SER`  $x$  with all terms of degree strictly less than  $n$  removed. Shallow version of `serchop`.

## 9.3 Constructors.

### 9.3.1 Clean constructors.

`GEN zeropadic(GEN p, long n)` creates a 0 `t_PADIC` equal to  $O(p^n)$ .

`GEN zeroser(long v, long n)` creates a 0 `t_SER` in variable  $v$  equal to  $O(X^n)$ .

`GEN scalarser(GEN x, long v, long prec)` creates a constant `t_SER` in variable  $v$  and precision `prec`, whose constant coefficient is (a copy of)  $x$ , in other words  $x + O(v^{\text{prec}})$ . Assumes that `prec`  $\geq 0$ .

`GEN pol_0(long v)` Returns the constant polynomial 0 in variable  $v$ .

`GEN pol_1(long v)` Returns the constant polynomial 1 in variable  $v$ .

`GEN pol_x(long v)` Returns the monomial of degree 1 in variable  $v$ .

`GEN pol_xn(long n, long v)` Returns the monomial of degree  $n$  in variable  $v$ ; assume that  $n \geq 0$ .

`GEN pol_xnall(long n, long v)` Returns the Laurent monomial of degree  $n$  in variable  $v$ ;  $n < 0$  is allowed.

`GEN pol_x_powers(long N, long v)` returns the powers of `pol_x(v)`, of degree 0 to  $N - 1$ , in a vector with  $N$  components.

`GEN scalarpol(GEN x, long v)` creates a constant `t_POL` in variable  $v$ , whose constant coefficient is (a copy of)  $x$ .

`GEN deg1pol(GEN a, GEN b, long v)` creates the degree 1 `t_POL`  $ap_1x(v) + b$   
`GEN zeropol(long v)` is identical `pol_0`.  
`GEN zerocol(long n)` creates a `t_COL` with `n` components set to `gen_0`.  
`GEN zerovec(long n)` creates a `t_VEC` with `n` components set to `gen_0`.  
`GEN zerovec_block(long n)` as `zerovec` but return a clone.  
`GEN col_ei(long n, long i)` creates a `t_COL` with `n` components set to `gen_0`, but for the `i`-th one which is set to `gen_1` (`i`-th vector in the canonical basis).  
`GEN vec_ei(long n, long i)` creates a `t_VEC` with `n` components set to `gen_0`, but for the `i`-th one which is set to `gen_1` (`i`-th vector in the canonical basis).  
`GEN trivial_fact(void)` returns the trivial (empty) factorization `Mat([ ]~, [ ]~)`  
`GEN prime_fact(GEN x)` returns the factorization `Mat([x]~, [1]~)`  
`GEN Rg_col_ei(GEN x, long n, long i)` creates a `t_COL` with `n` components set to `gen_0`, but for the `i`-th one which is set to `x`.  
`GEN vecsmall_ei(long n, long i)` creates a `t_VECSMALL` with `n` components set to 0, but for the `i`-th one which is set to 1 (`i`-th vector in the canonical basis).  
`GEN scalarcol(GEN x, long n)` creates a `t_COL` with `n` components set to `gen_0`, but the first one which is set to a copy of `x`. (The name comes from `RgV_isscalar`.)  
`GEN mkintmodu(ulong x, ulong y)` creates the `t_INTMOD` `Mod(x, y)`. The inputs must satisfy  $x < y$ .  
`GEN zeromat(long m, long n)` creates a `t_MAT` with `m x n` components set to `gen_0`. Note that the result allocates a *single* column, so modifying an entry in one column modifies it in all columns. To fully allocate a matrix initialized with zero entries, use `zeromatcopy`.  
`GEN zeromatcopy(long m, long n)` creates a `t_MAT` with `m x n` components set to `gen_0`.  
`GEN matid(long n)` identity matrix in dimension `n` (with components `gen_1` and `gen_0`).  
`GEN scalarmat(GEN x, long n)` scalar matrix, `x` times the identity.  
`GEN scalarmat_s(long x, long n)` scalar matrix, `stoi(x)` times the identity.  
`GEN vecrange(GEN a, GEN b)` returns the `t_VEC` `[a..b]`.  
`GEN vecrangess(long a, long b)` returns the `t_VEC` `[a..b]`.

See also next section for analogs of the following functions:

`GEN mkfracss(long x, long y)` creates the `t_FRAC`  $x/y$ . Assumes that  $y > 1$  and  $(x, y) = 1$ .  
`GEN sstoQ(long x, long y)` returns the `t_INT` or `t_FRAC`  $x/y$ ; no assumptions.  
`void Qtoss(GEN q, long *n, long *d)` given a `t_INT` or `t_FRAC`  $q$ , set `n` and `d` such that  $q = n/d$  with  $d \geq 1$  and  $(n, d) = 1$ . Overflow error if numerator or denominator do not fit into a long integer.  
`GEN mkfraccopy(GEN x, GEN y)` creates the `t_FRAC`  $x/y$ . Assumes that  $y > 1$  and  $(x, y) = 1$ .  
`GEN mkrfraccopy(GEN x, GEN y)` creates the `t_RFRAC`  $x/y$ . Assumes that  $y$  is a `t_POL`,  $x$  a compatible type whose variable has lower or same priority, with  $(x, y) = 1$ .

GEN `mkcolcopy`(GEN `x`) creates a 1-dimensional `t_COL` containing `x`.

GEN `mkmatcopy`(GEN `x`) creates a 1-by-1 `t_MAT` wrapping the `t_COL` `x`.

GEN `mkveccopy`(GEN `x`) creates a 1-dimensional `t_VEC` containing `x`.

GEN `mkvec2copy`(GEN `x`, GEN `y`) creates a 2-dimensional `t_VEC` equal to `[x,y]`.

GEN `mkcols`(long `x`) creates a 1-dimensional `t_COL` containing `stoi(x)`.

GEN `mkcol2s`(long `x`, long `y`) creates a 2-dimensional `t_COL` containing `[stoi(x), stoi(y)]` .

GEN `mkcol3s`(long `x`, long `y`, long `z`) creates a 3-dimensional `t_COL` containing `[stoi(x), stoi(y), stoi(z)]` .

GEN `mkcol4s`(long `x`, long `y`, long `z`, long `t`) creates a 4-dimensional `t_COL` containing `[stoi(x), stoi(y), stoi(z), stoi(t)]` .

GEN `mkvecs`(long `x`) creates a 1-dimensional `t_VEC` containing `stoi(x)`.

GEN `mkvec2s`(long `x`, long `y`) creates a 2-dimensional `t_VEC` containing `[stoi(x), stoi(y)]`.

GEN `mkmat22s`(long `a`, long `b`, long `c`, long `d`) creates the 2 by 2 `t_MAT` with successive rows `[stoi(a), stoi(b)]` and `[stoi(c), stoi(d)]`.

GEN `mkvec3s`(long `x`, long `y`, long `z`) creates a 3-dimensional `t_VEC` containing `[stoi(x), stoi(y), stoi(z)]`.

GEN `mkvec4s`(long `x`, long `y`, long `z`, long `t`) creates a 4-dimensional `t_VEC` containing `[stoi(x), stoi(y), stoi(z), stoi(t)]`.

GEN `mkvecsmall`(long `x`) creates a 1-dimensional `t_VECSMALL` containing `x`.

GEN `mkvecsmall2`(long `x`, long `y`) creates a 2-dimensional `t_VECSMALL` containing `[x, y]`.

GEN `mkvecsmall3`(long `x`, long `y`, long `z`) creates a 3-dimensional `t_VECSMALL` containing `[x, y, z]`.

GEN `mkvecsmall4`(long `x`, long `y`, long `z`, long `t`) creates a 4-dimensional `t_VECSMALL` containing `[x, y, z, t]`.

GEN `mkvecsmalln`(long `n`, ...) returns the `t_VECSMALL` whose `n` coefficients (long) follow.  
*Warning:* since this is a variadic function, C type promotion is not performed on the arguments by the compiler, thus you have to make sure that all the arguments are of type `long`, in particular integer constants need to be written with the `L` suffix: `mkvecsmalln(2, 1L, 2L)` is correct, but `mkvecsmalln(2, 1, 2)` is not.

### 9.3.2 Unclean constructors.

Contrary to the policy of general PARI functions, the functions in this subsection do *not* copy their arguments, nor do they produce an object a priori suitable for `gerepileupto`. In particular, they are faster than their clean equivalent (which may not exist). *If* you restrict their arguments to universal objects (e.g `gen_0`), then the above warning does not apply.

`GEN mkcomplex(GEN x, GEN y)` creates the `t_COMPLEX`  $x + iy$ .

`GEN mulcxI(GEN x)` creates the `t_COMPLEX`  $ix$ . The result in general contains data pointing back to the original  $x$ . Use `gcopy` if this is a problem. But in most cases, the result is to be used immediately, before  $x$  is subject to garbage collection.

`GEN mulcxmI(GEN x)`, as `mulcxI`, but returns  $-ix$ .

`GEN mulcxpowIs(GEN x, long k)`, as `mulcxI`, but returns  $x \cdot i^k$ .

`GEN mkquad(GEN n, GEN x, GEN y)` creates the `t_QUAD`  $x + yw$ , where  $w$  is a root of  $n$ , which is of the form `quadpoly(D)`.

`GEN mkfrac(GEN x, GEN y)` creates the `t_FRAC`  $x/y$ . Assumes that  $y > 1$  and  $(x, y) = 1$ .

`GEN mkrfrac(GEN x, GEN y)` creates the `t_RFRAC`  $x/y$ . Assumes that  $y$  is a `t_POL`,  $x$  a compatible type whose variable has lower or same priority, with  $(x, y) = 1$ .

`GEN mkcol(GEN x)` creates a 1-dimensional `t_COL` containing  $x$ .

`GEN mkcol2(GEN x, GEN y)` creates a 2-dimensional `t_COL` equal to  $[x, y]$ .

`GEN mkcol3(GEN x, GEN y, GEN z)` creates a 3-dimensional `t_COL` equal to  $[x, y, z]$ .

`GEN mkcol4(GEN x, GEN y, GEN z, GEN t)` creates a 4-dimensional `t_COL` equal to  $[x, y, z, t]$ .

`GEN mkcol5(GEN a1, GEN a2, GEN a3, GEN a4, GEN a5)` creates the 5-dimensional `t_COL` equal to  $[a_1, a_2, a_3, a_4, a_5]$ .

`GEN mkcol6(GEN x, GEN y, GEN z, GEN t, GEN u, GEN v)` creates the 6-dimensional column vector  $[x, y, z, t, u, v]$ .

`GEN mkintmod(GEN x, GEN y)` creates the `t_INTMOD`  $\text{Mod}(x, y)$ . The inputs must be `t_INTs` satisfying  $0 \leq x < y$ .

`GEN mkpolmod(GEN x, GEN y)` creates the `t_POLMOD`  $\text{Mod}(x, y)$ . The input must satisfy  $\deg x < \deg y$  with respect to the main variable of the `t_POL`  $y$ .  $x$  may be a scalar.

`GEN mkmat(GEN x)` creates a 1-column `t_MAT` with column  $x$  (a `t_COL`).

`GEN mkmat2(GEN x, GEN y)` creates a 2-column `t_MAT` with columns  $x, y$  (`t_COLS` of the same length).

`GEN mkmat22(GEN a, GEN b, GEN c, GEN d)` creates the 2 by 2 `t_MAT` with successive rows  $[a, b]$  and  $[c, d]$ .

`GEN mkmat3(GEN x, GEN y, GEN z)` creates a 3-column `t_MAT` with columns  $x, y, z$  (`t_COLS` of the same length).

`GEN mkmat4(GEN x, GEN y, GEN z, GEN t)` creates a 4-column `t_MAT` with columns  $x, y, z, t$  (`t_COLS` of the same length).

`GEN mkmat5(GEN x, GEN y, GEN z, GEN t, GEN u)` creates a 5-column `t_MAT` with columns  $x, y, z, t, u$  (`t_COLS` of the same length).

GEN `mkvec`(GEN `x`) creates a 1-dimensional `t_VEC` containing `x`.

GEN `mkvec2`(GEN `x`, GEN `y`) creates a 2-dimensional `t_VEC` equal to `[x,y]`.

GEN `mkvec3`(GEN `x`, GEN `y`, GEN `z`) creates a 3-dimensional `t_VEC` equal to `[x,y,z]`.

GEN `mkvec4`(GEN `x`, GEN `y`, GEN `z`, GEN `t`) creates a 4-dimensional `t_VEC` equal to `[x,y,z,t]`.

GEN `mkvec5`(GEN `a1`, GEN `a2`, GEN `a3`, GEN `a4`, GEN `a5`) creates the 5-dimensional `t_VEC` equal to `[a1,a2,a3,a4,a5]`.

GEN `mkqfi`(GEN `x`, GEN `y`, GEN `z`) creates `t_QFI` equal to `Qfb(x,y,z)`, assuming that  $y^2 - 4xz < 0$ .

GEN `mkerr`(long `n`) returns a `t_ERROR` with error code `n` (enum `err_list`).

It is sometimes useful to return such a container whose entries are not universal objects, but nonetheless suitable for `gerepileupto`. If the entries can be computed at the time the result is returned, the following macros achieve this effect:

GEN `retmkvec`(GEN `x`) returns a vector containing the single entry `x`, where the vector root is created just before the function argument `x` is evaluated. Expands to

```
{
    GEN res = cgetg(2, t_VEC);
    gel(res, 1) = x; /* or rather, the expansion of x */
    return res;
}
```

For instance, the `retmkvec(gcopy(x))` returns a clean object, just like `return mkveccopy(x)` would.

GEN `retmkvec2`(GEN `x`, GEN `y`) returns the 2-dimensional `t_VEC` `[x,y]`.

GEN `retmkvec3`(GEN `x`, GEN `y`, GEN `z`) returns the 3-dimensional `t_VEC` `[x,y,z]`.

GEN `retmkvec4`(GEN `x`, GEN `y`, GEN `z`, GEN `t`) returns the 4-dimensional `t_VEC` `[x,y,z,t]`.

GEN `retmkvec5`(GEN `x`, GEN `y`, GEN `z`, GEN `t`, GEN `u`) returns the 5-dimensional row vector `[x,y,z,t,u]`.

GEN `retconst_vec`(long `n`, GEN `x`) returns the `n`-dimensional `t_VEC` whose entries are constant and all equal to `x`.

GEN `retmkcol`(GEN `x`) returns the 1-dimensional `t_COL` `[x]` .

GEN `retmkcol2`(GEN `x`, GEN `y`) returns the 2-dimensional `t_COL` `[x,y]` .

GEN `retmkcol3`(GEN `x`, GEN `y`, GEN `z`) returns the 3-dimensional `t_COL` `[x,y,z]` .

GEN `retmkcol4`(GEN `x`, GEN `y`, GEN `z`, GEN `t`) returns the 4-dimensional `t_COL` `[x,y,z,t]` .

GEN `retmkcol5`(GEN `x`, GEN `y`, GEN `z`, GEN `t`, GEN `u`) returns the 5-dimensional column vector `[x,y,z,t,u]` .

GEN `retmkcol6`(GEN `x`, GEN `y`, GEN `z`, GEN `t`, GEN `u`, GEN `v`) returns the 6-dimensional column vector `[x,y,z,t,u,v]` .

GEN `retconst_col`(long `n`, GEN `x`) returns the `n`-dimensional `t_COL` whose entries are constant and all equal to `x`.

GEN `retmkmat`(GEN `x`) returns the 1-column `t_MAT` with column `x`.

GEN `retmkmat2`(GEN `x`, GEN `y`) returns the 2-column `t_MAT` with columns `x`, `y`.

GEN `retmkmat3`(GEN `x`, GEN `y`, GEN `z`) returns the 3-dimensional `t_MAT` with columns `x`, `y`, `z`.

GEN `retmkmat4`(GEN `x`, GEN `y`, GEN `z`, GEN `t`) returns the 4-dimensional `t_MAT` with columns `x`, `y`, `z`, `t`.

GEN `retmkmat5`(GEN `x`, GEN `y`, GEN `z`, GEN `t`, GEN `u`) returns the 5-dimensional `t_MAT` with columns `x`, `y`, `z`, `t`, `u`.

GEN `retmkcomplex`(GEN `x`, GEN `y`) returns the `t_COMPLEX`  $x + I*y$ .

GEN `retmkfrac`(GEN `x`, GEN `y`) returns the `t_FRAC`  $x / y$ . Assume  $x$  and  $y$  are coprime and  $y > 1$ .

GEN `retmkfrac`(GEN `x`, GEN `y`) returns the `t_RFRAC`  $x / y$ . Assume  $x$  and  $y$  are coprime and more generally that the rational function cannot be simplified.

GEN `retmkintmod`(GEN `x`, GEN `y`) returns the `t_INTMOD`  $\text{Mod}(x, y)$ .

GEN `retmkqfi`(GEN `a`, GEN `b`, GEN `c`).

GEN `retmkqfr`(GEN `a`, GEN `b`, GEN `c`, GEN `d`).

GEN `retmkquad`(GEN `n`, GEN `a`, GEN `b`).

GEN `retmkpolmod`(GEN `x`, GEN `y`) returns the `t_POLMOD`  $\text{Mod}(x, y)$ .

GEN `mkintn`(long `n`, ...) returns the non-negative `t_INT` whose development in base  $2^{32}$  is given by the following  $n$  32bit-words (`unsigned int`).

```
mkintn(3, a2, a1, a0);
```

returns  $a_2 2^{64} + a_1 2^{32} + a_0$ .

GEN `mkpoln`(long `n`, ...) Returns the `t_POL` whose  $n$  coefficients (GEN) follow, in order of decreasing degree.

```
mkpoln(3, gen_1, gen_2, gen_0);
```

returns the polynomial  $X^2 + 2X$  (in variable 0, use `setvarn` if you want other variable numbers). Beware that  $n$  is the number of coefficients, hence *one more* than the degree.

GEN `mkvecn`(long `n`, ...) returns the `t_VEC` whose  $n$  coefficients (GEN) follow.

GEN `mkcoln`(long `n`, ...) returns the `t_COL` whose  $n$  coefficients (GEN) follow.

GEN `scalarcol_shallow`(GEN `x`, long `n`) creates a `t_COL` with  $n$  components set to `gen_0`, but the first one which is set to a shallow copy of `x`. (The name comes from `RgV_isscalar`.)

GEN `scalarmat_shallow`(GEN `x`, long `n`) creates an  $n \times n$  scalar matrix whose diagonal is set to shallow copies of the scalar `x`.

GEN `RgX_sylvestermatrix`(GEN `f`, GEN `g`) return the Sylvester matrix attached to the two `t_POL` in the same variable  $f$  and  $g$ .

GEN `diagonal_shallow`(GEN `x`) returns a diagonal matrix whose diagonal is given by the vector  $x$ . Shallow function.

GEN `scalarpol_shallow`(GEN `a`, long `v`) returns the degree 0 `t_POL`  $\text{apol}_x(v)^0$ .



GEN `deg1pol_shallow`(GEN `a`, GEN `b`, long `v`) returns the degree 1 `t_POL`  $a\text{pol\_x}(v) + b$

GEN `deg2pol_shallow`(GEN `a`, GEN `b`, GEN `c`, long `v`) returns the degree 2 `t_POL`  $ax^2 + bx + c$  where  $x = \text{pol\_x}(v)$ .

GEN `zeropadic_shallow`(GEN `p`, long `n`) returns a (shallow) 0 `t_PADIC` equal to  $O(p^n)$ .

### 9.3.3 From roots to polynomials.

GEN `deg1_from_roots`(GEN `L`, long `v`) given a vector  $L$  of scalars, returns the vector of monic linear polynomials in variable  $v$  whose roots are the  $L[i]$ , i.e. the  $x - L[i]$ .

GEN `roots_from_deg1`(GEN `L`) given a vector  $L$  of monic linear polynomials, return their roots, i.e. the  $-L[i](0)$ .

GEN `roots_to_pol`(GEN `L`, long `v`) given a vector of scalars  $L$ , returns the monic polynomial in variable  $v$  whose roots are the  $L[i]$ . Leaves some garbage on stack, but suitable for `gerepileupto`.

GEN `roots_to_pol_r1`(GEN `L`, long `v`, long `r1`) as `roots_to_pol` assuming the first  $r_1$  roots are “real”, and the following ones are representatives of conjugate pairs of “complex” roots. So if  $L$  has  $r_1 + r_2$  elements, we obtain a polynomial of degree  $r_1 + 2r_2$ . In most applications, the roots are indeed real and complex, but the implementation assumes only that each “complex” root  $z$  introduces a quadratic factor  $X^2 - \text{trace}(z)X + \text{norm}(z)$ . Leaves some garbage on stack, but suitable for `gerepileupto`.

## 9.4 Integer parts.

GEN `gfloor`(GEN `x`) creates the floor of  $x$ , i.e. the (true) integral part.

GEN `gfrac`(GEN `x`) creates the fractional part of  $x$ , i.e.  $x$  minus the floor of  $x$ .

GEN `gceil`(GEN `x`) creates the ceiling of  $x$ .

GEN `ground`(GEN `x`) rounds towards  $+\infty$  the components of  $x$  to the nearest integers.

GEN `grndtoi`(GEN `x`, long `*e`) same as `ground`, but in addition sets `*e` to the binary exponent of  $x - \text{ground}(x)$ . If this is positive, all significant bits are lost. This kind of situation raises an error message in `ground` but not in `grndtoi`.

GEN `gtrunc`(GEN `x`) truncates  $x$ . This is the false integer part if  $x$  is a real number (i.e. the unique integer closest to  $x$  among those between 0 and  $x$ ). If  $x$  is a `t_SER`, it is truncated to a `t_POL`; if  $x$  is a `t_RFRAC`, this takes the polynomial part.

GEN `gtrunc2n`(GEN `x`, long `n`) creates the floor of  $2^n x$ , this is only implemented for `t_INT`, `t_REAL`, `t_FRAC` and `t_COMPLEX` of those.

GEN `gcvtoi`(GEN `x`, long `*e`) analogous to `grndtoi` for `t_REAL` inputs except that rounding is replaced by truncation. Also applies componentwise for vector or matrix inputs; otherwise, sets `*e` to `-HIGHEXPOBIT` (infinite real accuracy) and return `gtrunc(x)`.

## 9.5 Valuation and shift.

`GEN gshift[z](GEN x, long n[, GEN z])` yields the result of shifting (the components of)  $x$  left by  $n$  (if  $n$  is non-negative) or right by  $-n$  (if  $n$  is negative). Applies only to `t_INT` and vectors/matrices of such. For other types, it is simply multiplication by  $2^n$ .

`GEN gmul2n[z](GEN x, long n[, GEN z])` yields the product of  $x$  and  $2^n$ . This is different from `gshift` when  $n$  is negative and  $x$  is a `t_INT`: `gshift` truncates, while `gmul2n` creates a fraction if necessary.

`long gvaluation(GEN x, GEN p)` returns the greatest exponent  $e$  such that  $p^e$  divides  $x$ , when this makes sense.

`long gval(GEN x, long v)` returns the highest power of the variable number  $v$  dividing the `t_POL`  $x$ .

## 9.6 Comparison operators.

### 9.6.1 Generic.

`long gcmp(GEN x, GEN y)` comparison of  $x$  with  $y$ : returns 1 ( $x > y$ ), 0 ( $x = y$ ) or  $-1$  ( $x < y$ ). Two `t_STR` are compared using the standard lexicographic ordering; a `t_STR` cannot be compared to any non-string type. If neither  $x$  nor  $y$  is a `t_STR`, their allowed types are `t_INT`, `t_REAL`, `t_FRAC`, `t_QUAD` with positive discriminant (use the canonical embedding  $w \rightarrow \sqrt{D}/2$  or  $w \rightarrow (1 + \sqrt{D})/2$ ) or `t_INFINITY`. Use `cmp_universal` to compare arbitrary `GENs`.

`long lexcmp(GEN x, GEN y)` comparison of  $x$  with  $y$  for the lexicographic ordering; when comparing objects of different lengths whose components are all equal up to the smallest of their length, consider that the longest is largest. Consider scalars as 1-component vectors. Return `gcmp(x, y)` if both arguments are scalars.

`int gequalX(GEN x)` return 1 (true) if  $x$  is a variable (monomial of degree 1 with `t_INT` coefficients equal to 1 and 0), and 0 otherwise

`long gequal(GEN x, GEN y)` returns 1 (true) if  $x$  is equal to  $y$ , 0 otherwise. A priori, this makes sense only if  $x$  and  $y$  have the same type, in which case they are recursively compared componentwise. When the types are different, a `true` result means that  $x - y$  was successfully computed and that `gequal0` found it equal to 0. In particular

`gequal(cgetg(1, t_VEC), gen_0)`

is true, and the relation is not transitive. E.g. an empty `t_COL` and an empty `t_VEC` are not equal but are both equal to `gen_0`.

`long gidentical(GEN x, GEN y)` returns 1 (true) if  $x$  is identical to  $y$ , 0 otherwise. In particular, the types and length of  $x$  and  $y$  must be equal. This test is much stricter than `gequal`, in particular, `t_REAL` with different accuracies are tested different. This relation is transitive.

`GEN gmax(GEN x, GEN y)` returns a copy of the maximum of  $x$  and  $y$ , compared using `gcmp`.

`GEN gmin(GEN x, GEN y)` returns a copy of the minimum of  $x$  and  $y$ , compared using `gcmp`.

`GEN gmax_shallow(GEN x, GEN y)` shallow version of `gmax`.

`GEN gmin_shallow(GEN x, GEN y)` shallow version of `gmin`.

### 9.6.2 Comparison with a small integer.

`int isexactzero(GEN x)` returns 1 (true) if `x` is exactly equal to 0 (including `t_INTMOD`s like `Mod(0,2)`), and 0 (false) otherwise. This includes recursive objects, for instance vectors, whose components are 0.

`GEN gisexactzero(GEN x)` returns `NULL` unless `x` is exactly equal to 0 (as per `isexactzero`). When `x` is an exact zero return the attached scalar zero as a `t_INT` (`gen_0`), a `t_INTMOD` (`Mod(0,N)` for the largest possible `N`) or a `t_FFELT`.

`int isrationalzero(GEN x)` returns 1 (true) if `x` is equal to an integer 0 (excluding `t_INTMOD`s like `Mod(0,2)`), and 0 (false) otherwise. Contrary to `isintzero`, this includes recursive objects, for instance vectors, whose components are 0.

`int ismpzero(GEN x)` returns 1 (true) if `x` is a `t_INT` or a `t_REAL` equal to 0.

`int isintzero(GEN x)` returns 1 (true) if `x` is a `t_INT` equal to 0.

`int isint1(GEN x)` returns 1 (true) if `x` is a `t_INT` equal to 1.

`int isintm1(GEN x)` returns 1 (true) if `x` is a `t_INT` equal to  $-1$ .

`int equali1(GEN n)` Assuming that `x` is a `t_INT`, return 1 (true) if `x` is equal to 1, and return 0 (false) otherwise.

`int equalim1(GEN n)` Assuming that `x` is a `t_INT`, return 1 (true) if `x` is equal to  $-1$ , and return 0 (false) otherwise.

`int is_pm1(GEN x)`. Assuming that `x` is a *non-zero* `t_INT`, return 1 (true) if `x` is equal to  $-1$  or 1, and return 0 (false) otherwise.

`int gequal0(GEN x)` returns 1 (true) if `x` is equal to 0, 0 (false) otherwise.

`int gequal1(GEN x)` returns 1 (true) if `x` is equal to 1, 0 (false) otherwise.

`int gequalm1(GEN x)` returns 1 (true) if `x` is equal to  $-1$ , 0 (false) otherwise.

`long gcmpsg(long s, GEN x)`

`long gcmpgs(GEN x, long s)` comparison of `x` with the `long s`.

`GEN gmaxsg(long s, GEN x)`

`GEN gmaxgs(GEN x, long s)` returns the largest of `x` and the `long s` (converted to `GEN`)

`GEN gminsg(long s, GEN x)`

`GEN gmings(GEN x, long s)` returns the smallest of `x` and the `long s` (converted to `GEN`)

`long gequalsg(long s, GEN x)`

`long gequalgs(GEN x, long s)` returns 1 (true) if `x` is equal to the `long s`, 0 otherwise.

## 9.7 Miscellaneous Boolean functions.

`int isrationalzeroscalar(GEN x)` equivalent to, but faster than,

```
is_scalar_t(typ(x)) && isrationalzero(x)
```

`int isinexact(GEN x)` returns 1 (true) if  $x$  has an inexact component, and 0 (false) otherwise.

`int isinexactreal(GEN x)` return 1 if  $x$  has an inexact `t_REAL` component, and 0 otherwise.

`int isrealappr(GEN x, long e)` applies (recursively) to complex inputs; returns 1 if  $x$  is approximately real to the bit accuracy  $e$ , and 0 otherwise. This means that any `t_COMPLEX` component must have imaginary part  $t$  satisfying `gexpo(t) < e`.

`int isint(GEN x, GEN *n)` returns 0 (false) if  $x$  does not round to an integer. Otherwise, returns 1 (true) and set  $n$  to the rounded value.

`int issmall(GEN x, long *n)` returns 0 (false) if  $x$  does not round to a small integer (suitable for `itos`). Otherwise, returns 1 (true) and set  $n$  to the rounded value.

`long iscomplex(GEN x)` returns 1 (true) if  $x$  is a complex number (of component types embeddable into the reals) but is not itself real, 0 if  $x$  is a real (not necessarily of type `t_REAL`), or raises an error if  $x$  is not embeddable into the complex numbers.

### 9.7.1 Obsolete.

The following less convenient comparison functions and Boolean operators were used by the historical GP interpreter. They are provided for backward compatibility only and should not be used:

```
GEN gle(GEN x, GEN y)
```

```
GEN glt(GEN x, GEN y)
```

```
GEN gge(GEN x, GEN y)
```

```
GEN ggt(GEN x, GEN y)
```

```
GEN geq(GEN x, GEN y)
```

```
GEN gne(GEN x, GEN y)
```

```
GEN gor(GEN x, GEN y)
```

```
GEN gand(GEN x, GEN y)
```

```
GEN gnot(GEN x, GEN y)
```

## 9.8 Sorting.

### 9.8.1 Basic sort.

`GEN sort(GEN x)` sorts the vector `x` in ascending order using a mergesort algorithm, and `gcmp` as the underlying comparison routine (returns the sorted vector). This routine copies all components of `x`, use `gen_sort_inplace` for a more memory-efficient function.

`GEN lexsort(GEN x)`, as `sort`, using `lexcmp` instead of `gcmp` as the underlying comparison routine.

`GEN vecsort(GEN x, GEN k)`, as `sort`, but sorts the vector `x` in ascending *lexicographic* order, according to the entries of the `t_VECSMALL` `k`. For example, if `k = [2, 1, 3]`, sorting will be done with respect to the second component, and when these are equal, with respect to the first, and when these are equal, with respect to the third.

### 9.8.2 Indirect sorting.

`GEN indexsort(GEN x)` as `sort`, but only returns the permutation which, applied to `x`, would sort the vector. The result is a `t_VECSMALL`.

`GEN indexlexsort(GEN x)`, as `indexsort`, using `lexcmp` instead of `gcmp` as the underlying comparison routine.

`GEN indexvecsort(GEN x, GEN k)`, as `vecsort`, but only returns the permutation that would sort the vector `x`.

`long vecindexmin(GEN x)` returns the index for a maximal element of `x` (`t_VEC`, `t_COL` or `t_VECSMALL`).

`long vecindexmax(GEN x)` returns the index for a maximal element of `x` (`t_VEC`, `t_COL` or `t_VECSMALL`).

`long vecindexmax(GEN x)`

**9.8.3 Generic sort and search.** The following routines allow to use an arbitrary comparison function `int (*cmp)(void* data, GEN x, GEN y)`, such that `cmp(data,x,y)` returns a negative result if  $x < y$ , a positive one if  $x > y$  and 0 if  $x = y$ . The `data` argument is there in case your `cmp` requires additional context.

`GEN gen_sort(GEN x, void *data, int (*cmp)(void *, GEN, GEN))`, as `sort`, with an explicit comparison routine.

`GEN gen_sort_uniq(GEN x, void *data, int (*cmp)(void *, GEN, GEN))`, as `gen_sort`, removing duplicate entries.

`GEN gen_indexsort(GEN x, void *data, int (*cmp)(void*, GEN, GEN))`, as `indexsort`.

`GEN gen_indexsort_uniq(GEN x, void *data, int (*cmp)(void*, GEN, GEN))`, as `indexsort`, removing duplicate entries.

`void gen_sort_inplace(GEN x, void *data, int (*cmp)(void*, GEN, GEN), GEN *perm)` sort `x` in place, without copying its components. If `perm` is non-NULL, it is set to the permutation that would sort the original `x`.

`GEN gen_setminus(GEN A, GEN B, int (*cmp)(GEN, GEN))` given two sorted vectors `A` and `B`, returns the vector of elements of `A` not belonging to `B`.

`GEN sort_factor(GEN y, void *data, int (*cmp)(void *, GEN, GEN))`: assuming `y` is a factorization matrix, sorts its rows in place (no copy is made) according to the comparison function `cmp` applied to its first column.

`GEN merge_sort_uniq(GEN x, GEN y, void *data, int (*cmp)(void *, GEN, GEN))` assuming `x` and `y` are sorted vectors, with respect to the `cmp` comparison function, return a sorted concatenation, with duplicates removed.

`GEN merge_factor(GEN fx, GEN fy, void *data, int (*cmp)(void *, GEN, GEN))` let `fx` and `fy` be factorization matrices for  $X$  and  $Y$  sorted with respect to the comparison function `cmp` (see `sort_factor`), returns the factorization of  $X * Y$ .

`long gen_search(GEN v, GEN y, long flag, void *data, int (*cmp)(void*, GEN, GEN))`

. Let `v` be a vector sorted according to `cmp(data,a,b)`; look for an index  $i$  such that `v[i]` is equal to `y`. `flag` has the same meaning as in `setsearch`: if `flag` is 0, return  $i$  if it exists and 0 otherwise; if `flag` is non-zero, return 0 if  $i$  exists and the index where `y` should be inserted otherwise.

`long tablesearch(GEN T, GEN x, int (*cmp)(GEN, GEN))` is a faster implementation for the common case `gen_search(T,x,0,cmp,cmp_nodata)`.

### 9.8.4 Further useful comparison functions.

`int cmp_universal(GEN x, GEN y)` a somewhat arbitrary universal comparison function, devoid of sensible mathematical meaning. It is transitive, and returns 0 if and only if `gidentical(x,y)` is true. Useful to sort and search vectors of arbitrary data.

`int cmp_nodata(void *data, GEN x, GEN y)`. This function is a hack used to pass an existing basic comparison function lacking the `data` argument, i.e. with prototype `int (*cmp)(GEN x, GEN y)`. Instead of `gen_sort(x, NULL, cmp)` which may or may not work depending on how your compiler handles typecasts between incompatible function pointers, one should use `gen_sort(x, (void*)cmp, cmp_nodata)`.

Here are a few basic comparison functions, to be used with `cmp_nodata`:

`int ZV_cmp(GEN x, GEN y)` compare two ZV, which we assume have the same length (lexicographic order).

`int cmp_Flx(GEN x, GEN y)` compare two Flx, which we assume have the same main variable (lexicographic order).

`int cmp_RgX(GEN x, GEN y)` compare two polynomials, which we assume have the same main variable (lexicographic order). The coefficients are compared using `gcmp`.

`int cmp_prime_over_p(GEN x, GEN y)` compare two prime ideals, which we assume divide the same prime number. The comparison is ad hoc but orders according to increasing residue degrees.

`int cmp_prime_ideal(GEN x, GEN y)` compare two prime ideals in the same  $nf$ . Orders by increasing primes, breaking ties using `cmp_prime_over_p`.

`int cmp_padic(GEN x, GEN y)` compare two `t_PADIC` (for the same prime  $p$ ).

Finally a more elaborate comparison function:

`int gen_cmp_RgX(void *data, GEN x, GEN y)` compare two polynomials, ordering first by increasing degree, then according to the coefficient comparison function:

```
int (*cmp_coeff)(GEN,GEN) = (int (*)(GEN,GEN)) data;
```

## 9.9 Divisibility, Euclidean division.

GEN **gdivexact**(GEN *x*, GEN *y*) returns the quotient  $x/y$ , assuming *y* divides *x*. Not stack clean if  $y = 1$  (we return *x*, not a copy).

int **gdvd**(GEN *x*, GEN *y*) returns 1 (true) if *y* divides *x*, 0 otherwise.

GEN **gdiventres**(GEN *x*, GEN *y*) creates a 2-component vertical vector whose components are the true Euclidean quotient and remainder of *x* and *y*.

GEN **gdivent**[*z*](GEN *x*, GEN *y*[, GEN *z*]) yields the true Euclidean quotient of *x* and the *t\_INT* or *t\_POL* *y*, as per the `\ GP` operator.

GEN **gdiventsg**(long *s*, GEN *y*[, GEN *z*]), as **gdivent** except that *x* is a long.

GEN **gdiventgs**[*z*](GEN *x*, long *s*[, GEN *z*]), as **gdivent** except that *y* is a long.

GEN **gmod**[*z*](GEN *x*, GEN *y*[, GEN *z*]) yields the remainder of *x* modulo the *t\_INT* or *t\_POL* *y*, as per the `% GP` operator. A *t\_REAL* or *t\_FRAC* *y* is also allowed, in which case the remainder is the unique real  $r$  such that  $0 \leq r < |y|$  and  $y = qx + r$  for some (in fact unique) integer  $q$ .

GEN **gmodsg**(long *s*, GEN *y*[, GEN *z*]) as **gmod**, except *x* is a long.

GEN **gmodgs**(GEN *x*, long *s*[, GEN *z*]) as **gmod**, except *y* is a long.

GEN **gdivmod**(GEN *x*, GEN *y*, GEN *\*r*) If *r* is not equal to NULL or ONLY\_REM, creates the (false) Euclidean quotient of *x* and *y*, and puts (the address of) the remainder into *\*r*. If *r* is equal to NULL, do not create the remainder, and if *r* is equal to ONLY\_REM, create and output only the remainder. The remainder is created after the quotient and can be disposed of individually with a **cgiv**(*r*).

GEN **poldivrem**(GEN *x*, GEN *y*, GEN *\*r*) same as **gdivmod** but specifically for *t\_POL*s *x* and *y*, not necessarily in the same variable. Either of *x* and *y* may also be scalars, treated as polynomials of degree 0.

GEN **gdeuc**(GEN *x*, GEN *y*) creates the Euclidean quotient of the *t\_POL*s *x* and *y*. Either of *x* and *y* may also be scalars, treated as polynomials of degree 0.

GEN **grem**(GEN *x*, GEN *y*) creates the Euclidean remainder of the *t\_POL* *x* divided by the *t\_POL* *y*. Either of *x* and *y* may also be scalars, treated as polynomials of degree 0.

GEN **gdivround**(GEN *x*, GEN *y*) if *x* and *y* are real (*t\_INT*, *t\_REAL*, *t\_FRAC*), return the rounded Euclidean quotient of *x* and *y* as per the `\ GP` operator. Operate componentwise if *x* is a *t\_COL*, *t\_VEC* or *t\_MAT*. Otherwise as **gdivent**.

GEN **centermod\_i**(GEN *x*, GEN *y*, GEN *y2*), as **centermodii**, componentwise.

GEN **centermod**(GEN *x*, GEN *y*), as **centermod\_i**, except that *y2* is computed (and left on the stack for efficiency).

GEN **ginvmod**(GEN *x*, GEN *y*) creates the inverse of *x* modulo *y* when it exists. *y* must be of type *t\_INT* (in which case *x* is of type *t\_INT*) or *t\_POL* (in which case *x* is either a scalar type or a *t\_POL*).

## 9.10 GCD, content and primitive part.

### 9.10.1 Generic.

`GEN resultant(GEN x, GEN y)` creates the resultant of the `t_POLs` `x` and `y` computed using Sylvester's matrix (inexact inputs), a modular algorithm (inputs in  $\mathbf{Q}[X]$ ) or the subresultant algorithm, as optimized by Lazard and Ducos. Either of `x` and `y` may also be scalars (treated as polynomials of degree 0)

`GEN ggcd(GEN x, GEN y)` creates the GCD of `x` and `y`.

`GEN glcm(GEN x, GEN y)` creates the LCM of `x` and `y`.

`GEN gbezout(GEN x, GEN y, GEN *u, GEN *v)` returns the GCD of `x` and `y`, and puts (the addresses of) objects `u` and `v` such that  $ux + vy = \text{gcd}(x, y)$  into `*u` and `*v`.

`GEN subresex(GEN x, GEN y, GEN *U, GEN *V)` returns the resultant of `x` and `y`, and puts (the addresses of) polynomials `u` and `v` such that  $ux + vy = \text{Res}(x, y)$  into `*U` and `*V`.

`GEN content(GEN x)` returns the GCD of all the components of `x`.

`GEN primitive_part(GEN x, GEN *c)` sets `c` to `content(x)` and returns the primitive part  $x / c$ . A trivial content is set to `NULL`.

`GEN primpart(GEN x)` as above but the content is lost. (For efficiency, the content remains on the stack.)

`GEN denom_i(GEN x)` shallow version of `denom`.

`GEN numer_i(GEN x)` shallow version of `numer`.

### 9.10.2 Over the rationals.

`long Q_pval(GEN x, GEN p)` valuation at the `t_INT` `p` of the `t_INT` or `t_FRAC` `x`.

`long Q_pvalrem(GEN x, GEN p, GEN *r)` returns the valuation  $e$  at the `t_INT` `p` of the `t_INT` or `t_FRAC` `x`. The quotient  $x/p^e$  is returned in `*r`.

`GEN Q_abs(GEN x)` absolute value of the `t_INT` or `t_FRAC` `x`.

`GEN Qdivii(GEN x, GEN y)`, assuming  $x$  and  $y$  are both of type `t_INT`, return the quotient  $x/y$  as a `t_INT` or `t_FRAC`; marginally faster than `gdiv`.

`GEN Q_abs_shallow(GEN x)`  $x$  being a `t_INT` or a `t_FRAC`, returns a shallow copy of  $|x|$ , in particular returns  $x$  itself when  $x \geq 0$ , and `gneg(x)` otherwise.

`GEN Q_gcd(GEN x, GEN y)` gcd of the `t_INT` or `t_FRAC` `x` and `y`.

In the following functions, arguments belong to a  $M \otimes_{\mathbf{Z}} \mathbf{Q}$  for some natural  $\mathbf{Z}$ -module  $M$ , e.g. multivariate polynomials with integer coefficients (or vectors/matrices recursively built from such objects), and an element of  $M$  is said to be *integral*. We are interested in contents, denominators, etc. with respect to this canonical integral structure; in particular, contents belong to  $\mathbf{Q}$ , denominators to  $\mathbf{Z}$ . For instance the  $\mathbf{Q}$ -content of  $(1/2)xy$  is  $(1/2)$ , and its  $\mathbf{Q}$ -denominator is 2, whereas `content` would return  $y/2$  and `denom` 1.

`GEN Q_content(GEN x)` the  $\mathbf{Q}$ -content of  $x$ .

`GEN Z_content(GEN x)` as `Q_content` but assume that all rationals are in fact `t_INTs` and return `NULL` when the content is 1. This function returns as soon as the content is found to equal 1.



GEN `Q_content_safe(GEN x)` as `Q_content`, returning NULL when the **Q**-content is not defined (e.g. for a `t_REAL` or `t_INTMOD` component).

GEN `Q_denom(GEN x)` the **Q**-denominator of  $x$ . Shallow function. Raises an `e_TYPE` error out when the notion is meaningless, e.g. for a `t_REAL` or `t_INTMOD` component.

GEN `Q_denom_safe(GEN x)` the **Q**-denominator of  $x$ . Shallow function. Return NULL when the notion is meaningless.

GEN `Q_primitive_part(GEN x, GEN *c)` sets  $c$  to the **Q**-content of  $x$  and returns  $x / c$ , which is integral.

GEN `Q_primpart(GEN x)` as above but the content is lost. (For efficiency, the content remains on the stack.)

GEN `vec_Q_primpart(GEN x)` as above component-wise.

GEN `Q_remove_denom(GEN x, GEN *ptd)` sets  $d$  to the **Q**-denominator of  $x$  and returns  $x * d$ , which is integral. Shallow function.

GEN `Q_div_to_int(GEN x, GEN c)` returns  $x / c$ , assuming  $c$  is a rational number (`t_INT` or `t_FRAC`) and the result is integral.

GEN `Q_mul_to_int(GEN x, GEN c)` returns  $x * c$ , assuming  $c$  is a rational number (`t_INT` or `t_FRAC`) and the result is integral.

GEN `Q_muli_to_int(GEN x, GEN d)` returns  $x * c$ , assuming  $c$  is a `t_INT` and the result is integral.

GEN `mul_content(GEN cx, GEN cy)`  $cx$  and  $cy$  are as set by `primitive_part`: either a `GEN` or NULL representing the trivial content 1. Returns their product (either a `GEN` or NULL).

GEN `inv_content(GEN c)`  $c$  is as set by `primitive_part`: either a `GEN` or NULL representing the trivial content 1. Returns its inverse (either a `GEN` or NULL).

GEN `mul_denom(GEN dx, GEN dy)`  $dx$  and  $dy$  are as set by `Q_remove_denom`: either a `t_INT` or NULL representing the trivial denominator 1. Returns their product (either a `t_INT` or NULL).

## 9.11 Generic arithmetic operators.

### 9.11.1 Unary operators.

GEN `gneg[z](GEN x[, GEN z])` yields  $-x$ .

GEN `gneg_i(GEN x)` shallow function yielding  $-x$ .

GEN `gabs[z](GEN x[, GEN z])` yields  $|x|$ .

GEN `gsqr(GEN x)` creates the square of  $x$ .

GEN `ginv(GEN x)` creates the inverse of  $x$ .

### 9.11.2 Binary operators.

Let “*op*” be a binary operation among

*op*=**add**: addition ( $x + y$ ).

*op*=**sub**: subtraction ( $x - y$ ).

*op*=**mul**: multiplication ( $x * y$ ).

*op*=**div**: division ( $x / y$ ).

The names and prototypes of the functions corresponding to *op* are as follows:

GEN *gop*(GEN *x*, GEN *y*)

GEN *gopgs*(GEN *x*, long *s*)

GEN *gopsg*(long *s*, GEN *y*)

Explicitly

GEN *gadd*(GEN *x*, GEN *y*), GEN *gaddgs*(GEN *x*, long *s*), GEN *gaddsg*(long *s*, GEN *x*)

GEN *gmul*(GEN *x*, GEN *y*), GEN *gmulgs*(GEN *x*, long *s*), GEN *gmulsg*(long *s*, GEN *x*)

GEN *gsub*(GEN *x*, GEN *y*), GEN *gsubgs*(GEN *x*, long *s*), GEN *gsubsg*(long *s*, GEN *x*)

GEN *gdiv*(GEN *x*, GEN *y*), GEN *gdivgs*(GEN *x*, long *s*), GEN *gdivsg*(long *s*, GEN *x*)

GEN *gpow*(GEN *x*, GEN *y*, long *l*) creates  $x^y$ . If *y* is a *t\_INT*, return *powgi*(*x*,*y*) (the precision *l* is not taken into account). Otherwise, the result is  $\exp(y * \log(x))$  where exact arguments are converted to floats of precision *l* in case of need; if there is no need, for instance if *x* is a *t\_REAL*, *l* is ignored. Indeed, if *x* is a *t\_REAL*, the accuracy of  $\log x$  is determined from the accuracy of *x*, it is no problem to multiply by *y*, even if it is an exact type, and the accuracy of the exponential is determined, exactly as in the case of the initial  $\log x$ .

GEN *gpowgs*(GEN *x*, long *n*) creates  $x^n$  using binary powering. To treat the special case  $n = 0$ , we consider *gpowgs* as a series of *gmul*, so we follow the rule of returning result which is as exact as possible given the input. More precisely, we return

- *gen\_1* if *x* has type *t\_INT*, *t\_REAL*, *t\_FRAC*, or *t\_PADIC*
- *Mod*(1,*N*) if *x* is a *t\_INTMOD* modulo *N*.
- *gen\_1* for *t\_COMPLEX*, *t\_QUAD* unless one component is a *t\_INTMOD*, in which case we return *Mod*(1, *N*) for a suitable *N* (the gcd of the moduli that appear).
- *FF\_1*(*x*) for a *t\_FFELT*.
- *qfi\_1*(*x*) and *qfr\_1*(*x*) for *t\_QFI* and *t\_QFR*.
- the identity permutation for *t\_VECSMALL*.
- *Rg\_get\_1*(*x*) otherwise

Of course, the only practical use of this routine for  $n = 0$  is to obtain the multiplicative neutral element in the base ring (or to treat marginal cases that should be special cased anyway if there is the slightest doubt about what the result should be).

GEN *powgi*(GEN *x*, GEN *y*) creates  $x^y$ , where *y* is a *t\_INT*, using left-shift binary powering. The case where  $y = 0$  (as all cases where *y* is small) is handled by *gpowgs*(*x*, 0).

GEN `gpowers`(GEN `x`, long `n`) returns the vector  $[1, x, \dots, x^n]$ .

GEN `grootsof1`(long `n`, long `prec`) returns the vector  $[1, x, \dots, x^{n-1}]$ , where  $x$  is the  $n$ -th root of unity  $\exp(2i\pi/n)$ .

GEN `gsqrpowers`(GEN `x`, long `n`) returns the vector  $[x, x^4, \dots, x^{n^2}]$ .

In addition we also have the obsolete forms:

void `gaddz`(GEN `x`, GEN `y`, GEN `z`)

void `gsubz`(GEN `x`, GEN `y`, GEN `z`)

void `gmulz`(GEN `x`, GEN `y`, GEN `z`)

void `gdivz`(GEN `x`, GEN `y`, GEN `z`)

## 9.12 Generic operators: product, powering, factorback.

To describe the following functions, we use the following private typedefs to simplify the description:

```
typedef (*F0)(void *);
typedef (*F1)(void *, GEN);
typedef (*F2)(void *, GEN, GEN);
```

They correspond to generic functions with one and two arguments respectively (the `void*` argument provides some arbitrary evaluation context).

GEN `gen_product`(GEN `v`, void `*D`, F2 `op`) Given two objects  $x, y$ , assume that `op`(`D`,  $x$ ,  $y$ ) implements an associative binary operator. If  $v$  has  $k$  entries, return

$$v[1] \text{ op } v[2] \text{ op } \dots \text{ op } v[k];$$

returns `gen_1` if  $k = 0$  and a copy of  $v[1]$  if  $k = 1$ . Use divide and conquer strategy. Leave some garbage on stack, but suitable for `gerepileupto` if `mul` is.

GEN `gen_pow`(GEN `x`, GEN `n`, void `*D`, F1 `sqr`, F2 `mul`)  $n > 0$  a `t_INT`, returns  $x^n$ ; `mul`(`D`,  $x$ ,  $y$ ) implements the multiplication in the underlying monoid; `sqr` is a (presumably optimized) shortcut for `mul`(`D`,  $x$ ,  $x$ ).

GEN `gen_powu`(GEN `x`, ulong `n`, void `*D`, F1 `sqr`, F2 `mul`)  $n > 0$ , returns  $x^n$ . See `gen_pow`.

GEN `gen_pow_i`(GEN `x`, GEN `n`, void `*E`, F1 `sqr`, F2 `mul`) internal variant of `gen_pow`, not memory-clean.

GEN `gen_powu_i`(GEN `x`, ulong `n`, void `*E`, F1 `sqr`, F2 `mul`) internal variant of `gen_powu`, not memory-clean.

GEN `gen_pow_fold`(GEN `x`, GEN `n`, void `*D`, F1 `sqr`, F1 `msqr`) variant of `gen_pow`, where `mul` is replaced by `msqr`, with `msqr`(`D`,  $y$ ) returning  $xy^2$ . In particular `D` must implicitly contain  $x$ .

GEN `gen_pow_fold_i`(GEN `x`, GEN `n`, void `*E`, F1 `sqr`, F1 `msqr`) internal variant of the function `gen_pow_fold`, not memory-clean.

GEN `gen_powu_fold`(GEN `x`, ulong `n`, void `*D`, F1 `sqr`, F1 `msqr`), see `gen_pow_fold`.

GEN `gen_powu_fold_i`(GEN `x`, ulong `n`, void `*E`, F1 `sqr`, F1 `msqr`) see `gen_pow_fold_i`.

`GEN gen_pow_init(GEN x, GEN n, long k, void *E, GEN (*sqr)(void*, GEN), GEN (*mul)(void*, GEN, GEN))` Return a table  $R$  that can be used with `gen_pow_table` to compute the powers of  $x$  up to  $n$ . The table is of size  $2^k \log_2(n)$ .

`GEN gen_pow_table(GEN R, GEN n, void *E, GEN (*one)(void*), GEN (*mul)(void*, GEN, GEN))`

Return  $x^n$ , where  $R$  is as given by `gen_pow_init(x,m,k,E,sqr,mul)` for some integer  $m \geq n$ .

`GEN gen_powers(GEN x, long n, long usesqr, void *D, F1 sqr, F2 mul, F0 one)` returns  $[x^0, \dots, x^n]$  as a `t_VEC`; `mul(D, x, y)` implements the multiplication in the underlying monoid; `sqr` is a (presumably optimized) shortcut for `mul(D, x, x)`; `one` returns the monoid unit. The flag `usesqr` should be set to 1 if squaring are faster than multiplication by  $x$ .

`GEN gen_factorback(GEN L, GEN e, F2 mul, F2 pow, void *D)` generic form of `factorback`. The pair  $[L, e]$  is of the form

- `[fa, NULL]`,  $fa$  a two-column factorization matrix: expand it.
- `[v, NULL]`,  $v$  a vector of objects: return their product.
- or `[v, e]`,  $v$  a vector of objects,  $e$  a vector of integral exponents: return the product of the  $v[i]^{e[i]}$ .

`mul(D, x, y)` and `pow(D, x, n)` return  $xy$  and  $x^n$  respectively.

### 9.13 Matrix and polynomial norms.

This section concerns only standard norms of  $\mathbf{R}$  and  $\mathbf{C}$  vector spaces, not algebraic norms given by the determinant of some multiplication operator. We have already seen type-specific functions like `ZM_supnorm` or `RgM_fpnorml2` and limit ourselves to generic functions assuming nothing about their `GEN` argument; these functions allow the following scalar types: `t_INT`, `t_FRAC`, `t_REAL`, `t_COMPLEX`, `t_QUAD` and are defined recursively (in terms of norms of their components) for the following “container” types: `t_POL`, `t_VEC`, `t_COL` and `t_MAT`. They raise an error if some other type appears in the argument.

`GEN gnorml2(GEN x)` The norm of a scalar is the square of its complex modulus, the norm of a recursive type is the sum of the norms of its components. For polynomials, vectors or matrices of complex numbers one recovers the *square* of the usual  $L^2$  norm. In most applications, the missing square root computation can be skipped.

`GEN gnorml1(GEN x, long prec)` The norm of a scalar is its complex modulus, the norm of a recursive type is the sum of the norms of its components. For polynomials, vectors or matrices of complex numbers one recovers the usual  $L^1$  norm. One must include a real precision `prec` in case the inputs include `t_COMPLEX` or `t_QUAD` with exact rational components: a square root must be computed and we must choose an accuracy.

`GEN gnorml1_fake(GEN x)` as `gnorml1`, except that the norm of a `t_QUAD`  $x + wy$  or `t_COMPLEX`  $x + Iy$  is defined as  $|x| + |y|$ , where we use the ordinary real absolute value. This is still a norm of  $\mathbf{R}$  vector spaces, which is easier to compute than `gnorml1` and can often be used in its place.

`GEN gsupnorm(GEN x, long prec)` The norm of a scalar is its complex modulus, the norm of a recursive type is the max of the norms of its components. A precision `prec` must be included for the same reason as in `gnorml1`.

`void gsupnorm_aux(GEN x, GEN *m, GEN *m2, long prec)` is the low-level function underlying `gsupnorm`, used as follows:

```
GEN m = NULL, m2 = NULL;
gsupnorm_aux(x, &m, &m2);
```

After the call, the sup norm of  $x$  is the min of  $m$  and the square root of  $m2$ ; one or both of  $m$ ,  $m2$  may be NULL, in which case it must be omitted. You may initially set  $m$  and  $m2$  to non-NULL values, in which case, the above procedure yields the max of (the initial)  $m$ , the square root of (the initial)  $m2$ , and the sup norm of  $x$ .

The strange interface is due to the fact that  $|z|^2$  is easier to compute than  $|z|$  for a `t_QUAD` or `t_COMPLEX`  $z$ :  $m2$  is the max of those  $|z|^2$ , and  $m$  is the max of the other  $|z|$ .

## 9.14 Substitution and evaluation.

`GEN gsubst(GEN x, long v, GEN y)` substitutes the object  $y$  into  $x$  for the variable number  $v$ .

`GEN poleval(GEN q, GEN x)` evaluates the `t_POL` or `t_RFRAC`  $q$  at  $x$ . For convenience, a `t_VEC` or `t_COL` is also recognized as the `t_POL` `gtovectrev(q)`.

`GEN RgX_cxeval(GEN T, GEN x, GEN xi)` evaluate the `t_POL`  $T$  at  $x$  via Horner's scheme. If  $xi$  is not NULL it must be equal to  $1/x$  and we evaluate  $x^{\deg T}T(1/x)$  instead. This is useful when  $|x| > 1$  is a `t_REAL` or an inexact `t_COMPLEX` and  $T$  has "balanced" coefficients, since the evaluation becomes numerically stable.

`GEN RgX_RgM_eval(GEN q, GEN x)` evaluates the `t_POL`  $q$  at the square matrix  $x$ .

`GEN RgX_RgMV_eval(GEN f, GEN V)` returns the evaluation  $f(x)$ , assuming that  $V$  was computed by `FpXQ_powers(x, n)` for some  $n > 1$ .

`GEN qfeval(GEN q, GEN x)` evaluates the quadratic form  $q$  (symmetric matrix) at  $x$  (column vector of compatible dimensions).

`GEN qfevalb(GEN q, GEN x, GEN y)` evaluates the polar bilinear form attached to the quadratic form  $q$  (symmetric matrix) at  $x, y$  (column vectors of compatible dimensions).

`GEN hqfeval(GEN q, GEN x)` evaluates the Hermitian form  $q$  (a Hermitian complex matrix) at  $x$ .

`GEN qf_apply_RgM(GEN q, GEN M)`  $q$  is a symmetric  $n \times n$  matrix,  $M$  an  $n \times k$  matrix, return  $M'qM$ .

`GEN qf_apply_ZM(GEN q, GEN M)` as above assuming that both  $q$  and  $M$  have integer entries.



## Chapter 10: Miscellaneous mathematical functions

### 10.1 Fractions.

GEN `absfrac`(GEN `x`) returns the absolute value of the `t_FRAC` `x`.

GEN `absfrac_shallow`(GEN `x`) `x` being a `t_FRAC`, returns a shallow copy of  $|x|$ , in particular returns `x` itself when  $x \geq 0$ , and `gneg(x)` otherwise.

GEN `sqrfrac`(GEN `x`) returns the square of the `t_FRAC` `x`.

### 10.2 Binomials.

GEN `binomial`(GEN `x`, long `k`)

GEN `binomialuu`(ulong `n`, ulong `k`)

GEN `vecbinomial`(long `n`), which returns a vector `v` with  $n + 1$  `t_INT` components such that  $v[k + 1] = \text{binomial}(n, k)$  for  $k$  from 0 up to  $n$ .

### 10.3 Real numbers.

GEN `R_abs`(GEN `x`) `x` being a `t_INT`, a `t_REAL` or a `t_FRAC`, returns  $|x|$ .

GEN `R_abs_shallow`(GEN `x`) `x` being a `t_INT`, a `t_REAL` or a `t_FRAC`, returns a shallow copy of  $|x|$ , in particular returns `x` itself when  $x \geq 0$ , and `gneg(x)` otherwise.

GEN `modRr_safe`(GEN `x`, GEN `y`) let `x` be a `t_INT`, a `t_REAL` or `t_FRAC` and let `y` be a `t_REAL`. Return  $x\%y$  unless the input accuracy is insufficient to compute the floor or  $x/y$  in which case we return `NULL`.

## 10.4 Complex numbers.

GEN `gimag`(GEN `x`) returns a copy of the imaginary part of `x`.

GEN `greal`(GEN `x`) returns a copy of the real part of `x`. If `x` is a `t_QUAD`, returns the coefficient of 1 in the “canonical” integral basis  $(1, \omega)$ .

GEN `gconj`(GEN `x`) returns  $\text{greal}(x) - 2\text{gimag}(x)$ , which is the ordinary complex conjugate except for a real `t_QUAD`.

GEN `imag_i`(GEN `x`), shallow variant of `gimag`.

GEN `real_i`(GEN `x`), shallow variant of `greal`.

GEN `conj_i`(GEN `x`), shallow variant of `gconj`.

GEN `mulreal`(GEN `x`, GEN) returns the real part of  $xy$ ;  $x, y$  have type `t_INT`, `t_FRAC`, `t_REAL` or `t_COMPLEX`. See also `RgM_mulreal`.

GEN `cxnorm`(GEN `x`) norm of the `t_COMPLEX`  $x$  (modulus squared).

GEN `cxexpm1`(GEN `x`) returns  $\exp(x) - 1$ , for a `t_COMPLEX`  $x$ .

int `cx_approx_equal`(GEN `a`, GEN `b`) test whether (`t_INT`, `t_FRAC`, `t_REAL`, or `t_COMPLEX` of those)  $a$  and  $b$  are approximately equal. This returns 1 if and only if the division by  $a - b$  would produce a division by 0 (which is a less stringent test than testing whether  $a - b$  evaluates to 0).

## 10.5 Quadratic numbers and binary quadratic forms.

GEN `quad_disc`(GEN `x`) returns the discriminant of the `t_QUAD`  $x$ . Not stack-clean but suitable for `gerepileupto`.

GEN `quadnorm`(GEN `x`) norm of the `t_QUAD`  $x$ .

GEN `qfb_disc`(GEN `x`) returns the discriminant of the `t_QFI` or `t_QFR`  $x$ .

GEN `qfb_disc3`(GEN `x`, GEN `y`, GEN `z`) returns  $y^2 - 4xz$  assuming all inputs are `t_INTs`. Not stack-clean.

GEN `qfb_apply_ZM`(GEN `q`, GEN `g`) returns  $q \circ g$ .

GEN `qfbforms`(GEN `D`) given a discriminant  $D < 0$ , return the list of reduced forms of discriminant  $D$  as `t_VECSMALL` with 3 components. The primitive forms in the list enumerate the class group of the quadratic order of discriminant  $D$ ; if  $D$  is fundamental, all returned forms are automatically primitive.



## 10.6 Polynomials.

**GEN truecoef**(GEN *x*, long *n*) returns **polcoef**(*x*,*n*, -1), i.e. the coefficient of the term of degree *n* in the main variable. This is a safe but expensive function that must *copy* its return value so that it be *gerepile*-safe. Use **polcoef\_i** for a fast internal variant.

**GEN polcoef\_i**(GEN *x*, long *n*, long *v*) internal shallow function. Rewrite *x* as a Laurent polynomial in the variable *v* and returns its coefficient of degree *n* (**gen\_0** if this falls outside the coefficient array). Allow **t\_POL**, **t\_SER**, **t\_RFRAC** and scalars.

**long degree**(GEN *x*) returns **poldegree**(*x*, -1), the degree of *x* with respect to its main variable, with the usual meaning if the leading coefficient of *x* is non-zero. If the sign of *x* is 0, this function always returns  $-1$ . Otherwise, we return the index of the leading coefficient of *x*, i.e. the coefficient of largest index stored in *x*. For instance the “degrees” of

```
0. E-38 * x^4 + 0.E-19 * x + 1
Mod(0,2) * x^0    \\ sign is 0 !
```

are 4 and  $-1$  respectively.

**long degpol**(GEN *x*) is a simple macro returning **lg**(*x*) - 3. This is the degree of the **t\_POL** *x* with respect to its main variable, *if* its leading coefficient is non-zero (a rational 0 is impossible, but an inexact 0 is allowed, as well as an exact modular 0, e.g. **Mod**(0,2)). If *x* has no coefficients (rational 0 polynomial), its length is 2 and we return the expected  $-1$ .

**GEN characteristic**(GEN *x*) returns the characteristic of the base ring over which the polynomial is defined (as defined by **t\_INTMOD** and **t\_FFELT** components). The function raises an exception if incompatible primes arise from **t\_FFELT** and **t\_PADIC** components. Shallow function.

**GEN residual\_characteristic**(GEN *x*) returns a kind of “residual characteristic” of the base ring over which the polynomial is defined. This is defined as the gcd of all moduli **t\_INTMOD**s occurring in the structure, as well as primes *p* arising from **t\_PADIC**s or **t\_FFELT**s. The function raises an exception if incompatible primes arise from **t\_FFELT** and **t\_PADIC** components. Shallow function.

**GEN resultant**(GEN *x*, GEN *y*) resultant of *x* and *y*, with respect to the main variable of highest priority. Uses either the subresultant algorithm (generic case), a modular algorithm (inputs in **Q[X]**) or Sylvester’s matrix (inexact inputs).

**GEN resultant2**(GEN *x*, GEN *y*) resultant of *x* and *y*, with respect to the main variable of highest priority. Computes the determinant of Sylvester’s matrix.

**GEN cleanroots**(GEN *x*, long *prec*) returns the complex roots of the complex polynomial *x* (with coefficients **t\_INT**, **t\_FRAC**, **t\_REAL** or **t\_COMPLEX** of the above). The roots are returned as **t\_REAL** or **t\_COMPLEX** of **t\_REALS** of precision *prec* (guaranteeing a non-0 imaginary part). See **QX\_complex\_roots**.

**double fujiwara\_bound**(GEN *x*) return a quick upper bound for the logarithm in base 2 of the modulus of the largest complex roots of the polynomial *x* (complex coefficients).

**double fujiwara\_bound\_real**(GEN *x*, long *sign*) return a quick upper bound for the logarithm in base 2 of the absolute value of the largest real root of sign *sign* (1 or  $-1$ ), for the polynomial *x* (real coefficients).

**GEN polmod\_to\_embed**(GEN *x*, long *prec*) return the vector of complex embeddings of the **t\_POLMOD** *x* (with complex coefficients). Shallow function, simple complex variant of **conjvec**.

## 10.7 Power series.

**GEN** `sertoser(GEN x, long prec)` return the `t_SER`  $x$  truncated or extended (with zeros) to `prec` terms. Shallow function, assume that `prec`  $\geq 0$ .

**GEN** `derivser(GEN x)` returns the derivative of the `t_SER`  $x$  with respect to its main variable.

**GEN** `integser(GEN x)` returns the primitive of the `t_SER`  $x$  with respect to its main variable.

**GEN** `truecoef(GEN x, long n)` returns `polcoef(x,n, -1)`, i.e. the coefficient of the term of degree  $n$  in the main variable. This is a safe but expensive function that must *copy* its return value so that it be `gerepile`-safe. Use `polcoef_i` for a fast internal variant.

**GEN** `ser_unscale(GEN P, GEN h)` return  $P(hx)$ , not memory clean.

**GEN** `ser_normalize(GEN x)` divide  $x$  by its “leading term” so that the series is either 0 or equal to  $t^v(1 + O(t))$ . Shallow function if the “leading term” is 1.

**int** `ser_isexactzero(GEN x)` return 1 if  $x$  is a zero series, all of whose known coefficients are exact zeroes; this implies that `sign(x) = 0` and `lg(x)  $\leq 3$` .

**GEN** `ser_inv(GEN x)` return the inverse of the `t_SER`  $x$  using Newton iteration. This is in general slower than `ginv` unless the precision is huge (hundreds of terms, where the threshold depends strongly on the base field).

## 10.8 Functions to handle `t_FFELT`.

These functions define the public interface of the `t_FFELT` type to use in generic functions. However, in specific functions, it is better to use the functions class `FpXQ` and/or `F1xq` as appropriate.

**GEN** `FF_p(GEN a)` returns the characteristic of the definition field of the `t_FFELT` element  $a$ .

**long** `FF_f(GEN a)` returns the dimension of the definition field over its prime field; the cardinality of the dimension field is thus  $p^f$ .

**GEN** `FF_p_i(GEN a)` shallow version of `FF_p`.

**GEN** `FF_q(GEN a)` returns the cardinality of the definition field of the `t_FFELT` element  $a$ .

**GEN** `FF_mod(GEN a)` returns the polynomial (with reduced `t_INT` coefficients) defining the finite field, in the variable used to display  $a$ .

**GEN** `FF_gen(GEN a)` returns the standard generator of the definition field of the `t_FFELT` element  $a$ , see `ffgen`, that is  $x \pmod{T}$  where  $T$  is the polynomial over the prime field that define the finite field.

**GEN** `FF_to_FpXQ(GEN a)` converts the `t_FFELT`  $a$  to a polynomial  $P$  with reduced `t_INT` coefficients such that  $a = P(g)$  where  $g$  is the generator of the finite field returned by `ffgen`, in the variable used to display  $g$ .

**GEN** `FF_to_FpXQ_i(GEN a)` shallow version of `FF_to_FpXQ`.

**GEN** `FF_to_F2xq(GEN a)` converts the `t_FFELT`  $a$  to a `F2x`  $P$  such that  $a = P(g)$  where  $g$  is the generator of the finite field returned by `ffgen`, in the variable used to display  $g$ . This only work if the characteristic is 2.

**GEN** `FF_to_F2xq_i(GEN a)` shallow version of `FF_to_F2xq`.

GEN FF\_to\_Flxq(GEN a) converts the  $t\_FFELT$   $a$  to a  $Flx$   $P$  such that  $a = P(g)$  where  $g$  is the generator of the finite field returned by `ffgen`, in the variable used to display  $g$ . This only work if the characteristic is small enough.

GEN FF\_to\_Flxq\_i(GEN a) shallow version of FF\_to\_Flxq.

GEN p\_to\_FF(GEN p, long v) returns a  $t\_FFELT$  equal to 1 in the finite field  $\mathbf{Z}/p\mathbf{Z}$ . Useful for generic code that wants to handle (inefficiently)  $\mathbf{Z}/p\mathbf{Z}$  as if it were not a prime field.

GEN Tp\_to\_FF(GEN T, GEN p) returns a  $t\_FFELT$  equal to 1 in the finite field  $\mathbf{F}_p/(T)$ , where  $T$  is a  $ZX$ , assumed to be irreducible modulo  $p$ , or `NULL` in which case the routine acts as `p_to_FF(p,0)`. No checks.

GEN Fq\_to\_FF(GEN x, GEN ff) returns a  $t\_FFELT$  equal to  $x$  in the finite field defined by the  $t\_FFELT$   $ff$ , where  $x$  is an  $Fq$  (either a  $t\_INT$  or a  $ZX$ : a  $t\_POL$  with  $t\_INT$  coefficients). No checks.

GEN FqX\_to\_FFX(GEN x, GEN ff) given an  $FqX$   $x$ , return the polynomial with  $t\_FFELT$  coefficients obtained by applying `Fq_to_FF` coefficientwise. No checks, and no normalization if the leading coefficient maps to 0.

GEN FF\_1(GEN a) returns the unity in the definition field of the  $t\_FFELT$  element  $a$ .

GEN FF\_zero(GEN a) returns the zero element of the definition field of the  $t\_FFELT$  element  $a$ .

int FF\_equal0(GEN a) returns 1 if the  $t\_FFELT$   $a$  is equal to 0 else returns 0.

int FF\_equal1(GEN a) returns 1 if the  $t\_FFELT$   $a$  is equal to 1 else returns 0.

int FF\_equalm1(GEN a) returns  $-1$  if the  $t\_FFELT$   $a$  is equal to 1 else returns 0.

int FF\_equal(GEN a, GEN b) return 1 if the  $t\_FFELT$   $a$  and  $b$  have the same definition field and are equal, else 0.

int FF\_samefield(GEN a, GEN b) return 1 if the  $t\_FFELT$   $a$  and  $b$  have the same definition field, else 0.

int Rg\_is\_FF(GEN c, GEN \*ff) to be called successively on many objects, setting  $*ff = \text{NULL}$  (unset) initially. Returns 1 as long as  $c$  is a  $t\_FFELT$  defined over the same field as  $*ff$  (setting  $*ff = c$  if unset), and 0 otherwise.

int RgC\_is\_FFC(GEN x, GEN \*ff) apply `Rg_is_FF` successively to all components of the  $t\_VEC$  or  $t\_COL$   $x$ . Return 0 if one call fails, and 1 otherwise.

int RgM\_is\_FFM(GEN x, GEN \*ff) apply `Rg_is_FF` to all components of the  $t\_MAT$ . Return 0 if one call fails, and 1 otherwise.

GEN FF\_add(GEN a, GEN b) returns  $a + b$  where  $a$  and  $b$  are  $t\_FFELT$  having the same definition field.

GEN FF\_Z\_add(GEN a, GEN x) returns  $a + x$ , where  $a$  is a  $t\_FFELT$ , and  $x$  is a  $t\_INT$ , the computation being performed in the definition field of  $a$ .

GEN FF\_Q\_add(GEN a, GEN x) returns  $a + x$ , where  $a$  is a  $t\_FFELT$ , and  $x$  is a  $t\_RFRAC$ , the computation being performed in the definition field of  $a$ .

GEN FF\_sub(GEN a, GEN b) returns  $a - b$  where  $a$  and  $b$  are  $t\_FFELT$  having the same definition field.

GEN FF\_mul(GEN a, GEN b) returns  $ab$  where  $a$  and  $b$  are  $t\_FFELT$  having the same definition field.

GEN FF\_Z\_mul(GEN a, GEN b) returns  $ab$ , where  $a$  is a  $t\_FFELT$ , and  $b$  is a  $t\_INT$ , the computation being performed in the definition field of  $a$ .

GEN FF\_div(GEN a, GEN b) returns  $a/b$  where  $a$  and  $b$  are  $t\_FFELT$  having the same definition field.

GEN FF\_neg(GEN a) returns  $-a$  where  $a$  is a  $t\_FFELT$ .

GEN FF\_neg\_i(GEN a) shallow function returning  $-a$  where  $a$  is a  $t\_FFELT$ .

GEN FF\_inv(GEN a) returns  $a^{-1}$  where  $a$  is a  $t\_FFELT$ .

GEN FF\_sqr(GEN a) returns  $a^2$  where  $a$  is a  $t\_FFELT$ .

GEN FF\_mul2n(GEN a, long n) returns  $a2^n$  where  $a$  is a  $t\_FFELT$ .

GEN FF\_pow(GEN a, GEN n) returns  $a^n$  where  $a$  is a  $t\_FFELT$  and  $n$  is a  $t\_INT$ .

GEN FF\_Frobenius(GEN a, GEN n) returns  $a^{p^n}$  where  $a$  is a  $t\_FFELT$   $n$  is a  $t\_INT$ , and  $p$  is the characteristic of the definition field of  $a$ .

GEN FF\_Z\_Z\_muldiv(GEN a, GEN x, GEN y) returns  $ay/z$ , where  $a$  is a  $t\_FFELT$ , and  $x$  and  $y$  are  $t\_INT$ , the computation being performed in the definition field of  $a$ .

GEN Z\_FF\_div(GEN x, GEN a) return  $x/a$  where  $a$  is a  $t\_FFELT$ , and  $x$  is a  $t\_INT$ , the computation being performed in the definition field of  $a$ .

GEN FF\_norm(GEN a) returns the norm of the  $t\_FFELT$   $a$  with respect to its definition field.

GEN FF\_trace(GEN a) returns the trace of the  $t\_FFELT$   $a$  with respect to its definition field.

GEN FF\_conjvec(GEN a) returns the vector of conjugates  $[a, a^p, a^{p^2}, \dots, a^{p^{n-1}}]$  where the  $t\_FFELT$   $a$  belong to a field with  $p^n$  elements.

GEN FF\_charpoly(GEN a) returns the characteristic polynomial) of the  $t\_FFELT$   $a$  with respect to its definition field.

GEN FF\_minpoly(GEN a) returns the minimal polynomial of the  $t\_FFELT$   $a$ .

GEN FF\_sqrt(GEN a) returns an  $t\_FFELT$   $b$  such that  $a = b^2$  if it exist, where  $a$  is a  $t\_FFELT$ .

long FF\_issquareall(GEN x, GEN \*pt) returns 1 if  $x$  is a square, and 0 otherwise. If  $x$  is indeed a square, set  $pt$  to its square root.

long FF\_issquare(GEN x) returns 1 if  $x$  is a square and 0 otherwise.

long FF\_ispower(GEN x, GEN K, GEN \*pt) Given  $K$  a positive integer, returns 1 if  $x$  is a  $K$ -th power, and 0 otherwise. If  $x$  is indeed a  $K$ -th power, set  $pt$  to its  $K$ -th root.

GEN FF\_sqrtn(GEN a, GEN n, GEN \*zn) returns an  $n$ -th root of  $a$  if it exist. If  $zn$  is non-NULL set it to a primitive  $n$ -th root of the unity.

GEN FF\_log(GEN a, GEN g, GEN o) the  $t\_FFELT$   $g$  being a generator for the definition field of the  $t\_FFELT$   $a$ , returns a  $t\_INT$   $e$  such that  $a^e = g$ . If  $e$  does not exists, the result is currently undefined. If  $o$  is not NULL it is assumed to be a factorization of the multiplicative order of  $g$  (as set by `FF_primroot`)

GEN FF\_order(GEN a, GEN o) returns the order of the  $t\_FFELT$   $a$ . If  $o$  is non-NULL, it is assumed that  $o$  is a multiple of the order of  $a$ .

`GEN FF_primroot(GEN a, GEN *o)` returns a generator of the multiplicative group of the definition field of the `t_FFELT` `a`. If `o` is not `NULL`, set it to the factorization of the order of the primitive root (to speed up `FF_log`).

`GEN FF_map(GEN m, GEN a)` returns  $A(m)$  where  $A=a.pol$  assuming  $a$  and  $m$  belongs to fields having the same characteristic.

### 10.8.1 FFX.

The functions in this sections take polynomial arguments and a `t_FFELT` `a`. The coefficients of the polynomials must be of type `t_INT`, `t_INTMOD` or `t_FFELT` and compatible with `a`.

`GEN FFX_mul(GEN P, GEN Q, GEN a)` returns the product of the polynomials `P` and `Q` defined over the definition field of the `t_FFELT` `a`.

`GEN FFX_sqr(GEN P, GEN a)` returns the square of the polynomial `P` defined over the definition field of the `t_FFELT` `a`.

`GEN FFX_rem(GEN P, GEN Q, GEN a)` returns the remainder of the polynomial `P` modulo the polynomial `Q`, where `P` and `Q` are defined over the definition field of the `t_FFELT` `a`.

`GEN FFX_ispower(GEN P, ulong k, GEN a, GEN *py)` return 1 if the FFX `P` is a  $k$ -th power, 0 otherwise, where `P` is defined over the definition field of the `t_FFELT` `a`. If `py` is not `NULL`, set it to  $g$  such that  $g^k = f$ .

`GEN FFX_factor(GEN f, GEN a)` returns the factorization of the univariate polynomial `f` over the definition field of the `t_FFELT` `a`. The coefficients of `f` must be of type `t_INT`, `t_INTMOD` or `t_FFELT` and compatible with `a`.

`GEN FFX_factor_squarefree(GEN f, GEN a)` returns the squarefree factorization of the univariate polynomial `f` over the definition field of the `t_FFELT` `a`. This is a vector  $[u_1, \dots, u_k]$  of pairwise coprime FFX such that  $u_k \neq 1$  and  $f = \prod u_i^i$ .

`GEN FFX_ddf(GEN f, GEN a)` assuming that  $f$  is squarefree, returns the distinct degree factorization of  $f$  modulo  $p$ . The returned value `v` is a `t_VEC` with two components:  $F=v[1]$  is a vector of (FFX) factors, and  $E=v[2]$  is a `t_VECSMALL`, such that  $f$  is equal to the product of the  $F[i]$  and each  $F[i]$  is a product of irreducible factors of degree  $E[i]$ .

`GEN FFX_degfact(GEN f, GEN a)`, as `FFX_factor`, but the degrees of the irreducible factors are returned instead of the factors themselves (as a `t_VECSMALL`).

`GEN FFX_roots(GEN f, GEN a)` returns the roots (`t_FFELT`) of the univariate polynomial `f` over the definition field of the `t_FFELT` `a`. The coefficients of `f` must be of type `t_INT`, `t_INTMOD` or `t_FFELT` and compatible with `a`.

`GEN FFX_preimage(GEN F, GEN x, GEN a)` returns  $P\%F$  where  $P=x.pol$  assuming  $a$  and  $x$  belongs to fields having the same characteristic, and that the coefficients of  $F$  belong to the definition field of  $a$ .

### 10.8.2 FFM.

`GEN FFM_FFC_gauss(GEN M, GEN C, GEN ff)` given a matrix  $M$  (`t_MAT`) and a column vector  $C$  (`t_COL`) over the finite field given by `ff` (`t_FFELT`) such that  $M$  is invertible, return the unique column vector  $X$  such that  $MX = C$ .

`GEN FFM_FFC_invimage(GEN M, GEN C, GEN ff)` given a matrix  $M$  (`t_MAT`) and a column vector  $C$  (`t_COL`) over the finite field given by `ff` (`t_FFELT`), return a column vector  $X$  such that  $MX = C$ , or `NULL` if no such vector exists.

`GEN FFM_FFC_mul(GEN M, GEN C, GEN ff)` returns the product of the matrix  $M$  (`t_MAT`) and the column vector  $C$  (`t_COL`) over the finite field given by `ff` (`t_FFELT`).

`GEN FFM_deplin(GEN M, GEN ff)` returns a non-zero vector (`t_COL`) in the kernel of the matrix  $M$  over the finite field given by `ff`, or `NULL` if no such vector exists.

`GEN FFM_det(GEN M, GEN ff)` returns the determinant of the matrix  $M$  over the finite field given by `ff`.

`GEN FFM_gauss(GEN M, GEN N, GEN ff)` given two matrices  $M$  and  $N$  (`t_MAT`) over the finite field given by `ff` (`t_FFELT`) such that  $M$  is invertible, return the unique matrix  $X$  such that  $MX = N$ .

`GEN FFM_image(GEN M, GEN ff)` returns a matrix whose columns span the image of the matrix  $M$  over the finite field given by `ff`.

`GEN FFM_indexrank(GEN M, GEN ff)` given a matrix  $M$  of rank  $r$  over the finite field given by `ff`, returns a vector with two `t_VECSMALL` components  $y$  and  $z$  containing  $r$  row and column indices, respectively, such that the  $r \times r$ -matrix formed by the  $M[i, j]$  for  $i$  in  $y$  and  $j$  in  $z$  is invertible.

`GEN FFM_inv(GEN M, GEN ff)` returns the inverse of the square matrix  $M$  over the finite field given by `ff`, or `NULL` if  $M$  is not invertible.

`GEN FFM_invimage(GEN M, GEN N, GEN ff)` given two matrices  $M$  and  $N$  (`t_MAT`) over the finite field given by `ff` (`t_FFELT`), return a matrix  $X$  such that  $MX = N$ , or `NULL` if no such matrix exists.

`GEN FFM_ker(GEN M, GEN ff)` returns the kernel of the `t_MAT`  $M$  over the finite field given by the `t_FFELT` `ff`.

`GEN FFM_mul(GEN M, GEN N, GEN ff)` returns the product of the matrices  $M$  and  $N$  (`t_MAT`) over the finite field given by `ff` (`t_FFELT`).

`long FFM_rank(GEN M, GEN ff)` returns the rank of the matrix  $M$  over the finite field given by `ff`.

`GEN FFM_suppl(GEN M, GEN ff)` given a matrix  $M$  over the finite field given by `ff` whose columns are linearly independent, returns a square invertible matrix whose first columns are those of  $M$ .

### 10.8.3 FFXQ.

`GEN FFXQ_mul(GEN P, GEN Q, GEN T, GEN a)` returns the product of the polynomials  $P$  and  $Q$  modulo the polynomial  $T$ , where  $P$ ,  $Q$  and  $T$  are defined over the definition field of the `t_FFELT` `a`.

`GEN FFXQ_sqr(GEN P, GEN T, GEN a)` returns the square of the polynomial  $P$  modulo the polynomial  $T$ , where  $P$  and  $T$  are defined over the definition field of the `t_FFELT` `a`.

`GEN FFXQ_inv(GEN P, GEN Q, GEN a)` returns the inverse of the polynomial  $P$  modulo the polynomial  $Q$ , where  $P$  and  $Q$  are defined over the definition field of the `t_FFELT` `a`.

## 10.9 Transcendental functions.

The following two functions are only useful when interacting with `gp`, to manipulate its internal default precision (expressed as a number of decimal digits, not in words as used everywhere else):

`long getrealprecision(void)` returns `realprecision`.

`long setrealprecision(long n, long *prec)` sets the new `realprecision` to `n`, which is returned. As a side effect, set `prec` to the corresponding number of words `ndec2prec(n)`.

### 10.9.1 Transcendental functions with `t_REAL` arguments.

In the following routines,  $x$  is assumed to be a `t_REAL` and the result is a `t_REAL` (sometimes a `t_COMPLEX` with `t_REAL` components), with the largest accuracy which can be deduced from the input. The naming scheme is inconsistent here, since we sometimes use the prefix `mp` even though `t_INT` inputs are forbidden:

`GEN sqrtr(GEN x)` returns the square root of  $x$ .

`GEN cbrtr(GEN x)` returns the real cube root of  $x$ .

`GEN sqrtnr(GEN x, long n)` returns the  $n$ -th root of  $x$ , assuming  $n \geq 1$  and  $x \geq 0$ .

`GEN sqrtnr_abs(GEN x, long n)` returns the  $n$ -th root of  $|x|$ , assuming  $n \geq 1$  and  $x \neq 0$ .

`GEN mpcos[z](GEN x[, GEN z])` returns  $\cos(x)$ .

`GEN mpsin[z](GEN x[, GEN z])` returns  $\sin(x)$ .

`GEN mplog[z](GEN x[, GEN z])` returns  $\log(x)$ . We must have  $x > 0$  since the result must be a `t_REAL`. Use `glog` for the general case, where you want such computations as  $\log(-1) = I$ .

`GEN mpexp[z](GEN x[, GEN z])` returns  $\exp(x)$ .

`GEN mpexpm1(GEN x)` returns  $\exp(x) - 1$ , but is more accurate than `subrs(mpexp(x), 1)`, which suffers from catastrophic cancellation if  $|x|$  is very small.

`void mpsincosm1(GEN x, GEN *s, GEN *c)` sets  $s$  and  $c$  to  $\sin(x)$  and  $\cos(x) - 1$  respectively, where  $x$  is a `t_REAL`; the latter is more accurate than `subrs(mpcos(y), 1)`, which suffers from catastrophic cancellation if  $|x|$  is very small.

`GEN mpveceint1(GEN C, GEN eC, long n)` as `veceint1`; assumes that  $C > 0$  is a `t_REAL` and that `eC` is `NULL` or `mpexp(C)`.

`GEN mpeint1(GEN x, GEN expx)` returns `eint1(x)`, for a `t_REAL`  $x \geq 0$ , assuming that `expx` is `mpexp(x)`.

`GEN mplambertW(GEN y)` solution  $x$  of the implicit equation  $x \exp(x) = y$ , for  $y > 0$  a `t_REAL`.

Useful low-level functions which *disregard* the sign of  $x$ :

`GEN sqrtr_abs(GEN x)` returns  $\sqrt{|x|}$  assuming  $x \neq 0$ .

`GEN cbrtr_abs(GEN x)` returns  $|x|^{1/3}$  assuming  $x \neq 0$ .

`GEN exp1r_abs(GEN x)` returns  $\exp(|x|) - 1$ , assuming  $x \neq 0$ .

`GEN logr_abs(GEN x)` returns  $\log(|x|)$ , assuming  $x \neq 0$ .

### 10.9.2 Other complex transcendental functions.

GEN `szeta(long s, long prec)` returns the value of Riemann's zeta function at the (possibly negative) integer  $s \neq 1$ , in relative accuracy `prec`.

GEN `veczeta(GEN a, GEN b, long N, long prec)` returns in a vector all the  $\zeta(aj + b)$ , where  $j = 0, 1, \dots, N - 1$ , where  $a$  and  $b$  are real numbers (of arbitrary type, although `t_INT` is treated more efficiently) and  $b > 1$ . Assumes that  $N \geq 1$ .

GEN `ggamma1m1(GEN x, long prec)` return  $\Gamma(1 + x) - 1$  assuming  $|x| < 1$ . Guard against cancellation when  $x$  is small.

A few variants on `sin` and `cos`:

void `mpsincos(GEN x, GEN *s, GEN *c)` sets  $s$  and  $c$  to  $\sin(x)$  and  $\cos(x)$  respectively, where  $x$  is a `t_REAL`

GEN `expIi(GEN x)` returns  $\exp(ix)$ , where  $x$  is a `t_REAL`. The return type is `t_COMPLEX` unless the imaginary part is equal to 0 to the current accuracy (its sign is 0).

GEN `expIxy(GEN x, GEN y, long prec)` returns  $\exp(ixy)$ . Efficient when  $x$  is real and  $y$  pure imaginary.

void `gsincos(GEN x, GEN *s, GEN *c, long prec)` general case.

GEN `rootsof1_cx(GEN d, long prec)` return  $e(1/d)$  at precision `prec`,  $e(x) = \exp(2i\pi x)$ .

GEN `rootsof1u_cx(ulong d, long prec)` return  $e(1/d)$  at precision `prec`.

GEN `rootsof1q_cx(long a, long b, long prec)` return  $e(a/b)$  at precision `prec`.

GEN `rootsof1powinit(long a, long b, long prec)` precompute  $b$ -th roots of 1 for `rootsof1pow`, i.e. to later compute  $e(ac/b)$  for varying  $c$ .

GEN `rootsof1pow(GEN T, long c)` given  $T = \text{rootsof1powinit}(a, b, \text{prec})$ , return  $e(ac/b)$ .

A generalization of `affrr_fixlg`

GEN `affc_fixlg(GEN x, GEN res)` assume `res` was allocated using `cgetc`, and that  $x$  is either a `t_REAL` or a `t_COMPLEX` with `t_REAL` components. Assign  $x$  to `res`, first shortening the components of `res` if needed (in a `gerepile`-safe way). Further convert `res` to a `t_REAL` if  $x$  is a `t_REAL`.

GEN `trans_eval(const char *fun, GEN (*f) (GEN, long), GEN x, long prec)` evaluate the transcendental function  $f$  (named "fun" at the argument  $x$  and precision `prec`). This is a quick way to implement a transcendental function to be made available under GP, starting from a  $C$  function handling only `t_REAL` and `t_COMPLEX` arguments. This routine first converts  $x$  to a suitable type:

- `t_INT/t_FRAC` to `t_REAL` of precision `prec`, `t_QUAD` to `t_REAL` or `t_COMPLEX` of precision `prec`.
- `t_POLMOD` to a `t_COL` of complex embeddings (as in `conjvec`)

Then evaluates the function at `t_VEC`, `t_COL`, `t_MAT` arguments coefficientwise.



### 10.9.3 Modular functions.

GEN `cxredsl2`(GEN `z`, GEN `*g`) given  $t$  a `t_COMPLEX` belonging to the upper half-plane, find  $\gamma \in \mathrm{SL}_2(\mathbf{Z})$  such that  $\gamma \cdot z$  belongs to the standard fundamental domain and set `*g` to  $\gamma$ .

GEN `cxredsl2_i`(GEN `z`, GEN `*g`, GEN `*czd`) as `cxredsl2`; also sets `*czd` to  $cz+d$ , if  $\gamma = [a, b; c, d]$ .

GEN `cxEk`(GEN `tau`, long `k`, long `prec`) returns  $E_k(\tau)$  by direct evaluation of  $1 + 2/\zeta(1-k) \sum_n n^{k-1} q^n / (1 - q^n)$ ,  $q = e(\tau)$ . Assume that  $\Im \tau > 0$  and  $k$  even. Very slow unless  $\tau$  is already reduced modulo  $\mathrm{SL}_2(\mathbf{Z})$ . Not `gerepile`-clean but suitable for `gerepileupto`.

### 10.9.4 Transcendental functions with `t_PADIC` arguments.

GEN `Qp_exp`(GEN `x`) shortcut for `gexp(x, /*ignored*/prec)`

GEN `Qp_gamma`(GEN `x`) shortcut for `ggamma(x, /*ignored*/prec)`

GEN `Qp_log`(GEN `x`) shortcut for `glog(x, /*ignored*/prec)`

GEN `Qp_sqrt`(GEN `x`) shortcut for `gsqrt(x, /*ignored*/prec)` Return NULL if  $x$  is not a square.

GEN `Qp_sqrtn`(GEN `x`, GEN `n`, GEN `*z`) shortcut for `gsqrtn(x, n, z, /*ignored*/prec)`. Return NULL if  $x$  is not an  $n$ -th power.

GEN `Qp_agm2_sequence`(GEN `a1`, GEN `b1`) assume  $a_1/b_1 = 1 \bmod p$  if  $p$  odd and  $\bmod 2^4$  if  $p = 2$ . Let  $A_1 = a_1/p^v$  and  $B_1 = b_1/p^v$  with  $v = v_p(a_1) = v_p(b_1)$ ; let further  $A_{n+1} = (A_n + B_n + 2B_{n+1})/4$ ,  $B_{n+1} = B_n \sqrt{A_n/B_n}$  (the square root of  $A_n B_n$  congruent to  $B_n \bmod p$ ) and  $R_n = p^v(A_n - B_n)$ . We stop when  $R_n$  is 0 at the given  $p$ -adic accuracy. This function returns in a triplet `t_VEC` the three sequences  $(A_n)$ ,  $(B_n)$  and  $(R_n)$ , corresponding to a sequence of 2-isogenies on the Tate curve  $y^2 = x(x - a_1)(x + a_1 - b_1)$ . The common limit of  $A_n$  and  $B_n$  is the  $M_2(a_1, b_1)$ , the square of the  $p$ -adic AGM of  $\sqrt{a_1}$  and  $\sqrt{b_1}$ . This is given by `ellQp_Ei` and is used by corresponding ascending and descending  $p$ -adic Landen transforms:

void `Qp_ascending_Landen`(GEN `ABR`, GEN `*ptx`, GEN `*pty`)

void `Qp_descending_Landen`(GEN `ABR`, GEN `*ptx`, GEN `*pty`)

### 10.9.5 Cached constants.

The cached constant is returned at its current precision, which may be larger than `prec`. One should always use the `mpxxx` variant: `mppi`, `mpeuler`, or `mplog2`.

GEN `consteuler`(long `prec`) precomputes Euler-Mascheroni's constant at precision `prec`.

GEN `constcatalan`(long `prec`) precomputes Catalan's constant at precision `prec`.

GEN `constpi`(long `prec`) precomputes  $\pi$  at precision `prec`.

GEN `constlog2`(long `prec`) precomputes  $\log(2)$  at precision `prec`.

void `mpbern`(long `n`, long `prec`) precomputes the  $n$  even Bernoulli numbers  $B_2, \dots, B_{2n}$  as `t_FRAC` or `t_REALs` of precision `prec`. For any  $2 \leq k \leq 2n$ , if a floating point approximation of  $B_k$  to accuracy `prec` is enough to reconstruct it exactly, a `t_FRAC` is stored; otherwise a `t_REAL` at the requested accuracy. No more than  $n$  Bernoulli numbers will ever be stored (by `bernfrac` or `bernreal`), unless a subsequent call to `mpbern` increases the cache. If `prec` is 0, the  $B_k$  are computed exactly.

The following functions use cached data if `prec` is smaller than the precision of the cached value; otherwise the newly computed data replaces the old cache.

GEN `mppi(long prec)` returns  $\pi$  at precision `prec`.

GEN `Pi2n(long n, long prec)` returns  $2^n\pi$  at precision `prec`.

GEN `PiI2(long n, long prec)` returns the complex number  $2\pi i$  at precision `prec`.

GEN `PiI2n(long n, long prec)` returns the complex number  $2^n\pi i$  at precision `prec`.

GEN `mpeuler(long prec)` returns Euler-Mascheroni's constant at precision `prec`.

GEN `mpeuler(long prec)` returns Catalan's number at precision `prec`.

GEN `mplog2(long prec)` returns  $\log 2$  at precision `prec`.

GEN `bernreal(long i, long prec)` returns the Bernoulli number  $B_i$  as a `t_REAL` at precision `prec`. If `mpbern(n, p)` was called previously with  $n \geq i$  and  $p \geq \text{prec}$ , then the cached value is (converted to a `t_REAL` of accuracy `prec` then) returned. Otherwise, the missing value is computed. In the latter case, if  $n \geq i$ , the cached table is updated.

GEN `bernfrac(long i)` returns the Bernoulli number  $B_i$  as a rational number (`t_FRAC` or `t_INT`). If a cached table includes  $B_i$  as a rational number, the latter is returned. Otherwise, the missing value is computed. In the latter case, the cached Bernoulli table may be updated.

## 10.10 Permutations .

Permutations are represented in two different ways

- (`perm`) a `t_VECSMALL`  $p$  representing the bijection  $i \mapsto p[i]$ ; unless mentioned otherwise, this is the form used in the functions below for both input and output,

- (`cyc`) a `t_VEC` of `t_VECSMALL`s representing a product of disjoint cycles.

GEN `identity_perm(long n)` return the identity permutation on  $n$  symbols.

GEN `cyclic_perm(long n, long d)` return the cyclic permutation mapping  $i$  to  $i + d \pmod{n}$  in  $S_n$ . Assume that  $d \leq n$ .

GEN `perm_mul(GEN s, GEN t)` multiply  $s$  and  $t$  (composition  $s \circ t$ )

GEN `perm_conj(GEN s, GEN t)` return  $sts^{-1}$ .

int `perm_commute(GEN p, GEN q)` return 1 if  $p$  and  $q$  commute, 0 otherwise.

GEN `perm_inv(GEN p)` returns the inverse of  $p$ .

GEN `perm_pow(GEN p, long n)` returns  $p^n$

GEN `cyc_pow_perm(GEN p, long n)` the permutation  $p$  is given as a product of disjoint cycles (`cyc`); return  $p^n$  (as a `perm`).

GEN `cyc_pow(GEN p, long n)` the permutation  $p$  is given as a product of disjoint cycles (`cyc`); return  $p^n$  (as a `cyc`).

GEN `perm_cycles(GEN p)` return the cyclic decomposition of  $p$ .

long `perm_order(GEN p)` returns the order of the permutation  $p$  (as the lcm of its cycle lengths).

`long perm_sign(GEN p)` returns the sign of the permutation  $p$ .

`GEN vecperm_orbits(GEN p, long n)` the permutation  $p \in S_n$  being given as a product of disjoint cycles, return the orbits of the subgroup generated by  $p$  on  $\{1, 2, \dots, n\}$ .

`GEN Z_to_perm(long n, GEN x)` as `numtoperm`, returning a `t_VECSMALL`.

`GEN perm_to_Z(GEN v)` as `permtonum` for a `t_VECSMALL` input.

## 10.11 Small groups.

The small (finite) groups facility is meant to deal with subgroups of Galois groups obtained by `galoisinit` and thus is currently limited to weakly super-solvable groups.

A group  $grp$  of order  $n$  is represented by its regular representation (for an arbitrary ordering of its element) in  $S_n$ . A subgroup of such group is represented by the restriction of the representation to the subgroup. A *small group* can be either a group or a subgroup. Thus it is embedded in some  $S_n$ , where  $n$  is the multiple of the order. Such an  $n$  is called the *domain* of the small group. The domain of a trivial subgroup cannot be derived from the subgroup data, so some functions require the subgroup domain as argument.

The small group  $grp$  is represented by a `t_VEC` with two components:

$grp[1]$  is a generating subset  $[s_1, \dots, s_g]$  of  $grp$  expressed as a vector of permutations of length  $n$ .

$grp[2]$  contains the relative orders  $[o_1, \dots, o_g]$  of the generators  $grp[1]$ .

See `galoisinit` for the technical details.

`GEN checkgroup(GEN gal, GEN *elts)` check whether  $gal$  is a small group or a Galois group. Returns the underlying small group and set  $elts$  to the list of elements or to `NULL` if it is not known.

`GEN checkgrouplets(GEN gal)` check whether  $gal$  is a small group or a Galois group, or a vector of permutations listing the group elements. Returns the list of group elements as permutations.

`GEN galois_group(GEN gal)` return the underlying small group of the Galois group  $gal$ .

`GEN cyclicgroup(GEN g, long s)` return the cyclic group with generator  $g$  of order  $s$ .

`GEN trivialgroup(void)` return the trivial group.

`GEN dicyclicgroup(GEN g1, GEN g2, long s1, long s2)` returns the group with generators  $g1, g2$  with respecting relative orders  $s1, s2$ .

`GEN abelian_group(GEN v)` let  $v$  be a `t_VECSMALL` seen as the SNF of a small abelian group, return its regular representation.

`long group_domain(GEN grp)` returns the domain of the *non-trivial* small group  $grp$ . Return an error if  $grp$  is trivial.

`GEN group_elts(GEN grp, long n)` returns the list of elements of the small group  $grp$  of domain  $n$  as permutations.

`GEN group_set(GEN grp, long n)` returns a  $F2v$   $b$  such that  $b[i]$  is set if and only if the small group  $grp$  of domain  $n$  contains a permutation sending 1 to  $i$ .

`GEN grouplets_set(GEN elts, long n)`, where  $elts$  is the list of elements of a small group of domain  $n$ , returns a  $F2v$   $b$  such that  $b[i]$  is set if and only if the small group contains a permutation sending 1 to  $i$ .

GEN `groupelts_conjclasses`(GEN `elts`, long `*pn`), where `elts` is the list of elements of a small group (sorted with respect to `vecsmall_lexcmp`), return a `t_VECSMALL` `conj` of the same length such that `conj[i]` is the index in  $\{1, \dots, n\}$  of the conjugacy class of `elts[i]` for some unspecified but deterministic ordering of the classes, where  $n$  is the number of conjugacy classes. If `pn` is non NULL, `*pn` is set to  $n$ .

GEN `conjclasses_repr`(GEN `conj`, long `nb`), where `conj` and `nb` are as returned by the call `groupelts_conjclasses(elts)`, return `t_VECSMALL` of length `nb` which gives the indices in `elts` of a representative of each conjugacy class.

GEN `group_to_cc`(GEN `G`), where `G` is a small group or a Galois group, returns a `cc` (conjugacy classes) structure [`elts,conj,rep,flag`], as obtained by `alggroupcenter`, where `conj` is `groupelts_conjclasses(elts)` and `rep` is the attached `conjclasses_repr`. `flag` is 1 if the permutation representation is transitive (in which case an element  $g$  of  $G$  is characterized by  $g[1]$ ), and 0 otherwise. Shallow function.

long `group_order`(GEN `grp`) returns the order of the small group `grp` (which is the product of the relative orders).

long `group_isabelian`(GEN `grp`) returns 1 if the small group `grp` is Abelian, else 0.

GEN `group_abelianHNF`(GEN `grp`, GEN `elts`) if `grp` is not Abelian, returns NULL, else returns the HNF matrix of `grp` with respect to the generating family `grp[1]`. If `elts` is no NULL, it must be the list of elements of `grp`.

GEN `group_abelianSNF`(GEN `grp`, GEN `elts`) if `grp` is not Abelian, returns NULL, else returns its cyclic decomposition. If `elts` is no NULL, it must be the list of elements of `grp`.

long `group_subgroup_isnormal`(GEN `G`, GEN `H`),  $H$  being a subgroup of the small group  $G$ , returns 1 if  $H$  is normal in  $G$ , else 0.

long `group_isA4S4`(GEN `grp`) returns 1 if the small group `grp` is isomorphic to  $A_4$ , 2 if it is isomorphic to  $S_4$  and 0 else. This is mainly to deal with the idiosyncrasy of the format.

GEN `group_leftcoset`(GEN `G`, GEN `g`) where  $G$  is a small group and  $g$  a permutation of the same domain, the left coset  $gG$  as a vector of permutations.

GEN `group_rightcoset`(GEN `G`, GEN `g`) where  $G$  is a small group and  $g$  a permutation of the same domain, the right coset  $Gg$  as a vector of permutations.

long `group_perm_normalize`(GEN `G`, GEN `g`) where  $G$  is a small group and  $g$  a permutation of the same domain, return 1 if  $gGg^{-1} = G$ , else 0.

GEN `group_quotient`(GEN `G`, GEN `H`), where  $G$  is a small group and  $H$  is a subgroup of  $G$ , returns the quotient map  $G \rightarrow G/H$  as an abstract data structure.

GEN `quotient_perm`(GEN `C`, GEN `g`) where  $C$  is the quotient map  $G \rightarrow G/H$  for some subgroup  $H$  of  $G$  and  $g$  an element of  $G$ , return the image of  $g$  by  $C$  (i.e. the coset  $gH$ ).

GEN `quotient_group`(GEN `C`, GEN `G`) where  $C$  is the quotient map  $G \rightarrow G/H$  for some normal subgroup  $H$  of  $G$ , return the quotient group  $G/H$  as a small group.

GEN `quotient_subgroup_lift`(GEN `C`, GEN `H`, GEN `S`) where  $C$  is the quotient map  $G \rightarrow G/H$  for some group  $G$  normalizing  $H$  and  $S$  is a subgroup of  $G/H$ , return the inverse image of  $S$  by  $C$ .

GEN `group_subgroups`(GEN `grp`) returns the list of subgroups of the small group `grp` as a `t_VEC`.

`GEN subgroups_tableset(GEN S, long n)` where  $S$  is a vector of subgroups of domain  $n$ , returns a table which matches the set of elements of the subgroups against the index of the subgroups.

`long tableset_find_index(GEN tbl, GEN set)` searches the set `set` in the table `tbl` and returns its attached index, or 0 if not found.

`GEN grouperelts_abelian_group(GEN elts)` where  $elts$  is the list of elements of an *Abelian* small group, returns the corresponding small group.

`long grouperelts_exponent(GEN elts)` where  $elts$  is the list of elements of a small group, returns the exponent the group (the LCM of the order of the elements of the group).

`GEN grouperelts_center(GEN elts)` where  $elts$  is the list of elements of a small group, returns the list of elements of the center of the group.

`GEN group_export(GEN grp, long format)` convert a small group to another format, as a `t_STR` describing the group for the given syntax, see `galoisexport`.

`GEN group_export_GAP(GEN G)` export a small group to GAP format.

`GEN group_export_MAGMA(GEN G)` export a small group to MAGMA format.

`long group_ident(GEN grp, GEN elts)` returns the index of the small group  $grp$  in the GAP4 Small Group library, see `galoisidentify`. If  $elts$  is not `NULL`, it must be the list of elements of  $grp$ .

`long group_ident_trans(GEN grp, GEN elts)` returns the index of the regular representation of the small group  $grp$  in the GAP4 Transitive Group library, see `polgalois`. If  $elts$  is no `NULL`, it must be the list of elements of  $grp$ .



## Chapter 11:

### Standard data structures

#### 11.1 Character strings.

##### 11.1.1 Functions returning a char \*.

`char* pari_strdup(const char *s)` returns a malloc'ed copy of *s* (uses `pari_malloc`).

`char* pari_strndup(const char *s, long n)` returns a malloc'ed copy of at most *n* chars from *s* (uses `pari_malloc`). If *s* is longer than *n*, only *n* characters are copied and a terminal null byte is added.

`char* stack_strdup(const char *s)` returns a copy of *s*, allocated on the PARI stack (uses `stack_malloc`).

`char* stack_strcat(const char *s, const char *t)` returns the concatenation of *s* and *t*, allocated on the PARI stack (uses `stack_malloc`).

`char* stack_sprintf(const char *fmt, ...)` runs `pari_sprintf` on the given arguments, returning a string allocated on the PARI stack.

`char* uordinal(ulong x)` return the ordinal number attached to *x* (i.e. 1st, 2nd, etc.) as a `stack_malloc`'ed string.

`char* itostr(GEN x)` writes the `t_INT` *x* to a `stack_malloc`'ed string.

`char* GENTostr(GEN x)`, using the current default output format (`GP_DATA->fmt`, which contains the output style and the number of significant digits to print), converts *x* to a malloc'ed string. Simple variant of `pari_sprintf`.

`char* GENTostr_raw(GEN x)` as `GENTostr` with the following differences: 1) the output format is `f_RAW`; 2) the result is allocated on the stack and *must not* be freed.

`char* GENTostr_unquoted(GEN x)` as `GENTostr_raw` with the following additional difference: a `t_STR` *x* is printed without enclosing quotes (to be used by `print`).

`char* GENToTeXstr(GEN x)`, as `GENTostr`, except that `f_TEX` overrides the output format from `GP_DATA->fmt`.

`char* RgV_to_str(GEN g, long flag)` *g* being a vector of `GEN`s, returns a malloc'ed string, the concatenation of the `GENTostr` applied to its elements, except that `t_STR` are printed without enclosing quotes. `flag` determines the output format: `f_RAW`, `f_PRETTYMAT` or `f_TEX`.

### 11.1.2 Functions returning a `t_STR`.

`GEN strtogenstr(const char *s)` returns a `t_STR` with content  $s$ .

`GEN strntogenstr(const char *s, long n)` returns a `t_STR` containing the first  $n$  characters of  $s$ .

`GEN chartogenstr(char c)` returns a `t_STR` containing the character  $c$ .

`GEN GENTogenstr(GEN x)` returns a `t_STR` containing the printed form of  $x$  (in `raw` format). This is often easier to use than `GENTostr` (which returns a malloc-ed `char*`) since there is no need to free the string after use.

`GEN GENTogenstr_nospace(GEN x)` as `GENTogenstr`, removing all spaces from the output.

`GEN Str(GEN g)` as `RgV_to_str` with output format `f_RAW`, but returns a `t_STR`, not a malloc'ed string.

`GEN Strtex(GEN g)` as `RgV_to_str` with output format `f_TEX`, but returns a `t_STR`, not a malloc'ed string.

`GEN Strexpend(GEN g)` as `RgV_to_str` with output format `f_RAW`, performing tilde and environment expansion on the result. Returns a `t_STR`, not a malloc'ed string.

`GEN gsprintf(const char *fmt, ...)` equivalent to `pari_sprintf(fmt, ...)`, followed by `strtoGENstr`. Returns a `t_STR`, not a malloc'ed string.

`GEN gvsprintf(const char *fmt, va_list ap)` variadic version of `gsprintf`

### 11.1.3 Dynamic strings.

A `pari_str` is a dynamic string which grows dynamically as needed. This structure contains private data and two public members `char *string`, which is the string itself and `use_stack` which tells whether the string lives

- on the PARI stack (value 1), meaning that it will be destroyed by any manipulation of the stack, e.g. a `gerepile` call or resetting `avma`;
- in malloc'ed memory (value 0), in which case it is impervious to stack manipulation but will need to be explicitly freed by the user after use, via `pari_free(s.string)`.

`void str_init(pari_str *S, int use_stack)` initializes a dynamic string; if `use_stack` is 0, then the string is malloc'ed, else it lives on the PARI stack.

`void str_printf(pari_str *S, const char *fmt, ...)` write to the end of  $S$  the remaining arguments according to PARI format `fmt`.

`void str_putc(pari_str *S, char c)` write the character  $c$  to the end of  $S$ .

`void str_puts(pari_str *S, const char *s)` write the string  $s$  to the end of  $S$ .



## 11.2 Output.

### 11.2.1 Output contexts.

An output context, of type `PariOUT`, is a `struct` that models a stream and contains the following function pointers:

```
void (*putch)(char);          /* fputc()-alike */
void (*puts)(const char*);    /* fputs()-alike */
void (*flush)(void);          /* fflush()-alike */
```

The methods `putch` and `puts` are used to print a character or a string respectively. The method `flush` is called to finalize a messages.

The generic functions `pari_putc`, `pari_puts`, `pari_flush` and `pari_printf` print according to a *default output context*, which should be sufficient for most purposes. Lower level functions are available, which take an explicit output context as first argument:

`void out_putc(PariOUT *out, char c)` essentially equivalent to `out->putc(c)`. In addition, registers whether the last character printed was a `\n`.

`void out_puts(PariOUT *out, const char *s)` essentially equivalent to `out->puts(s)`. In addition, registers whether the last character printed was a `\n`.

`void out_printf(PariOUT *out, const char *fmt, ...)`

`void out_vprintf(PariOUT *out, const char *fmt, va_list ap)`

N.B. The function `out_flush` does not exist since it would be identical to `out->flush()`

`int pari_last_was_newline(void)` returns a non-zero value if the last character printed via `out_putc` or `out_puts` was `\n`, and 0 otherwise.

`void pari_set_last_newline(int last)` sets the boolean value to be returned by the function `pari_last_was_newline` to *last*.

**11.2.2 Default output context.** They are defined by the global variables `pariOut` and `pariErr` for normal outputs and warnings/errors, and you probably do not want to change them. If you *do* change them, diverting output in non-trivial ways, this probably means that you are rewriting `gp`. For completeness, we document in this section what the default output contexts do.

**pariOut.** writes output to the `FILE*` `pari_outfile`, initialized to `stdout`. The low-level methods are actually the standard `putc` / `fputs`, plus some magic to handle a log file if one is open.

**pariErr.** prints to the `FILE*` `pari_errfile`, initialized to `stderr`. The low-level methods are as above.

You can stick with the default `pariOut` output context and change PARI's standard output, redirecting `pari_outfile` to another file, using

`void switchout(const char *name)` where `name` is a character string giving the name of the file you want to write to; the output is *appended* at the end of the file. To close the file and revert to outputting to `stdout`, call `switchout(NULL)`.

**11.2.3 PARI colors.** In this section we describe the low-level functions used to implement GP's color scheme, attached to the `colors` default. The following symbolic names are attached to gp's output strings:

- `c_ERR` an error message
- `c_HIST` a history number (as in `%1 = ...`)
- `c_PROMPT` a prompt
- `c_INPUT` an input line (minus the prompt part)
- `c_OUTPUT` an output
- `c_HELP` a help message
- `c_TIME` a timer
- `c_NONE` everything else

If the `colors` default is set to a non-empty value, before gp outputs a string, it first outputs an ANSI colors escape sequence — understood by most terminals —, according to the `colors` specifications. As long as this is in effect, the following strings are rendered in color, possibly in bold or underlined.

`void term_color(long c)` prints (as if using `pari_puts`) the ANSI color escape sequence attached to output object `c`. If `c` is `c_NONE`, revert to default printing style.

`void out_term_color(PariOUT *out, long c)` as `term_color`, using output context `out`.

`char* term_get_color(char *s, long c)` returns as a character string the ANSI color escape sequence attached to output object `c`. If `c` is `c_NONE`, the value used to revert to default printing style is returned. The argument `s` is either `NULL` (string allocated on the PARI stack), or preallocated storage (in which case, it must be able to hold at least 16 chars, including the final `\0`).

#### 11.2.4 Obsolete output functions.

These variants of `void output(GEN x)`, which prints `x`, followed by a newline and a buffer flush are complicated to use and less flexible than what we saw above, or than the `pari_printf` variants. They are provided for backward compatibility and are scheduled to disappear.

`void brute(GEN x, char format, long dec)`

`void matbrute(GEN x, char format, long dec)`

`void texe(GEN x, char format, long dec)`

## 11.3 Files.

The following routines are trivial wrappers around system functions (possibly around one of several functions depending on availability). They are usually integrated within PARI's diagnostics system, printing messages if `DEBUGFILES` is high enough.

`int pari_is_dir(const char *name)` returns 1 if `name` points to a directory, 0 otherwise.

`int pari_is_file(const char *name)` returns 1 if `name` points to a file, 0 otherwise.

`int file_is_binary(FILE *f)` returns 1 if the file `f` is a binary file (in the `writebin` sense), 0 otherwise.

`void pari_unlink(const char *s)` deletes the file named `s`. Warn if the operation fails.

`void pari_fread_chars(void *b, size_t n, FILE *f)` read `n` chars from stream `f`, storing the result in pre-allocated buffer `b` (assumed to be large enough).

`char* path_expand(const char *s)` perform tilde and environment expansion on `s`. Returns a malloc'ed buffer.

`void strftime_expand(const char *s, char *buf, long max)` perform time expansion on `s`, storing the result (at most `max` chars) in buffer `buf`. Trivial wrapper around

```
time_t t = time(NULL);
strftime(buf, max, s, localtime(&t));
```

`char* pari_get_homedir(const char *user)` expands `~user` constructs, returning the home directory of user `user`, or NULL if it could not be determined (in particular if the operating system has no such concept). The return value may point to static area and may be overwritten by subsequent system calls: use immediately or `strdup` it.

`int pari_stdin_isatty(void)` returns 1 if our standard input `stdin` is attached to a terminal. Trivial wrapper around `isatty`.

### 11.3.1 pariFILE.

PARI maintains a linked list of open files, to reclaim resources (file descriptors) on error or interrupts. The corresponding data structure is a `pariFILE`, which is a wrapper around a standard `FILE*`, containing further the file name, its type (regular file, pipe, input or output file, etc.). The following functions create and manipulate this structure; they are integrated within PARI's diagnostics system, printing messages if `DEBUGFILES` is high enough.

`pariFILE* pari_fopen(const char *s, const char *mode)` wrapper around `fopen(s, mode)`, return NULL on failure.

`pariFILE* pari_fopen_or_fail(const char *s, const char *mode)` simple wrapper around `fopen(s, mode)`; error on failure.

`pariFILE* pari_fopengz(const char *s)` opens the file whose name is `s`, and associates a (read-only) `pariFILE` with it. If `s` is a compressed file (`.gz` suffix), it is uncompressed on the fly. If `s` cannot be opened, also try to open `s.gz`. Returns NULL on failure.

`void pari_fclose(pariFILE *f)` closes the underlying file descriptor and deletes the `pariFILE` struct.

`pariFILE* pari_safefopen(const char *s, const char *mode)` creates a *new* file `s` (a priori for writing) with 600 permissions. Error if the file already exists. To avoid symlink attacks, a symbolic link exists, regardless of where it points to.

### 11.3.2 Temporary files.

PARI has its own idea of the system temp directory derived from an environment variable (`$GPTMPDIR`, else `$TMPDIR`), or the first writable directory among `/tmp`, `/var/tmp` and `..`.

`char* pari_unique_dir(const char *s)` creates a “unique directory” and return its name built from the string *s*, the user id and process pid (on Unix systems). This directory is itself located in the temp directory mentioned above. The name returned is `malloc`’ed.

`char* pari_unique_filename(const char *s)` creates a *new* empty file in the temp directory, whose name contains the id-string *s* (truncated to its first 8 chars), followed by a system-dependent suffix (incorporating the ids of both the user and the running process, for instance). The function returns the tempfile name and creates an empty file with that name. The name returned is `malloc`’ed.

`char* pari_unique_filename_suffix(const char *s, const char *suf)` analogous to above `pari_unique_filename`, creating a (previously non-existent) tempfile whose name ends with suffix *suf*.

## 11.4 Errors.

This section documents the various error classes, and the corresponding arguments to `pari_err`. The general syntax is

```
void pari_err(numerr, ...)
```

In the sequel, we mostly use sequences of arguments of the form

```
const char *s
const char *fmt, ...
```

where *fmt* is a PARI format, producing a string *s* from the remaining arguments. Since providing the correct arguments to `pari_err` is quite error-prone, we also provide specialized routines `pari_err_ERRORCLASS(...)` instead of `pari_err(e_ERRORCLASS, ...)` so that the C compiler can check their arguments.

We now inspect the list of valid keywords (error classes) for `numerr`, and the corresponding required arguments.

### 11.4.1 Internal errors, “system” errors.

**11.4.1.1 `e_ARCH`.** A requested feature *s* is not available on this architecture or operating system.

```
pari_err(e_ARCH)
```

prints the error message: `sorry, 's' not available on this system`.

**11.4.1.2 `e_BUG`.** A bug in the PARI library, in function *s*.

```
pari_err(e_BUG, const char *s)
pari_err_BUG(const char *s)
```

prints the error message: `Bug in s, please report`.

**11.4.1.3 e\_FILE.** Error while trying to open a file.

```
pari_err(e_FILE, const char *what, const char *name)
pari_err_FILE(const char *what, const char *name)
```

prints the error message: error opening *what*: '*name*'.

**11.4.1.4 e\_FILEDESC.** Error while handling a file descriptor.

```
pari_err(e_FILEDESC, const char *where, long n)
pari_err_FILEDESC(const char *where, long n)
```

prints the error message: invalid file descriptor in *where*: '*name*'.

**11.4.1.5 e\_IMPL.** A requested feature *s* is not implemented.

```
pari_err(e_IMPL, const char *s)
pari_err_IMPL(const char *s)
```

prints the error message: sorry, *s* is not yet implemented.

**11.4.1.6 e\_PACKAGE.** Missing optional package *s*.

```
pari_err(e_PACKAGE, const char *s)
pari_err_PACKAGE(const char *s)
```

prints the error message: package *s* is required, please install it

**11.4.2 Syntax errors, type errors.**

**11.4.2.1 e\_DIM.** arguments submitted to function *s* have inconsistent dimensions. E.g., when solving a linear system, or trying to compute the determinant of a non-square matrix.

```
pari_err(e_DIM, const char *s)
pari_err_DIM(const char *s)
```

prints the error message: inconsistent dimensions in *s*.

**11.4.2.2 e\_FLAG.** A flag argument is out of bounds in function *s*.

```
pari_err(e_FLAG, const char *s)
pari_err_FLAG(const char *s)
```

prints the error message: invalid flag in *s*.

**11.4.2.3 e\_NOTFUNC.** Generated by the PARI evaluator; tried to use a GEN which is not a t\_CLOSURE in a function call syntax (as in `f = 1; f(2);`).

```
pari_err(e_NOTFUNC, GEN fun)
```

prints the error message: not a function in a function call.

**11.4.2.4 e\_OP.** Impossible operation between two objects than cannot be typecast to a sensible common domain for deeper reasons than a type mismatch, usually for arithmetic reasons. As in  $0(2) + 0(3)$ : it is valid to add two t\_PADICs, provided the underlying prime is the same; so the addition is not forbidden a priori for type reasons, it only becomes so when inspecting the objects and trying to perform the operation.

```
pari_err(e_OP, const char *op, GEN x, GEN y)
pari_err_OP(const char *op, GEN x, GEN y)
```

As e\_TYPE2, replacing forbidden by inconsistent.

**11.4.2.5 e\_PRIORITY.** object  $o$  in function  $s$  contains variables whose priority is incompatible with the expected operation. E.g. `Pol([x,1], 'y')`: this raises an error because it's not possible to create a polynomial whose coefficients involve variables with higher priority than the main variable.

```
pari_err(e_PRIORITY, const char *s, GEN o, const char *op, long v)
pari_err_PRIORITY(const char *s, GEN o, const char *op, long v)
```

prints the error message: `incorrect priority in s, variable  $v_o$  op  $v$ , were  $v_o$  is gvar(o).`

**11.4.2.6 e\_SYNTAX.** Syntax error, generated by the PARI parser.

```
pari_err(e_SYNTAX, const char *msg, const char *e, const char *entry)
```

where `msg` is a complete error message, and `e` and `entry` point into the *same* character string, which is the input that was incorrectly parsed: `e` points to the character where the parser failed, and `entry`  $\leq$  `e` points somewhat before.

Prints the error message: `msg`, followed by a colon, then a part of the input character string (in general `entry` itself, but an initial segment may be truncated if `e - entry` is large); a caret points at `e`, indicating where the error took place.

**11.4.2.7 e\_TYPE.** An argument  $x$  of function  $s$  had an unexpected type. (As in `factor("blah").`)

```
pari_err(e_TYPE, const char *s, GEN x)
pari_err_TYPE(const char *s, GEN x)
```

prints the error message: `incorrect type in s ( $t_x$ )`, where  $t_x$  is the type of  $x$ .

**11.4.2.8 e\_TYPE2.** Forbidden operation between two objects than cannot be typecast to a sensible common domain, because their types do not match up. (As in `Mod(1,2) + Pi.`)

```
pari_err(e_TYPE2, const char *op, GEN x, GEN y)
pari_err_TYPE2(const char *op, GEN x, GEN y)
```

prints the error message: `forbidden  $s$   $t_x$  op  $t_y$` , where  $t_z$  denotes the type of  $z$ . Here,  $s$  denotes the spelled out name of the operator  $op \in \{+, *, /, \%, =\}$ , e.g. *addition* for "+" or *assignment* for "=". If  $op$  is not in the above operator, list, it is taken to be the already spelled out name of a function, e.g. "gcd", and the error message becomes `forbidden op  $t_x$ ,  $t_y$` .

**11.4.2.9 e\_VAR.** polynomials  $x$  and  $y$  submitted to function  $s$  have inconsistent variables. E.g., considering the algebraic number `Mod(t,t^2+1)` in `nfini(x^2+1)`.

```
pari_err(e_VAR, const char *s, GEN x, GEN y)
pari_err_VAR(const char *s, GEN x, GEN y)
```

prints the error message: `inconsistent variables in s  $X \neq Y$` , where  $X$  and  $Y$  are the names of the variables of  $x$  and  $y$ , respectively.

### 11.4.3 Overflows.

**11.4.3.1 e\_COMPONENT.** Trying to access an inexistent component of a vector/matrix/list: the index is less than 1 or greater than the allowed length.

```
pari_err(e_COMPONENT, const char *f, const char *op, GEN lim, GEN x)
pari_err_COMPONENT(const char *f, const char *op, GEN lim, GEN x)
```

prints the error message: `non-existent component in f: index op lim`. Special case: if  $f$  is the empty string (no meaningful public function name can be used), we ignore it and print the message: `non-existent component: index op lim`.

**11.4.3.2 e\_DOMAIN.** An argument  $x$  is not in the function's domain (as in `moebius(0)` or `zeta(1)`).

```
pari_err(e_DOMAIN, char *f, char *v, char *op, GEN lim, GEN x)
pari_err_DOMAIN(char *f, char *v, char *op, GEN lim, GEN x)
```

prints the error message: `domain error in f: v op lim`. Special case: if `op` is the empty string, we ignore `lim` and print the error message: `domain error in f: v out of range`.

**11.4.3.3 e\_MAXPRIME.** A function using the precomputed list of prime numbers ran out of primes.

```
pari_err(e_MAXPRIME, ulong c)
pari_err_MAXPRIME(ulong c)
```

prints the error message: `not enough precomputed primes, need primelimit ~c` if  $c$  is non-zero. And simply `not enough precomputed primes` otherwise.

**11.4.3.4 e\_MEM.** A call to `pari_malloc` or `pari_realloc` failed.

```
pari_err(e_MEM)
```

prints the error message: `not enough memory`.

**11.4.3.5 e\_OVERFLOW.** An object in function  $s$  becomes too large to be represented within PARI's hardcoded limits. (As in `2^2^2^10` or `exp(1e100)`, which overflow in `lg` and `expo`.)

```
pari_err(e_OVERFLOW, const char *s)
pari_err_OVERFLOW(const char *s)
```

prints the error message: `overflow in s`.

**11.4.3.6 e\_PREC.** Function  $s$  fails because input accuracy is too low. (As in `floor(1e100)` at default accuracy.)

```
pari_err(e_PREC, const char *s)
pari_err_PREC(const char *s)
```

prints the error message: `precision too low in s`.

**11.4.3.7 e\_STACK.** The PARI stack overflows.

```
pari_err(e_STACK)
```

prints the error message: `the PARI stack overflows !` as well as some statistics concerning stack usage.

#### 11.4.4 Errors triggered intentionally.

**11.4.4.1 e\_ALARM.** A timeout, generated by the `alarm` function.

```
pari_err(e_ALARM, const char *fmt, ...)
```

prints the error message: `s`.

**11.4.4.2 e\_USER.** A user error, as triggered by `error( $g_1, \dots, g_n$ )` in GP.

```
pari_err(e_USER, GEN g)
```

prints the error message: `user error:`, then the entries of the vector  $g$ .

### 11.4.5 Mathematical errors.

**11.4.5.1 e\_CONSTPOL.** An argument of function  $s$  is a constant polynomial, which does not make sense. (As in `galoisinit(Pol(1))`.)

```
pari_err(e_CONSTPOL, const char *s)
pari_err_CONSTPOL(const char *s)
```

prints the error message: `constant polynomial in s`.

**11.4.5.2 e\_COPRIME.** Function  $s$  expected two coprime arguments, and did receive  $x, y$  which were not.

```
pari_err(e_COPRIME, const char *s, GEN x, GEN y)
pari_err_COPRIME(const char *s, GEN x, GEN y)
```

prints the error message: `elements not coprime in s: x,y`.

**11.4.5.3 e\_INV.** Tried to invert a non-invertible object  $x$ .

```
pari_err(e_INV, const char *s, GEN x)
pari_err_INV(const char *s, GEN x)
```

prints the error message: `impossible inverse in s: x`. If  $x = \text{Mod}(a, b)$  is a `t_INTMOD` and  $a$  is not 0 mod  $b$ , this allows to factor the modulus, as  $\text{gcd}(a, b)$  is a non-trivial divisor of  $b$ .

**11.4.5.4 e\_IRREDPOL.** Function  $s$  expected an irreducible polynomial, and did not receive one. (As in `nfinit(x^2-1)`.)

```
pari_err(e_IRREDPOL, const char *s, GEN x)
pari_err_IRREDPOL(const char *s, GEN x)
```

prints the error message: `not an irreducible polynomial in s: x`.

**11.4.5.5 e\_MISC.** Generic uncategorized error.

```
pari_err(e_MISC, const char *fmt, ...)
```

prints the error message: `s`.

**11.4.5.6 e\_MODULUS.** moduli  $x$  and  $y$  submitted to function  $s$  are inconsistent. E.g., considering the algebraic number  $\text{Mod}(t, t^2+1)$  in `nfinit(t^3-2)`.

```
pari_err(e_MODULUS, const char *s, GEN x, GEN y)
pari_err_MODULUS(const char *s, GEN x, GEN y)
```

prints the error message: `inconsistent moduli in s, then the moduli`.

**11.4.5.7 e\_PRIME.** Function  $s$  expected a prime number, and did receive  $p$ , which was not. (As in `idealprimedec(nf, 4)`.)

```
pari_err(e_PRIME, const char *s, GEN x)
pari_err_PRIME(const char *s, GEN x)
```

prints the error message: `not a prime in s: x`.



**11.4.5.8 e\_ROOTS0.** An argument of function  $s$  is a zero polynomial, and we need to consider its roots. (As in `polroots(0)`.)

```
pari_err(e_ROOTS0, const char *s)
pari_err_ROOTS0(const char *s)
```

prints the error message: `zero polynomial in s`.

**11.4.5.9 e\_SQRTN.** Tried to compute an  $n$ -th root of  $x$ , which does not exist, in function  $s$ . (As in `sqrt(Mod(-1,3))`.)

```
pari_err(e_SQRTN, GEN x)
pari_err_SQRTN(GEN x)
```

prints the error message: `not an n-th power residue in s: x`.

## 11.4.6 Miscellaneous functions.

`long name_numerr(const char *s)` return the error number corresponding to an error name. E.g. `name_numerr("e_DIM")` returns `e_DIM`.

`const char* numerr_name(long errnum)` returns the error name corresponding to an error number. E.g. `name_numerr(e_DIM)` returns `"e_DIM"`.

`char* pari_err2str(GEN err)` returns the error message that would be printed on `t_ERROR err`. The name is allocated on the PARI stack and must not be freed.

## 11.5 Hashtables.

A **hashtable**, or associative array, is a set of pairs  $(k, v)$  of keys and values. PARI implements general extensible hashtables for fast data retrieval: when creating a table, we may either choose to use the PARI stack, or `malloc` so as to be stack-independent. A hashtable is implemented as a table of linked lists, each list containing all entries sharing the same hash value. The table length is a prime number, which roughly doubles as the table overflows by gaining new entries; both the current number of entries and the threshold before the table grows are stored in the table. Finally the table remembers the functions used to hash the entries's keys and to test for equality two entries hashed to the same value.

An entry, or **hashentry**, contains

- a key/value pair  $(k, v)$ , both of type `void*` for maximal flexibility,
- the hash value of the key, for the table hash function. This hash is mapped to a table index (by reduction modulo the table length), but it contains more information, and is used to bypass costly general equality tests if possible,
- a link pointer to the next entry sharing the same table cell.

```
typedef struct {
    void *key, *val;
    ulong hash; /* hash(key) */
    struct hashentry *next;
} hashentry;

typedef struct {
    ulong len; /* table length */
```

```

    hashentry **table; /* the table */
    ulong nb, maxnb; /* number of entries stored and max nb before enlarging */
    ulong pindex; /* prime index */
    ulong (*hash) (void *k); /* hash function */
    int (*eq) (void *k1, void *k2); /* equality test */
    int use_stack; /* use the PARI stack, resp. malloc */
} hashtable;

```

```

hashtable* hash_create(size, hash, eq, use_stack)

```

```

    ulong size;
    ulong (*hash)(void*);
    int (*eq)(void*,void*);
    int use_stack;

```

creates a hashtable with enough room to contain `size` entries. The functions `hash` and `eq` compute the hash value of keys and test keys for equality, respectively. If `use_stack` is non zero, the resulting table will use the PARI stack; otherwise, we use `malloc`.

`hashtable* hash_create_ulong(ulong size, long stack)` special case when the keys are `ulong`s with ordinary equality test.

`hashtable* hash_create_str(ulong size, long stack)` special case when the keys are character strings with string equality test (and `hash_str` hash function).

`void hash_init_GEN(hashtable *h, ulong size, int (*eq)(GEN, GEN), use_stack)` Initialize `h` for an hashtable with enough room to contain `size` entries of type `GEN`. The functions `eq` test keys for equality. If `use_stack` is non zero, the resulting table will use the PARI stack; otherwise, we use `malloc`. The hash used is `hash_GEN`.

`void hash_insert(hashtable *h, void *k, void *v)` inserts  $(k, v)$  in hashtable `h`. No copy is made: `k` and `v` themselves are stored. The implementation does not prevent one to insert two entries with equal keys `k`, but which of the two is affected by later commands is undefined.

`void hash_insert2(hashtable *h, void *k, void *v, ulong hash)` as `hash_insert`, assuming `h->hash(k)` is `hash`.

`void hash_insert_long(hashtable *h, void *k, long v)` as `hash_insert` but `v` is a `long`.

`hashentry* hash_search(hashtable *h, void *k)` look for an entry with key `k` in `h`. Return it if it one exists, and `NULL` if not.

`hashentry* hash_search2(hashtable *h, void *k, ulong hash)` as `hash_search` assuming `h->hash(k)` is `hash`.

`int hash_haskey_long(hashtable *h, void *k, long *v)` returns 1 if the key `k` belongs to the hash and set `v` to its value, otherwise returns 0.

`hashentry * hash_select(hashtable *h, void *k, void *E, int (*select)(void *, hashentry *))` variant of `hash_search`, useful when entries with identical keys are inserted: among the entries attached to key `k`, return one satisfying the selection criterion (such that `select(E, e)` is non-zero), or `NULL` if none exist.

`hashentry* hash_remove(hashtable *h, void *k)` deletes an entry  $(k, v)$  with key `k` from `h` and return it. (Return `NULL` if none was found.) Only the linking structures are freed, memory attached to `k` and `v` is not reclaimed.

`hashentry* hash_remove_select(hashtable *h, void *k, void *E, int(*select)(void*, hashentry *))` a variant of `hash_remove`, useful when entries with identical keys are inserted: among the entries attached to key  $k$ , return one satisfying the selection criterion (such that `select(E,e)` is non-zero) and delete it, or `NULL` if none exist. Only the linking structures are freed, memory attached to  $k$  and  $v$  is not reclaimed.

`GEN hash_keys(hashtable *h)` return in a `t_VECSMALL` the keys stored in hashtable  $h$ .

`GEN hash_values(hashtable *h)` return in a `t_VECSMALL` the values stored in hashtable  $h$ .

`void hash_destroy(hashtable *h)` deletes the hashtable, by removing all entries.

`void hash_dbg(hashtable *h)` print statistics for hashtable  $h$ , allows to evaluate the attached hash function performance on actual data.

Some interesting hash functions are available:

`ulong hash_str(const char *s)`

`ulong hash_str2(const char *s)` is the historical PARI string hashing function and seems to be generally inferior to `hash_str`.

`ulong hash_GEN(GEN x)`

## 11.6 Dynamic arrays.

A **dynamic array** is a generic way to manage stacks of data that need to grow dynamically. It allocates memory using `pari_malloc`, and is independent of the PARI stack; it even works before the `pari_init` call.

### 11.6.1 Initialization.

To create a stack of objects of type `foo`, we proceed as follows:

```
foo *t_foo;
pari_stack s_foo;
pari_stack_init(&s_foo, sizeof(*t_foo), (void**)&t_foo);
```

Think of `s_foo` as the controlling interface, and `t_foo` as the (dynamic) array tied to it. The value of `t_foo` may be changed as you add more elements.

**11.6.2 Adding elements.** The following function pushes an element on the stack.

```
/* access globals t_foo and s_foo */
void push_foo(foo x)
{
    long n = pari_stack_new(&s_foo);
    t_foo[n] = x;
}
```

### 11.6.3 Accessing elements.

Elements are accessed naturally through the `t_foo` pointer. For example this function swaps two elements:

```
void swapfoo(long a, long b)
{
    foo x;
    if (a > s_foo.n || b > s_foo.n) pari_err_BUG("swapfoo");
    x          = t_foo[a];
    t_foo[a] = t_foo[b];
    t_foo[b] = x;
}
```

**11.6.4 Stack of stacks.** Changing the address of `t_foo` is not supported in general. In particular `realloc()`'ed array of stacks and stack of stacks are not supported.

**11.6.5 Public interface.** Let `s` be a `pari_stack` and `data` the data linked to it. The following public fields are defined:

- `s.alloc` is the number of elements allocated for `data`.
- `s.n` is the number of elements in the stack and `data[s.n-1]` is the topmost element of the stack. `s.n` can be changed as long as  $0 \leq s.n \leq s.alloc$  holds.

`void pari_stack_init(pari_stack *s, size_t size, void **data)` links `*s` to the data pointer `*data`, where `size` is the size of data element. The pointer `*data` is set to `NULL`, `s->n` and `s->alloc` are set to 0: the array is empty.

`void pari_stack_alloc(pari_stack *s, long nb)` makes room for `nb` more elements, i.e. makes sure that  $s.alloc \geq s.n + nb$ , possibly reallocating `data`.

`long pari_stack_new(pari_stack *s)` increases `s.n` by one unit, possibly reallocating `data`, and returns `s.n - 1`.

**Caveat.** The following construction is incorrect because `stack_new` can change the value of `t_foo`:

```
t_foo[ pari_stack_new(&s_foo) ] = x;
```

`void pari_stack_delete(pari_stack *s)` frees `data` and resets the stack to the state immediately following `stack_init` (`s->n` and `s->alloc` are set to 0).

`void * pari_stack_pushp(pari_stack *s, void *u)` This function assumes that `*data` is of pointer type. Pushes the element `u` on the stack `s`.

`void ** pari_stack_base(pari_stack *s)` returns the address of `data`, typecast to a `void **`.

## 11.7 Vectors and Matrices.

**11.7.1 Access and extract.** See Section 9.3.1 and Section 9.3.2 for various useful constructors. Coefficients are accessed and set using `gel`, `gcoeff`, see Section 5.2.7. There are many internal functions to extract or manipulate subvectors or submatrices but, like the accessors above, none of them are suitable for `gerepileupto`. Worse, there are no type verification, nor bound checking, so use at your own risk.

`GEN shallowcopy(GEN x)` returns a GEN whose components are the components of  $x$  (no copy is made). The result may now be used to compute in place without destroying  $x$ . This is essentially equivalent to

```
GEN y = cgetg(lg(x), typ(x));
for (i = 1; i < lg(x); i++) y[i] = x[i];
return y;
```

except that `t_MAT` is treated specially since shallow copies of all columns are made. The function also works for non-recursive types, but is useless in that case since it makes a deep copy. If  $x$  is known to be a `t_MAT`, you may call `RgM_shallowcopy` directly; if  $x$  is known not to be a `t_MAT`, you may call `leafcopy` directly.

`GEN RgM_shallowcopy(GEN x)` returns `shallowcopy(x)`, where  $x$  is a `t_MAT`.

`GEN shallowtrans(GEN x)` returns the transpose of  $x$ , *without* copying its components, i. e., it returns a GEN whose components are (physically) the components of  $x$ . This is the internal function underlying `gtrans`.

`GEN shallowconcat(GEN x, GEN y)` concatenate  $x$  and  $y$ , *without* copying components, i. e., it returns a GEN whose components are (physically) the components of  $x$  and  $y$ .

`GEN shallowconcat1(GEN x)`  $x$  must be `t_VEC` or `t_LIST`, concatenate its elements from left to right. Shallow version of `gconcat1`.

`GEN shallowmatconcat(GEN v)` shallow version of `matconcat`.

`GEN shallowextract(GEN x, GEN y)` extract components of the vector or matrix  $x$  according to the selection parameter  $y$ . This is the shallow analog of `extract0(x, y, NULL)`, see `vecextract`.

`GEN shallowmatextract(GEN M, GEN l1, GEN l2)` extract components of the matrix  $M$  according to the `t_VECSMALL`  $l1$  (list of lines indices) and  $l2$  (list of columns indices). This is the shallow analog of `extract0(x, l1, l2)`, see `vecextract`.

`GEN RgM_minor(GEN A, long i, long j)` given a square `t_MAT`  $A$ , return the matrix with  $i$ -th row and  $j$ -th column removed.

`GEN vconcat(GEN A, GEN B)` concatenate vertically the two `t_MAT`  $A$  and  $B$  of compatible dimensions. A NULL pointer is accepted for an empty matrix. See `shallowconcat`.

`GEN matslice(GEN A, long a, long b, long c, long d)` returns the submatrix  $A[a..b, c..d]$ . Assume  $a \leq b$  and  $c \leq d$ .

`GEN row(GEN A, long i)` return  $A[i, ]$ , the  $i$ -th row of the `t_MAT`  $A$ .

`GEN row_i(GEN A, long i, long j1, long j2)` return part of the  $i$ -th row of `t_MAT`  $A$ :  $A[i, j_1], A[i, j_1 + 1] \dots, A[i, j_2]$ . Assume  $j_1 \leq j_2$ .

GEN rowcopy(GEN A, long i) return the row  $A[i,]$  of the  $\mathbf{t\_MAT}$   $A$ . This function is memory clean and suitable for `gerepileupto`. See `row` for the shallow equivalent.

GEN rowslice(GEN A, long i1, long i2) return the  $\mathbf{t\_MAT}$  formed by the  $i_1$ -th through  $i_2$ -th rows of  $\mathbf{t\_MAT}$   $A$ . Assume  $i_1 \leq i_2$ .

GEN rowsplice(GEN A, long i) return the  $\mathbf{t\_MAT}$  formed from the coefficients of  $\mathbf{t\_MAT}$   $A$  with  $j$ -th row removed.

GEN rowpermute(GEN A, GEN p),  $p$  being a  $\mathbf{t\_VECSMALL}$  representing a list  $[p_1, \dots, p_n]$  of rows of  $\mathbf{t\_MAT}$   $A$ , returns the matrix whose rows are  $A[p_1,], \dots, A[p_n,]$ .

GEN rowslicepermute(GEN A, GEN p, long x1, long x2), short for

`rowslice(rowpermute(A,p), x1, x2)`

(more efficient).

GEN vecslice(GEN A, long j1, long j2), return  $A[j_1], \dots, A[j_2]$ . If  $A$  is a  $\mathbf{t\_MAT}$ , these correspond to *columns* of  $A$ . The object returned has the same type as  $A$  ( $\mathbf{t\_VEC}$ ,  $\mathbf{t\_COL}$  or  $\mathbf{t\_MAT}$ ). Assume  $j_1 \leq j_2$ .

GEN vecsplice(GEN A, long j) return  $A$  with  $j$ -th entry removed ( $\mathbf{t\_VEC}$ ,  $\mathbf{t\_COL}$ ) or  $j$ -th column removed ( $\mathbf{t\_MAT}$ ).

GEN vecreverse(GEN A). Returns a GEN which has the same type as  $A$  ( $\mathbf{t\_VEC}$ ,  $\mathbf{t\_COL}$  or  $\mathbf{t\_MAT}$ ), and whose components are the  $A[n], \dots, A[1]$ . If  $A$  is a  $\mathbf{t\_MAT}$ , these are the *columns* of  $A$ .

void vecreverse\_inplace(GEN A) as `vecreverse`, but reverse  $A$  in place.

GEN vecpermute(GEN A, GEN p)  $p$  is a  $\mathbf{t\_VECSMALL}$  representing a list  $[p_1, \dots, p_n]$  of indices. Returns a GEN which has the same type as  $A$  ( $\mathbf{t\_VEC}$ ,  $\mathbf{t\_COL}$  or  $\mathbf{t\_MAT}$ ), and whose components are  $A[p_1], \dots, A[p_n]$ . If  $A$  is a  $\mathbf{t\_MAT}$ , these are the *columns* of  $A$ .

GEN vecsmallpermute(GEN A, GEN p) as `vecpermute` when  $A$  is a  $\mathbf{t\_VECSMALL}$ .

GEN vecslicepermute(GEN A, GEN p, long y1, long y2) short for

`vecslice(vecpermute(A,p), y1, y2)`

(more efficient).

### 11.7.2 Componentwise operations.

The following convenience routines automate trivial loops of the form

`for (i = 1; i < lg(a); i++) gel(v,i) = f(gel(a,i), gel(b,i))`

for suitable  $f$ :

GEN vecinv(GEN a). Given a vector  $a$ , returns the vector whose  $i$ -th component is  $\text{ginv}(a[i])$ .

GEN vecmul(GEN a, GEN b). Given  $a$  and  $b$  two vectors of the same length, returns the vector whose  $i$ -th component is  $\text{gmul}(a[i], b[i])$ .

GEN vecdiv(GEN a, GEN b). Given  $a$  and  $b$  two vectors of the same length, returns the vector whose  $i$ -th component is  $\text{gdiv}(a[i], b[i])$ .

GEN vecpow(GEN a, GEN n). Given  $n$  a  $\mathbf{t\_INT}$ , returns the vector whose  $i$ -th component is  $a[i]^n$ .

`GEN vecmodii(GEN a, GEN b)`. Assuming  $a$  and  $b$  are two ZV of the same length, returns the vector whose  $i$ -th component is `modii(a[i], b[i])`.

`GEN vecmoduu(GEN a, GEN b)`. Assuming  $a$  and  $b$  are two `t_VECSMALL` of the same length, returns the vector whose  $i$ -th component is `a[i] % b[i]`.

Note that `vecadd` or `vecsub` do not exist since `gadd` and `gsub` have the expected behavior. On the other hand, `ginv` does not accept vector types, hence `vecinv`.

### 11.7.3 Low-level vectors and columns functions.

These functions handle `t_VEC` as an abstract container type of GENs. No specific meaning is attached to the content. They accept both `t_VEC` and `t_COL` as input, but `col` functions always return `t_COL` and `vec` functions always return `t_VEC`.

**Note.** All the functions below are shallow.

`GEN const_col(long n, GEN x)` returns a `t_COL` of  $n$  components equal to  $x$ .

`GEN const_vec(long n, GEN x)` returns a `t_VEC` of  $n$  components equal to  $x$ .

`int vec_isconst(GEN v)` Returns 1 if all the components of  $v$  are equal, else returns 0.

`void vec_setconst(GEN v, GEN x)`  $v$  a pre-existing vector. Set all its components to  $x$ .

`int vec_is1to1(GEN v)` Returns 1 if the components of  $v$  are pair-wise distinct, i.e. if  $i \mapsto v[i]$  is a 1-to-1 mapping, else returns 0.

`GEN vec_append(GEN V, GEN s)` append  $s$  to the vector  $V$ .

`GEN vec_prepend(GEN V, GEN s)` prepend  $s$  to the vector  $V$ .

`GEN vec_shorten(GEN v, long n)` shortens the vector  $v$  to  $n$  components.

`GEN vec_lengthen(GEN v, long n)` lengthens the vector  $v$  to  $n$  components. The extra components are not initialized.

`GEN vec_insert(GEN v, long n, GEN x)` inserts  $x$  at position  $n$  in the vector  $v$ .

## 11.8 Vectors of small integers.

### 11.8.1 t\_VECSMALL.

These functions handle `t_VECSMALL` as an abstract container type of small signed integers. No specific meaning is attached to the content.

`GEN const_vecsmall(long n, long c)` returns a `t_VECSMALL` of  $n$  components equal to  $c$ .

`GEN vec_to_vecsmall(GEN z)` identical to `ZV_to_zv(z)`.

`GEN vecsmall_to_vec(GEN z)` identical to `zv_to_ZV(z)`.

`GEN vecsmall_to_col(GEN z)` identical to `zv_to_ZC(z)`.

`GEN vecsmall_to_vec_inplace(GEN z)` apply `stoi` to all entries of  $z$  and set its type to `t_VEC`.

`GEN vecsmall_copy(GEN x)` makes a copy of  $x$  on the stack.

`GEN vecsmall_shorten(GEN v, long n)` shortens the `t_VECSMALL`  $v$  to  $n$  components.

`GEN vecsmall_lengthen(GEN v, long n)` lengthens the `t_VECSMALL v` to `n` components. The extra components are not initialized.

`GEN vecsmall_indexsort(GEN x)` performs an indirect sort of the components of the `t_VECSMALL x` and return a permutation stored in a `t_VECSMALL`.

`void vecsmall_sort(GEN v)` sorts the `t_VECSMALL v` in place.

`void vecsmall_reverse(GEN v)` as `vecreverse` for a `t_VECSMALL v`.

`long vecsmall_max(GEN v)` returns the maximum of the elements of `t_VECSMALL v`, assumed non-empty.

`long vecsmall_indexmax(GEN v)` returns the index of the largest element of `t_VECSMALL v`, assumed non-empty.

`long vecsmall_min(GEN v)` returns the minimum of the elements of `t_VECSMALL v`, assumed non-empty.

`long vecsmall_indexmin(GEN v)` returns the index of the smallest element of `t_VECSMALL v`, assumed non-empty.

`long vecsmall_isin(GEN v, long x)` returns the first index  $i$  such that  $v[i]$  is equal to  $x$ . Naive search in linear time, does not assume that  $v$  is sorted.

`GEN vecsmall_uniq(GEN v)` given a `t_VECSMALL v`, return the vector of unique occurrences.

`GEN vecsmall_uniq_sorted(GEN v)` same as `vecsmall_uniq`, but assumes  $v$  sorted.

`long vecsmall_duplicate(GEN v)` given a `t_VECSMALL v`, return 0 if there is no duplicates, or the index of the first duplicate (`vecsmall_duplicate([1,1])` returns 2).

`long vecsmall_duplicate_sorted(GEN v)` same as `vecsmall_duplicate`, but assume  $v$  sorted.

`int vecsmall_lexcmp(GEN x, GEN y)` compares two `t_VECSMALL` lexically.

`int vecsmall_prefixcmp(GEN x, GEN y)` truncate the longest `t_VECSMALL` to the length of the shortest and compares them lexicographically.

`GEN vecsmall_prepend(GEN V, long s)` prepend  $s$  to the `t_VECSMALL V`.

`GEN vecsmall_append(GEN V, long s)` append  $s$  to the `t_VECSMALL V`.

`GEN vecsmall_concat(GEN u, GEN v)` concat the `t_VECSMALL u` and  $v$ .

`long vecsmall_coincidence(GEN u, GEN v)` returns the numbers of indices where  $u$  and  $v$  agree.

`long vecsmall_pack(GEN v, long base, long mod)` handles the `t_VECSMALL v` as the digit of a number in base  $base$  and return this number modulo  $mod$ . This can be used as an hash function.

`GEN vecsmall_prod(GEN v)` given a `t_VECSMALL v`, return the product of its entries.



**11.8.2 Vectors of `t_VECSMALL`.** These functions manipulate vectors of `t_VECSMALL` (`vecvecsmall`).

`GEN vecvecsmall_sort(GEN x)` sorts lexicographically the components of the vector `x`.

`GEN vecvecsmall_sort_uniq(GEN x)` sorts lexicographically the components of the vector `x`, removing duplicates entries.

`GEN vecvecsmall_indexsort(GEN x)` performs an indirect lexicographic sorting of the components of the vector `x` and return a permutation stored in a `t_VECSMALL`.

`long vecvecsmall_search(GEN x, GEN y, long flag)` `x` being a sorted `vecvecsmall` and `y` a `t_VECSMALL`, search `y` inside `x`. `flag` has the same meaning as for `setsearch`.

`GEN vecvecsmall_max(GEN x)` returns the largest entry in all `x[i]`, assumed non-empty.



## Chapter 12:

### Functions related to the GP interpreter

#### 12.1 Handling closures.

##### 12.1.1 Functions to evaluate `t_CLOSURE`.

`void closure_disassemble(GEN C)` print the `t_CLOSURE C` in GP assembly format.

`GEN closure_callgenall(GEN C, long n, ...)` evaluate the `t_CLOSURE C` with the `n` arguments (of type `GEN`) following `n` in the function call. Assumes `C` has arity  $\geq n$ .

`GEN closure_callgenvec(GEN C, GEN args)` evaluate the `t_CLOSURE C` with the arguments supplied in the vector `args`. Assumes `C` has arity  $\geq \text{lg}(\text{args}) - 1$ .

`GEN closure_callgenvecprec(GEN C, GEN args, long prec)` as `closure_callgenvec` but set the precision locally to `prec`.

`GEN closure_callgen1(GEN C, GEN x)` evaluate the `t_CLOSURE C` with argument `x`. Assumes `C` has arity  $\geq 1$ .

`GEN closure_callgen1prec(GEN C, GEN x, long prec)` as `closure_callgen1`, but set the precision locally to `prec`.

`GEN closure_callgen2(GEN C, GEN x, GEN y)` evaluate the `t_CLOSURE C` with argument `x, y`. Assumes `C` has arity  $\geq 2$ .

`void closure_callvoid1(GEN C, GEN x)` evaluate the `t_CLOSURE C` with argument `x` and discard the result. Assumes `C` has arity  $\geq 1$ .

The following technical functions are used to evaluate *inline* closures and closures of arity 0.

The control flow statements (`break`, `next` and `return`) will cause the evaluation of the closure to be interrupted; this is called below a *flow change*. When that occurs, the functions below generally return `NULL`. The caller can then adopt three positions:

- raises an exception (`closure_evalnobrk`).
- passes through (by returning `NULL` itself).
- handles the flow change.

`GEN closure_evalgen(GEN code)` evaluates a closure and returns the result, or `NULL` if a flow change occurred.

`GEN closure_evalnobrk(GEN code)` as `closure_evalgen` but raise an exception if a flow change occurs. Meant for iterators where interrupting the closure is meaningless, e.g. `intnum` or `sumnum`.

`void closure_evalvoid(GEN code)` evaluates a closure whose return value is ignored. The caller has to deal with eventual flow changes by calling `loop_break`.

The remaining functions below are for exceptional situations:

`GEN closure_evalres(GEN code)` evaluates a closure and returns the result. The difference with `closure_evalgen` being that, if the flow end by a `return` statement, the result will be the returned value instead of `NULL`. Used by the main GP loop.

`GEN closure_evalbrk(GEN code, long *status)` as `closure_evalres` but set `status` to a non-zero value if a flow change occurred. This variant is not stack clean. Used by the break loop.

`GEN closure_trapgen(long numerr, GEN code)` evaluates closure, while trapping error `numerr`. Return `(GEN)1L` if error trapped, and the result otherwise, or `NULL` if a flow change occurred. Used by trap.

### 12.1.2 Functions to handle control flow changes.

`long loop_break(void)` processes an eventual flow changes inside an iterator. If this function return 1, the iterator should stop.

### 12.1.3 Functions to deal with lexical local variables.

Function using the prototype code ‘V’ need to manually create and delete a lexical variable for each code ‘V’, which will be given a number  $-1, -2, \dots$

`void push_lex(GEN a, GEN code)` creates a new lexical variable whose initial value is  $a$  on the top of the stack. This variable get the number  $-1$ , and the number of the other variables is decreased by one unit. When the first variable of a closure is created, the argument `code` must be the closure that references this lexical variable. The argument `code` must be `NULL` for all subsequent variables (if any). (The closure contains the debugging data for the variable).

`void pop_lex(long n)` deletes the  $n$  topmost lexical variables, increasing the number of other variables by  $n$ . The argument  $n$  must match the number of variables allocated through `push_lex`.

`GEN get_lex(long vn)` get the value of the variable with number  $vn$ .

`void set_lex(long vn, GEN x)` set the value of the variable with number  $vn$ .

### 12.1.4 Functions returning new closures.

`GEN compile_str(const char *s)` returns the closure corresponding to the GP expression  $s$ .

`GEN closure_deriv(GEN code)` returns a closure corresponding to the numerical derivative of the closure `code`.

`GEN snm_closure(entree *ep, GEN data)` Let `data` be a vector of length  $m$ , `ep` be an `entree` pointing to a C function  $f$  of arity  $n + m$ , returns a `t_CLOSURE` object  $g$  of arity  $n$  such that  $g(x_1, \dots, x_n) = f(x_1, \dots, x_n, gel(data, 1), \dots, gel(data, m))$ . If `data` is `NULL`, then  $m = 0$  is assumed. This function has a low overhead since it does not copy `data`.

`GEN strtofunction(char *str)` returns a closure corresponding to the built-in or install'ed function named `str`.

`GEN strtoclosure(char *str, long n, ...)` returns a closure corresponding to the built-in or install'ed function named `str` with the  $n$  last parameters set to the  $n$  GENs following  $n$ , see `snm_closure`. This function has an higher overhead since it copies the parameters and does more input validation.

In the example code below, `agm1` is set to the function `x->agm(x,1)` and `res` is set to `agm(2,1)`.

```
GEN agm1 = strtoclosure("agm",1, gen_1);
GEN res = closure_callgen1(agm1, gen_2);
```

**12.1.5 Functions used by the gp debugger (break loop).** `long closure_context(long s)` restores the compilation context starting at frame `s+1`, and returns the index of the topmost frame. This allow to compile expressions in the topmost lexical scope.

`void closure_err(long level)` prints a backtrace of the last 20 stack frames, starting at frame `level`, the numbering starting at 0.

**12.1.6 Standard wrappers for iterators.** Two families of standard wrappers are provided to interface iterators like `intnum` or `sumnum` with GP.

**12.1.6.1 Standard wrappers for inline closures.** These wrappers are used to implement GP functions taking inline closures as input. The object `(GEN)E` must be an inline closure which is evaluated with the lexical variable number `-1` set to  $x$ .

`GEN gp_eval(void *E, GEN x)` is used for the prototype code ‘E’.

`GEN gp_evalprec(void *E, GEN x, long prec)` as `gp_eval`, but set the precision locally to `prec`.

`long gp_evalvoid(void *E, GEN x)` is used for the prototype code ‘I’. The resulting value is discarded. Return a non-zero value if a control-flow instruction request the iterator to terminate immediately.

`long gp_evalbool(void *E, GEN x)` returns the boolean `gp_eval(E, x)` evaluates to (i.e. true iff the value is non-zero).

`GEN gp_evalupto(void *E, GEN x)` memory-safe version of `gp_eval`, `gcopy`-ing the result, when the evaluator returns components of previously allocated objects (e.g. member functions).

**12.1.6.2 Standard wrappers for true closures.** These wrappers are used to implement GP functions taking true closures as input.

`GEN gp_call(void *E, GEN x)` evaluates the closure `(GEN)E` on  $x$ .

`GEN gp_callprec(void *E, GEN x, long prec)` as `gp_call`, but set the precision locally to `prec`.

`GEN gp_call2(void *E, GEN x, GEN y)` evaluates the closure `(GEN)E` on  $(x, y)$ .

`long gp_callbool(void *E, GEN x)` evaluates the closure `(GEN)E` on  $x$ , returns 1 if its result is non-zero, and 0 otherwise.

`long gp_callvoid(void *E, GEN x)` evaluates the closure `(GEN)E` on  $x$ , discarding the result. Return a non-zero value if a control-flow instruction request the iterator to terminate immediately.

## 12.2 Defaults.

`entree* pari_is_default(const char *s)` return the `entree` structure attached to `s` if it is the name of a default, `NULL` otherwise.

`GEN setdefault(const char *s, const char *v, long flag)` is the low-level function underlying `default0`. If `s` is `NULL`, call all default setting functions with string argument `NULL` and flag `d_ACKNOWLEDGE`. Otherwise, check whether `s` corresponds to a default and call the corresponding default setting function with arguments `v` and `flag`.

We shall describe these functions below: if `v` is `NULL`, we only look at the default value (and possibly print or return it, depending on `flag`); otherwise the value of the default to `v`, possibly after some translation work. The flag is one of

- `d_INITRC` called while reading the `gprc`: print and return `gnil`, possibly defer until `gp` actually starts.

- `d_RETURN` return the current value, as a `t_INT` if possible, as a `t_STR` otherwise.

- `d_ACKNOWLEDGE` print the current value, return `gnil`.

- `d_SILENT` print nothing, return `gnil`.

Low-level functions called by `setdefault`:

`GEN sd_TeXstyle(const char *v, long flag)`

`GEN sd_breakloop(const char *v, long flag)`

`GEN sd_colors(const char *v, long flag)`

`GEN sd_compatible(const char *v, long flag)`

`GEN sd_datadir(const char *v, long flag)`

`GEN sd_debug(const char *v, long flag)`

`GEN sd_debugfiles(const char *v, long flag)`

`GEN sd_debugmem(const char *v, long flag)`

`GEN sd_echo(const char *v, long flag)`

`GEN sd_factor_add_primes(const char *v, long flag)`

`GEN sd_factor_proven(const char *v, long flag)`

`GEN sd_format(const char *v, long flag)`

`GEN sd_graphcolormap(const char *v, long flag)`

`GEN sd_graphcolors(const char *v, long flag)`

`GEN sd_help(const char *v, long flag)`

`GEN sd_histfile(const char *v, long flag)`

`GEN sd_histsize(const char *v, long flag)`

`GEN sd_lines(const char *v, long flag)`

`GEN sd_linewrap(const char *v, long flag)`

```

GEN sd_log(const char *v, long flag)
GEN sd_logfile(const char *v, long flag)
GEN sd_nbthreads(const char *v, long flag)
GEN sd_new_galois_format(const char *v, long flag)
GEN sd_output(const char *v, long flag)
GEN sd_parisize(const char *v, long flag)
GEN sd_parisizemax(const char *v, long flag)
GEN sd_path(const char *v, long flag)
GEN sd_plothsizes(const char *v, long flag)
GEN sd_prettyprinter(const char *v, long flag)
GEN sd_primelimit(const char *v, long flag)
GEN sd_prompt(const char *v, long flag)
GEN sd_prompt_cont(const char *v, long flag)
GEN sd_psfile(const char *v, long flag) The psfile default is obsolete, don't use this func-
tion.
GEN sd_readline(const char *v, long flag)
GEN sd_realbitprecision(const char *v, long flag)
GEN sd_realprecision(const char *v, long flag)
GEN sd_recover(const char *v, long flag)
GEN sd_secure(const char *v, long flag)
GEN sd_seriesprecision(const char *v, long flag)
GEN sd_simplify(const char *v, long flag)
GEN sd_sopath(const char *v, int flag)
GEN sd_strictargs(const char *v, long flag)
GEN sd_strictmatch(const char *v, long flag)
GEN sd_timer(const char *v, long flag)
GEN sd_threadsize(const char *v, long flag)
GEN sd_threadsizemax(const char *v, long flag)

```

Generic functions used to implement defaults: most of the above routines are implemented in terms of the following generic ones. In all routines below

- **v** and **flag** are the arguments passed to **default**: **v** is a new value (or the empty string: no change), and **flag** is one of **d\_INITRC**, **d\_RETURN**, etc.
- **s** is the name of the default being changed, used to display error messages or acknowledgements.

GEN sd\_toggle(const char \*v, long flag, const char \*s, int \*ptn)

- if *v* is neither "0" nor "1", an error is raised using `pari_err`.
- *ptn* points to the current numerical value of the toggle (1 or 0), and is set to the new value (when *v* is non-empty).

For instance, here is how the timer default is implemented internally:

```
GEN
sd_timer(const char *v, long flag)
{ return sd_toggle(v,flag,"timer", &(GP_DATA->chrono)); }
```

The exact behavior and return value depends on *flag*:

- *d\_RETURN*: returns the new toggle value, as a GEN.
- *d\_ACKNOWLEDGE*: prints a message indicating the new toggle value and return `gnil`.
- other cases: print nothing and return `gnil`.

GEN sd\_ulong(const char \*v, long flag, const char \*s, ulong \*ptn, ulong Min, ulong Max, const char \*\*msg)

- *ptn* points to the current numerical value of the toggle, and is set to the new value (when *v* is non-empty).

- *Min* and *Max* point to the minimum and maximum values allowed for the default.

• *v* must translate to an integer in the allowed ranger, a suffix among *k*/*K* ( $\times 10^3$ ), *m*/*M* ( $\times 10^6$ ), or *g*/*G* ( $\times 10^9$ ) is allowed, but no arithmetic expression.

• *msg* is a [NULL]-terminated array of messages or NULL (ignored). If *msg* is not NULL, *msg*[*i*] contains a message attached to the value *i* of the default. The last entry in the *msg* array is used as a message attached to all subsequent ones.

The exact behavior and return value depends on *flag*:

- *d\_RETURN*: returns the new value, as a GEN.
- *d\_ACKNOWLEDGE*: prints a message indicating the new value, possibly a message attached to it via the *msg* argument, and return `gnil`.
- other cases: print nothing and return `gnil`.

GEN sd\_intarray(const char \*v, long flag, const char \*s, GEN \*pz)

- records a *t\_VECSMALL* array of non-negative integers.
- *pz* points to the current *t\_VECSMALL* value, and is set to the new value (when *v* is non-empty).

The exact return value depends on *flag*:

- *d\_RETURN*: returns the new value, as a *t\_VEC* (converted via *zv\_to\_ZV*)
- *d\_ACKNOWLEDGE*: prints a message indicating the new value, (as a *t\_VEC*) and return `gnil`.
- other cases: print nothing and return `gnil`.

GEN sd\_string(const char \*v, long flag, const char \*s, char \*\*pstr) • *v* is subject to environment expansion, then time expansion.

- *pstr* points to the current string value, and is set to the new value (when *v* is non-empty).



## 12.3 Records and Lazy vectors.

The functions in this section are used to implement `ell` structures and analogous objects, which are vectors some of whose components are initialized to dummy values, later computed on demand. We start by initializing the structure:

`GEN obj_init(long d, long n)` returns an *obj*  $S$ , a `t_VEC` with  $d$  regular components, accessed as `gel(S,1), ..., gel(S,d)`; together with a record of  $n$  members, all initialized to 0. The arguments  $d$  and  $n$  must be non-negative.

After  $S = \text{obj\_init}(d, n)$ , the prototype of our other functions are of the form

`GEN obj_do(GEN S, long tag, ...)`

The first argument  $S$  holds the structure to be managed. The second argument *tag* is the index of the struct member (from 1 to  $n$ ) we operate on. We recommend to define an `enum` and use descriptive names instead of hardcoded numbers. For instance, if  $n = 3$ , after defining

```
enum { TAG_p = 1, TAG_list, TAG_data };
```

one may use `TAG_list` or 2 indifferently as a tag. The former being preferred, of course.

**Technical note.** In the current implementation,  $S$  is a `t_VEC` with  $d + 1$  entries. The first  $d$  components are ordinary `t_GEN` entries, which you can read or assign to in the customary way. But the last component `gel(S,d + 1)`, a `t_VEC` of length  $n$  initialized to `zerovec(n)`, must be handled in a special way: you should never access or modify its components directly, only through the API we are about to describe. Indeed, its entries are meant to contain dynamic data, which will be stored, retrieved and replaced (for instance by a value computed to a higher accuracy), while interacting safely with intermediate `gerepile` calls. This mechanism allows to simulate C `structs`, in a simpler way than with general hashtables, while remaining compatible with the GP language, which knows neither `structs` nor hashtables. It also serialize the structure in an ordinary `GEN`, which facilitates copies and garbage collection (use `gcopy` or `gerepile`), rather than having to deal with individual components of actual C `structs`.

`GEN obj_reinit(GEN S)` make a shallow copy of  $S$ , re-initializing all dynamic components. This allows “forking” a lazy vector while avoiding both a memory leak, and storing pointers to the same data in different objects (with risks of a double free later).

`GEN obj_check(GEN S, long tag)` if the *tag*-component in  $S$  is non empty, return it. Otherwise return `NULL`. The `t_INT 0` (initial value) is used as a sentinel to indicated an empty component.

`GEN obj_insert(GEN S, long tag, GEN O)` insert (a clone of)  $O$  as *tag*-component of  $S$ . Any previous value is deleted, and data pointing to it become invalid.

`GEN obj_insert_shallow(GEN S, long K, GEN O)` as `obj_insert`, inserting  $O$  as-is, not via a clone.

`GEN obj_checkbuild(GEN S, long tag, GEN (*build)(GEN))` if the *tag*-component of  $S$  is non empty, return it. Otherwise insert (a clone of) `build(S)` as *tag*-component in  $S$ , and return it.

`GEN obj_checkbuild_padicprec(GEN S, long tag, GEN (*build)(GEN, long), long prec)` if the *tag*-component of  $S$  is non empty *and* has relative  $p$ -adic precision  $\geq \text{prec}$ , return it. Otherwise insert (a clone of) `build(S, prec)` as *tag*-component in  $S$ , and return it.

`GEN obj_checkbuild_realprec(GEN S, long tag, GEN (*build)(GEN, long), long prec)` if the *tag*-component of  $S$  is non empty *and* satisfies `gprecision`  $\geq \text{prec}$ , return it. Otherwise insert (a clone of) `build(S, prec)` as *tag*-component in  $S$ , and return it.

`GEN obj_checkbuild_prec(GEN S, long tag, GEN (*build)(GEN, long), GEN (*gpr)(GEN), long prec)` if the *tag*-component of *S* is non empty *and* has precision  $\text{gpr}(x) \geq \text{prec}$ , return it. Otherwise insert (a clone of) `build(S, prec)` as *tag*-component in *S*, and return it.

`void obj_free(GEN S)` destroys all clones stored in the *n* tagged components, and replace them by the initial value 0. The regular entries of *S* are unaffected, and *S* remains a valid object. This is used to avoid memory leaks.

# Chapter 13:

## Algebraic Number Theory

### 13.1 General Number Fields.

#### 13.1.1 Number field types.

None of the following routines thoroughly check their input: they distinguish between *bona fide* structures as output by PARI routines, but designing perverse data will easily fool them. To give an example, a square matrix will be interpreted as an ideal even though the  $\mathbf{Z}$ -module generated by its columns may not be an  $\mathbf{Z}_K$ -module (i.e. the expensive `nfisideal` routine will *not* be called).

`long nftyp(GEN x)`. Returns the type of number field structure stored in `x`, `typ_NF`, `typ_BNF`, or `typ_BNR`. Other answers are possible, meaning `x` is not a number field structure.

`GEN get_nf(GEN x, long *t)`. Extract an *nf* structure from `x` if possible and return it, otherwise return `NULL`. Sets `t` to the `nftyp` of `x` in any case.

`GEN get_bnf(GEN x, long *t)`. Extract a *bnf* structure from `x` if possible and return it, otherwise return `NULL`. Sets `t` to the `nftyp` of `x` in any case.

`GEN get_nfpol(GEN x, GEN *nf)` try to extract an *nf* structure from `x`, and sets `*nf` to `NULL` (failure) or to the *nf*. Returns the (monic, integral) polynomial defining the field.

`GEN get_bnfpol(GEN x, GEN *bnf, GEN *nf)` try to extract a *bnf* and an *nf* structure from `x`, and sets `*bnf` and `*nf` to `NULL` (failure) or to the corresponding structure. Returns the (monic, integral) polynomial defining the field.

`GEN checknf(GEN x)` if an *nf* structure can be extracted from `x`, return it; otherwise raise an exception. The more general `get_nf` is often more flexible.

`GEN checkbnf(GEN x)` if an *bnf* structure can be extracted from `x`, return it; otherwise raise an exception. The more general `get_bnf` is often more flexible.

`GEN checkbnf_i(GEN bnf)` same as `checkbnf` but return `NULL` instead of raising an exception.

`void checkbnr(GEN bnr)` Raise an exception if the argument is not a *bnr* structure.

`GEN checknf_i(GEN nf)` same as `checknf` but return `NULL` instead of raising an exception.

`void checkbnrgen(GEN bnr)` Raise an exception if the argument is not a *bnr* structure, complete with explicit generators for the ray class group. This is normally useless and `checkbnr` should be instead, unless you are absolutely certain that the generators will be needed at a later point, and you are about to embark in a costly intermediate computation. PARI functions do check that generators are present in *bnr* before accessing them: they will raise an error themselves; many functions that may require them, e.g. `bnrconductor`, often do not actually need them.

`void checkrnf(GEN rnf)` Raise an exception if the argument is not an *rnf* structure.

`int checkrnf_i(GEN rnf)` same as `checkrnf` but return 0 on failure and 1 on success.

`void checkbid(GEN bid)` Raise an exception if the argument is not a *bid* structure.

GEN `checkbid_i`(GEN `bid`) same as `checkbid` but return NULL instead of raising an exception and return `bid` on success.

GEN `checkznstar_i`(GEN `G`) return `G` if it is a *znstar*; else return NULL on failure.

GEN `checkgal`(GEN `x`) if a *galoisinit* structure can be extracted from `x`, return it; otherwise raise an exception.

void `checksqmat`(GEN `x`, long `N`) check whether `x` is a square matrix of dimension `N`. May be used to check for ideals if `N` is the field degree.

void `checkprid`(GEN `pr`) Raise an exception if the argument is not a prime ideal structure.

int `checkprid_i`(GEN `pr`) same as `checkprid` but return 0 instead of raising an exception and return 1 on success.

int `is_nf_factor`(GEN `F`) return 1 if `F` is an ideal factorization and 0 otherwise.

int `is_nf_extfactor`(GEN `F`) return 1 if `F` is an extended ideal factorization (allowing 0 or negative exponents) and 0 otherwise.

GEN `get_prid`(GEN `ideal`) return the underlying prime ideal structure if one can be extracted from `ideal` (ideal or extended ideal), and return NULL otherwise.

void `checkabgrp`(GEN `v`) Raise an exception if the argument is not an abelian group structure, i.e. a `t_VEC` with either 2 or 3 entries: `[N, cyc]` or `[N, cyc, gen]`.

GEN `abgrp_get_no`(GEN `x`) extract the cardinality `N` from an abelian group structure.

GEN `abgrp_get_cyc`(GEN `x`) extract the elementary divisors `cyc` from an abelian group structure.

GEN `abgrp_get_gen`(GEN `x`) extract the generators `gen` from an abelian group structure.

void `checkmodpr`(GEN `modpr`) Raise an exception if the argument is not a `modpr` structure (from `nfmodprinit`).

GEN `get_modpr`(GEN `x`) return `x` if it is a `modpr` structure and NULL otherwise.

GEN `checknfelt_mod`(GEN `nf`, GEN `x`, const char `*s`) given an *nf* structure `nf` and a `t_POLMOD` `x`, return the attached polynomial representative (shallow) if `x` and `nf` are compatible. Raise an exception otherwise. Set `s` to the name of the caller for a meaningful error message.

void `check_ZKmodule`(GEN `x`, const char `*s`) check whether `x` looks like  $\mathbf{Z}_K$ -module (a pair `[A, I]`, where `A` is a matrix and `I` is a list of ideals; `A` has as many columns as `I` has elements. Otherwise raises an exception. Set `s` to the name of the caller for a meaningful error message.

long `idealtyp`(GEN `*ideal`, GEN `*fa`) The input is `ideal`, a pointer to an ideal (or extended ideal), which is usually modified, `fa` being set as a side-effect. Returns the type of the underlying ideal among `id_PRINCIPAL` (a number field element), `id_PRIME` (a prime ideal) `id_MAT` (an ideal in matrix form).

If `ideal` pointed to an ideal, set `fa` to NULL, and possibly simplify `ideal` (for instance the zero ideal is replaced by `gen_0`). If it pointed to an extended ideal, replace `ideal` by the underlying ideal and set `fa` to the factorization matrix component.

### 13.1.2 Extracting info from a nf structure.

These functions expect a true *nf* argument attached to a number field  $K = \mathbf{Q}[x]/(T)$ , e.g. a *bnf* will not work. Let  $n = [K : \mathbf{Q}]$  be the field degree.

`GEN nf_get_pol(GEN nf)` returns the polynomial  $T$  (monic, in  $\mathbf{Z}[x]$ ).

`long nf_get_varn(GEN nf)` returns the variable number of the number field defining polynomial.

`long nf_get_r1(GEN nf)` returns the number of real places  $r_1$ .

`long nf_get_r2(GEN nf)` returns the number of complex places  $r_2$ .

`void nf_get_sign(GEN nf, long *r1, long *r2)` sets  $r_1$  and  $r_2$  to the number of real and complex places respectively. Note that  $r_1 + 2r_2$  is the field degree.

`long nf_get_degree(GEN nf)` returns the number field degree,  $n = r_1 + 2r_2$ .

`GEN nf_get_disc(GEN nf)` returns the field discriminant.

`GEN nf_get_index(GEN nf)` returns the index of  $T$ , i.e. the index of the order generated by the power basis  $(1, x, \dots, x^{n-1})$  in the maximal order of  $K$ .

`GEN nf_get_zk(GEN nf)` returns a basis  $(w_1, w_2, \dots, w_n)$  for the maximal order of  $K$ . Those are polynomials in  $\mathbf{Q}[x]$  of degree  $< n$ ; it is guaranteed that  $w_1 = 1$ .

`GEN nf_get_zkden(GEN nf)` returns the denominator of `nf_get_zk`, as a positive `t_INT`.

`GEN nf_get_zkprimpart(GEN nf)` returns `nf_get_zk` times its denominator.

`GEN nf_get_invzk(GEN nf)` returns the matrix  $(m_{i,j}) \in M_n(\mathbf{Z})$  giving the power basis  $(x^i)$  in terms of the  $(w_j)$ , i.e. such that  $x^{j-1} = \sum_{i=1}^n m_{i,j} w_i$  for all  $1 \leq j \leq n$ ; since  $w_1 = 1 = x^0$ , we have  $m_{i,1} = \delta_{i,1}$  for all  $i$ . The conversion functions in the `algtobasis` family essentially amount to a left multiplication by this matrix.

`GEN nf_get_roots(GEN nf)` returns the  $r_1$  real roots of the polynomial defining the number fields: first the  $r_1$  real roots (as `t_REALs`), then the  $r_2$  representatives of the pairs of complex conjugates.

`GEN nf_get_allroots(GEN nf)` returns all the complex roots of  $T$ : first the  $r_1$  real roots (as `t_REALs`), then the  $r_2$  pairs of complex conjugates.

`GEN nf_get_M(GEN nf)` returns the  $(r_1 + r_2) \times n$  matrix  $M$  giving the embeddings of  $K$ :  $M[i, j]$  contains  $w_j(\alpha_i)$ , where  $\alpha_i$  is the  $i$ -th element of `nf_get_roots(nf)`. In particular, if  $v$  is an  $n$ -th dimensional `t_COL` representing the element  $\sum_{i=1}^n v[i] w_i$  of  $K$ , then `RgM_RgC_mul(M, v)` represents the embeddings of  $v$ .

`GEN nf_get_G(GEN nf)` returns a  $n \times n$  real matrix  $G$  such that  $Gv \cdot Gv = T_2(v)$ , where  $v$  is an  $n$ -th dimensional `t_COL` representing the element  $\sum_{i=1}^n v[i] w_i$  of  $K$  and  $T_2$  is the standard Euclidean form on  $K \otimes \mathbf{R}$ , i.e.  $T_2(v) = \sum_{\sigma} |\sigma(v)|^2$ , where  $\sigma$  runs through all  $n$  complex embeddings of  $K$ .

`GEN nf_get_roundG(GEN nf)` returns a rescaled version of  $G$ , rounded to nearest integers, specifically `RM_round_maxrank(G)`.

`GEN nf_get_ramified_primes(GEN nf)` returns the vector of ramified primes.

`GEN nf_get_Tr(GEN nf)` returns the matrix of the Trace quadratic form on the basis  $(w_1, \dots, w_n)$ : its  $(i, j)$  entry is  $\text{Tr} w_i w_j$ .

`GEN nf_get_diff(GEN nf)` returns the primitive part of the inverse of the above Trace matrix.

`long nf_get_prec(GEN nf)` returns the precision (in words) to which the *nf* was computed.

### 13.1.3 Extracting info from a bnf structure.

These functions expect a true *bnf* argument, e.g. a *bnr* will not work.

GEN `bnf_get_nf`(GEN `bnf`) returns the underlying *nf*.

GEN `bnf_get_clgp`(GEN `bnf`) returns the class group in *bnf*, which is a 3-component vector  $[h, cyc, gen]$ .

GEN `bnf_get_cyc`(GEN `bnf`) returns the elementary divisors of the class group (cyclic components)  $[d_1, \dots, d_k]$ , where  $d_k \mid \dots \mid d_1$ .

GEN `bnf_get_gen`(GEN `bnf`) returns the generators  $[g_1, \dots, g_k]$  of the class group. Each  $g_i$  has order  $d_i$ , and the full module of relations between the  $g_i$  is generated by the  $d_i g_i = 0$ .

GEN `bnf_get_no`(GEN `bnf`) returns the class number.

GEN `bnf_get_reg`(GEN `bnf`) returns the regulator.

GEN `bnf_get_logfu`(GEN `bnf`) returns (complex floating point approximations to) the logarithms of the complex embeddings of our system of fundamental units.

GEN `bnf_get_fu`(GEN `bnf`) returns the fundamental units. Raise an error if the *bnf* does not contain units in algebraic form.

GEN `bnf_get_fu_nocheck`(GEN `bnf`) as `bnf_get_fu` without checking whether units are present. Do not use this unless you initialize the *bnf* yourself!

GEN `bnf_get_tuU`(GEN `bnf`) returns a generator of the torsion part of  $\mathbf{Z}_K^*$ .

long `bnf_get_tuN`(GEN `bnf`) returns the order of the torsion part of  $\mathbf{Z}_K^*$ , i.e. the number of roots of unity in  $K$ .

### 13.1.4 Extracting info from a bnr structure.

These functions expect a true *bnr* argument.

GEN `bnr_get_bnf`(GEN `bnr`) returns the underlying *bnf*.

GEN `bnr_get_nf`(GEN `bnr`) returns the underlying *nf*.

GEN `bnr_get_clgp`(GEN `bnr`) returns the ray class group.

GEN `bnr_get_no`(GEN `bnr`) returns the ray class number.

GEN `bnr_get_cyc`(GEN `bnr`) returns the elementary divisors of the ray class group (cyclic components)  $[d_1, \dots, d_k]$ , where  $d_k \mid \dots \mid d_1$ .

GEN `bnr_get_gen`(GEN `bnr`) returns the generators  $[g_1, \dots, g_k]$  of the ray class group. Each  $g_i$  has order  $d_i$ , and the full module of relations between the  $g_i$  is generated by the  $d_i g_i = 0$ . Raise a generic error if the *bnr* does not contain the ray class group generators.

GEN `bnr_get_gen_nocheck`(GEN `bnr`) as `bnr_get_gen` without checking whether generators are present. Do not use this unless you initialize the *bnr* yourself!

GEN `bnr_get_bid`(GEN `bnr`) returns the *bid* attached to the *bnr* modulus.

GEN `bnr_get_mod`(GEN `bnr`) returns the modulus attached to the *bnr*.

### 13.1.5 Extracting info from an rnf structure.

These functions expect a true *rnf* argument, attached to an extension  $L/K$ ,  $K = \mathbf{Q}[y]/(T)$ ,  $L = K[x]/(P)$ .

`long rnf_get_degree(GEN rnf)` returns the *relative* degree  $[L : K]$ .

`long rnf_get_absdegree(GEN rnf)` returns the absolute degree  $[L : \mathbf{Q}]$ .

`long rnf_get_nfdegree(GEN rnf)` returns the degree of the base field  $[K : \mathbf{Q}]$ .

`GEN rnf_get_nf(GEN rnf)` returns the base field  $K$ , an *nf* structure.

`GEN rnf_get_nfpol(GEN rnf)` returns the polynomial  $T$  defining the base field  $K$ .

`long rnf_get_nfvarn(GEN rnf)` returns the variable  $y$  attached to the base field  $K$ .

`GEN rnf_get_nfzk(GEN rnf)` returns the integer basis of the base field  $K$ .

`GEN rnf_get_pol(GEN rnf)` returns the relative polynomial defining  $L/K$ .

`long rnf_get_varn(GEN rnf)` returns the variable  $x$  attached to  $L$ .

`GEN rnf_get_zk(GEN rnf)` returns the relative integer basis generating  $\mathbf{Z}_L$  as a  $\mathbf{Z}_K$ -module, as a pseudo-matrix  $(A, I)$  in HNF.

`GEN rnf_get_disc(GEN rnf)` is the output  $[\mathfrak{d}, s]$  of `rnfdisc`.

`GEN rnf_get_idealdisc(GEN rnf)` is the ideal discriminant  $\mathfrak{d}$  from `rnfdisc`.

`GEN rnf_get_index(GEN rnf)` is the index ideal  $\mathfrak{f}$

`GEN rnf_get_polabs(GEN rnf)` returns an absolute polynomial defining  $L/\mathbf{Q}$ .

`GEN rnf_get_alpha(GEN rnf)` a root  $\alpha$  of the polynomial defining the base field, modulo `polabs` (cf. `rnfequation`)

`GEN rnf_get_k(GEN rnf)` a small integer  $k$  such that  $\theta = \beta + k\alpha$  is a root of `polabs`, where  $\beta$  is a root of `pol` and  $\alpha$  a root of the polynomial defining the base field, as in `rnf_get_alpha` (cf. also `rnfequation`).

`GEN rnf_get_invzk(GEN rnf)` contains  $A^{-1}$ , where  $(A, I)$  is the chosen pseudo-basis for  $\mathbf{Z}_L$  over  $\mathbf{Z}_K$ .

`GEN rnf_get_map(GEN rnf)` returns technical data attached to the map  $K \rightarrow L$ . Currently, this contains data from `rnfequation`, as well as the polynomials  $T$  and  $P$ .

### 13.1.6 Extracting info from a bid structure.

These functions expect a true *bid* argument, attached to a modulus  $I = I_0 I_\infty$  in a number field  $K$ .

`GEN bid_get_mod(GEN bid)` returns the modulus attached to the *bid*.

`GEN bid_get_grp(GEN bid)` returns the Abelian group attached to  $(\mathbf{Z}_K/I)^*$ .

`GEN bid_get_ideal(GEN bid)` return the finite part  $I_0$  of the *bid* modulus (an integer ideal).

`GEN bid_get_arch(GEN bid)` return the Archimedean part  $I_\infty$  of the *bid* modulus as a vector of real places in `vec01` format, see Section 13.1.17.

GEN `bid_get_archp`(GEN `bid`) return the Archimedean part  $I_\infty$  of the *bid* modulus, as a vector of real places in indices format see Section 13.1.17.

GEN `bid_get_fact`(GEN `bid`) returns the ideal factorization  $I_0 = \prod_i \mathfrak{p}_i^{e_i}$ .

`bid_get_ideal`(*bid*), via `idealfactor`.

GEN `bid_get_no`(GEN `bid`) returns the cardinality of the group  $(\mathbf{Z}_K/I)^*$ .

GEN `bid_get_cyc`(GEN `bid`) returns the elementary divisors of the group  $(\mathbf{Z}_K/I)^*$  (cyclic components)  $[d_1, \dots, d_k]$ , where  $d_k \mid \dots \mid d_1$ .

GEN `bid_get_gen`(GEN `bid`) returns the generators of  $(\mathbf{Z}_K/I)^*$  contained in *bid*. Raise a generic error if *bid* does not contain generators.

GEN `bid_get_gen_nocheck`(GEN `bid`) as `bid_get_gen` without checking whether generators are present. Do not use this unless you initialize the *bid* yourself!

GEN `bid_get_sprk`(GEN `bid`) return a list of structures attached to the  $(\mathbf{Z}_K/\mathfrak{p}^e)^*$  where  $\mathfrak{p}^e$  divides  $I_0$  exactly.

GEN `bid_get_sarch`(GEN `bid`) return the structure attached to  $(\mathbf{Z}_K/I_\infty)^*$ , by `nfarchstar`.

GEN `bid_get_U`(GEN `bid`) return the matrix with integral coefficients relating the local generators (from chinese remainders) to the global SNF generators (*bid.gen*).

### 13.1.7 Extracting info from a znstar structure.

These functions expect an argument *G* as returned by `znstar0`(*N*, 1), attached to a positive *N* and the abelian group  $(\mathbf{Z}/N\mathbf{Z})^*$ . Let  $(g_i)$  be the SNF generators, where  $g_i$  has order  $d_i$ ; we call  $(g'_i)$  the (canonical) Conrey generators, where  $g'_i$  has order  $d'_i$ . Both sets of generators have the same cardinality.

GEN `znstar_get_N`(GEN `bid`) return *N*.

GEN `znstar_get_faN`(GEN *G*) return the factorization `factor`(*N*),  $N = \prod_j p_j^{e_j}$ .

GEN `znstar_get_pe`(GEN *G*) return the vector of primary factors  $(p_j^{e_j})$ .

GEN `znstar_get_no`(GEN *G*) the cardinality  $\phi(N)$  of *G*.

GEN `znstar_get_cyc`(GEN *G*) elementary divisors  $(d_i)$  of  $(\mathbf{Z}/N\mathbf{Z})^*$ .

GEN `znstar_get_gen`(GEN *G*) SNF generators divisors  $(g_i)$  of  $(\mathbf{Z}/N\mathbf{Z})^*$ .

GEN `znstar_get_conreycyc`(GEN *G*) orders  $(d'_i)$  of Conrey generators.

GEN `znstar_get_conreygen`(GEN *G*) Conrey generators  $(g'_i)$ .

GEN `znstar_get_U`(GEN *G*) a square matrix *U* such that  $(g_i) = U(g'_i)$ .

GEN `znstar_get_Ui`(GEN *G*) a square matrix *U'* such that  $U'(g_i) = (g'_i)$ . In general,  $UU'$  will not be the identity.



### 13.1.8 Inserting info in a number field structure.

If the required data is not part of the structure, it is computed then inserted, and the new value is returned.

These functions expect a **bnf** argument:

**GEN bnf\_build\_cycgen**(**GEN bnf**) the *bnf* contains generators  $[g_1, \dots, g_k]$  of the class group, each with order  $d_i$ . Then  $g_i^{d_i} = (x_i)$  is a principal ideal. This function returns the  $x_i$  as a factorization matrix (**famat**) giving the element in factored form as a product of *S*-units.

**GEN bnf\_build\_matalpha**(**GEN bnf**) the class group was computed using a factorbase *S* of prime ideals  $\mathfrak{p}_i$ ,  $i \leq r$ . They satisfy relations of the form  $\prod_j \mathfrak{p}_i^{e_{i,j}} = (\alpha_j)$ , where the  $e_{i,j}$  are given by the matrices *bnf*[1] (*W*, singling out a minimal set of generators in *S*) and *bnf*[2] (*B*, expressing the rest of *S* in terms of the singled out generators). This function returns the  $\alpha_j$  in factored form as a product of *S*-units.

**GEN bnf\_build\_units**(**GEN bnf**) returns a minimal set of generators for the unit group. The first element is a torsion unit, the others have infinite order.

These functions expect a **rnf** argument:

**GEN rnf\_build\_nfabs**(**GEN rnf**, **long prec**) given a *rnf* structure attached to  $L/K$ , (compute and) return an *nf* structure attached to *L* at precision **prec**.

**void rnfcomplete**(**GEN rnf**) as **rnf\_build\_nfabs** using the precision of *K* for **prec**.

**GEN rnf\_zkabs**(**GEN rnf**) returns a **Z**-basis in HNF for  $\mathbf{Z}_L$  as a pair  $[T, v]$ , where *T* is **rnf\_get\_polabs**(*rnf*) and *v* a vector of elements lifted from  $\mathbf{Q}[X]/(T)$ . Note that the function **rnf\_build\_nfabs** essentially applies **nfinit** to the output of this function.

### 13.1.9 Increasing accuracy.

**GEN nfnewprec**(**GEN x**, **long prec**). Raise an exception if **x** is not a number field structure (*nf*, *bnf* or *bnr*). Otherwise, sets its accuracy to **prec** and return the new structure. This is mostly useful with **prec** larger than the accuracy to which **x** was computed, but it is also possible to decrease the accuracy of **x** (truncating relevant components, which may speed up later computations). This routine may modify the original **x** (see below).

This routine is straightforward for *nf* structures, but for the other ones, it requires all principal ideals corresponding to the *bnf* relations in algebraic form (they are originally only available via floating point approximations). This in turn requires many calls to **bnfisprincipal0**, which is often slow, and may fail if the initial accuracy was too low. In this case, the routine will not actually fail but recomputes a *bnf* from scratch!

Since this process may be very expensive, the corresponding data is cached (as a *clone*) in the *original* **x** so that later precision increases become very fast. In particular, the copy returned by **nfnewprec** also contains this additional data.

**GEN bnfnewprec**(**GEN x**, **long prec**). As **nfnewprec**, but extracts a *bnf* structure from **x** before increasing its accuracy, and returns only the latter.

**GEN bnrnewprec**(**GEN x**, **long prec**). As **nfnewprec**, but extracts a *bnr* structure from **x** before increasing its accuracy, and returns only the latter.

**GEN nfnewprec\_shallow**(**GEN nf**, **long prec**)

GEN bnfnewprec\_shallow(GEN bnf, long prec)

GEN bnrnewprec\_shallow(GEN bnr, long prec) Shallow functions underlying the above, except that the first argument must now have the corresponding number field type. I.e. one cannot call `nfnewprec_shallow(nf, prec)` if `nf` is actually a *bnf*.

**13.1.10 Number field arithmetic.** The number field  $K = \mathbf{Q}[X]/(T)$  is represented by an `nf` (or `bnf` or `bnr` structure). An algebraic number belonging to  $K$  is given as

- a `t_INT`, `t_FRAC` or `t_POL` (implicitly modulo  $T$ ), or
- a `t_POLMOD` (modulo  $T$ ), or
- a `t_COL` `v` of dimension  $N = [K : \mathbf{Q}]$ , representing the element in terms of the computed integral basis  $(e_i)$ , as

`sum(i = 1, N, v[i] * nf.zk[i])`

The preferred forms are `t_INT` and `t_COL` of `t_INT`. Routines can handle denominators but it is much more efficient to remove denominators first (`Q_remove_denom`) and take them into account at the end.

**Safe routines.** The following routines do not assume that their `nf` argument is a true *nf* (it can be any number field type, e.g. a *bnf*), and accept number field elements in all the above forms. They return their result in `t_COL` form.

GEN nfadd(GEN nf, GEN x, GEN y) returns  $x + y$ .

GEN nfsub(GEN nf, GEN x, GEN y) returns  $x - y$ .

GEN nfdiv(GEN nf, GEN x, GEN y) returns  $x/y$ .

GEN nfinv(GEN nf, GEN x) returns  $x^{-1}$ .

GEN nfmul(GEN nf, GEN x, GEN y) returns  $xy$ .

GEN nfpow(GEN nf, GEN x, GEN k) returns  $x^k$ ,  $k$  is in  $\mathbf{Z}$ .

GEN nfpow\_u(GEN nf, GEN x, ulong k) returns  $x^k$ ,  $k \geq 0$ .

GEN nfsqr(GEN nf, GEN x) returns  $x^2$ .

long nfval(GEN nf, GEN x, GEN pr) returns the valuation of  $x$  at the maximal ideal  $\mathfrak{p}$  attached to the *prid* `pr`. Returns `LONG_MAX` if  $x$  is 0.

GEN nfnorm(GEN nf, GEN x) absolute norm of  $x$ .

GEN nftrace(GEN nf, GEN x) absolute trace of  $x$ .

GEN nfpoleval(GEN nf, GEN pol, GEN a) evaluate the `t_POL` `pol` (with coefficients in `nf`) on the algebraic number  $a$  (also in *nf*).

GEN FpX\_FpC\_nfpoleval(GEN nf, GEN pol, GEN a, GEN p) evaluate the `FpX` `pol` on the algebraic number  $a$  (also in *nf*).

The following three functions implement trivial functionality akin to Euclidean division for which we currently have no real use. Of course, even if the number field is actually Euclidean, these do not in general implement a true Euclidean division.

GEN `nfdivuuc`(GEN `nf`, GEN `a`, GEN `b`) returns the algebraic integer closest to  $x/y$ . Functionally identical to `ground( nfdiv(nf,x,y) )`.

GEN `nfdivrem`(GEN `nf`, GEN `a`, GEN `b`) returns the vector  $[q, r]$ , where

```
q = nfdivuuc(nf, a, b);
r = nfsub(nf, a, nfmul(nf,q,b));    \\ or r = nfmod(nf,a,b);
```

GEN `nfmod`(GEN `nf`, GEN `a`, GEN `b`) returns  $r$  such that

```
q = nfdivuuc(nf, a, b);
r = nfsub(nf, a, nfmul(nf,q,b));
```

GEN `nf_to_scalar_or_basis`(GEN `nf`, GEN `x`) let  $x$  be a number field element. If it is a rational scalar, i.e. can be represented by a `t_INT` or `t_FRAC`, return the latter. Otherwise returns its basis representation (`nfalgtobasis`). Shallow function.

GEN `nf_to_scalar_or_alg`(GEN `nf`, GEN `x`) let  $x$  be a number field element. If it is a rational scalar, i.e. can be represented by a `t_INT` or `t_FRAC`, return the latter. Otherwise returns its lifted `t_POLMOD` representation (`lifted nfbasistoalg`). Shallow function.

GEN `RgX_to_nfX`(GEN `nf`, GEN `x`) let  $x$  be a `t_POL` whose coefficients are number field elements; apply `nf_to_scalar_or_basis` to each coefficient and return the resulting new polynomial. Shallow function.

GEN `RgM_to_nfM`(GEN `nf`, GEN `x`) let  $x$  be a `t_MAT` whose coefficients are number field elements; apply `nf_to_scalar_or_basis` to each coefficient and return the resulting new matrix. Shallow function.

GEN `RgC_to_nfC`(GEN `nf`, GEN `x`) let  $x$  be a `t_COL` or `t_VEC` whose coefficients are number field elements; apply `nf_to_scalar_or_basis` to each coefficient and return the resulting new `t_COL`. Shallow function.

**Unsafe routines.** The following routines assume that their `nf` argument is a true *nf* (e.g. a *bnf* is not allowed) and their argument are restricted in various ways, see the precise description below.

GEN `nfinvmodideal`(GEN `nf`, GEN `x`, GEN `A`) given an algebraic integer  $x$  and a non-zero integral ideal  $A$  in HNF, returns a  $y$  such that  $xy \equiv 1$  modulo  $A$ .

GEN `nfpowmodideal`(GEN `nf`, GEN `x`, GEN `n`, GEN `ideal`) given an algebraic integer  $x$ , an integer  $n$ , and a non-zero integral ideal  $A$  in HNF, returns an algebraic integer congruent to  $x^n$  modulo  $A$ .

GEN `nfmuli`(GEN `nf`, GEN `x`, GEN `y`) returns  $x \times y$  assuming that both  $x$  and  $y$  are either `t_INTs` or `ZVs` of the correct dimension.

GEN `nfsqri`(GEN `nf`, GEN `x`) returns  $x^2$  assuming that  $x$  is a `t_INT` or a `ZV` of the correct dimension.

GEN `nfC_nf_mul`(GEN `nf`, GEN `v`, GEN `x`) given a `t_VEC` or `t_COL`  $v$  of elements of  $K$  in `t_INT`, `t_FRAC` or `t_COL` form, multiply it by the element  $x$  (arbitrary form). This is faster than multiplying coordinatewise since pre-computations related to  $x$  (computing the multiplication table) are done only once. The components of the result are in most cases `t_COLs` but are allowed to be `t_INTs` or `t_FRACs`. Shallow function.

GEN `nfC_multable_mul`(GEN `v`, GEN `mx`) same as `nfC_nf_mul`, where the argument  $x$  is replaced by its multiplication table `mx`.

GEN `zkC_multable_mul`(GEN `v`, GEN `x`) same as `nfC_nf_mul`, where  $v$  is a vector of algebraic integers,  $x$  is an algebraic integer, and  $x$  is replaced by `zk_multable(x)`.

GEN `zk_multable`(GEN `nf`, GEN `x`) given a ZC  $x$  (implicitly representing an algebraic integer), returns the ZM giving the multiplication table by  $x$ . Shallow function (the first column of the result points to the same data as  $x$ ).

GEN `zk_inv`(GEN `nf`, GEN `x`) given a ZC  $x$  (implicitly representing an algebraic integer), returns the QC giving the inverse  $x^{-1}$ . Return NULL if  $x$  is 0. Not memory clean but safe for `gerepileupto`.

GEN `zkmultable_inv`(GEN `mx`) as `zk_inv`, where the argument given is `zk_multable`( $x$ ).

GEN `zkmultable_capZ`(GEN `mx`) given a non-zero *zkmultable*  $mx$  attached to  $x \in \mathbf{Z}_K$ , return the positive generator of  $(x) \cap \mathbf{Z}$ .

GEN `zk_scalar_or_multable`(GEN `nf`, GEN `x`) given a `t_INT` or ZC  $x$ , returns a `t_INT` equal to  $x$  if the latter is a scalar (`t_INT` or `ZV_isscalar`( $x$ ) is 1) and `zk_multable`( $nf, x$ ) otherwise. Shallow function.

The following routines implement multiplication in a commutative  $R$ -algebra, generated by  $(e_1 = 1, \dots, e_n)$ , and given by a multiplication table  $M$ : elements in the algebra are  $n$ -dimensional `t_COLs`, and the matrix  $M$  is such that for all  $1 \leq i, j \leq n$ , its column with index  $(i-1)n + j$ , say  $(c_k)$ , gives  $e_i \cdot e_j = \sum c_k e_k$ . It is assumed that  $e_1$  is the neutral element for the multiplication (a convenient optimization, true in practice for all multiplications we needed to implement). If  $x$  has any other type than `t_COL` where an algebra element is expected, it is understood as  $x e_1$ .

GEN `multable`(GEN `M`, GEN `x`) given a column vector  $x$ , representing the quantity  $\sum_{i=1}^N x_i e_i$ , returns the multiplication table by  $x$ . Shallow function.

GEN `ei_multable`(GEN `M`, long `i`) returns the multiplication table by the  $i$ -th basis element  $e_i$ . Shallow function.

GEN `tablemul`(GEN `M`, GEN `x`, GEN `y`) returns  $x \cdot y$ .

GEN `tablesqr`(GEN `M`, GEN `x`) returns  $x^2$ .

GEN `tablemul_ei`(GEN `M`, GEN `x`, long `i`) returns  $x \cdot e_i$ .

GEN `tablemul_ei_ej`(GEN `M`, long `i`, long `j`) returns  $e_i \cdot e_j$ .

GEN `tablemulvec`(GEN `M`, GEN `x`, GEN `v`) given a vector  $v$  of elements in the algebra, returns the  $x \cdot v[i]$ .

The following routines implement naive linear algebra using the *black box field* mechanism:

GEN `nfM_det`(GEN `nf`, GEN `M`)

GEN `nfM_inv`(GEN `nf`, GEN `M`)

GEN `nfM_mul`(GEN `nf`, GEN `A`, GEN `B`)

GEN `nfM_nfC_mul`(GEN `nf`, GEN `A`, GEN `B`)

The following routines implement modular algorithms in cyclotomic fields. In the prototypes,  $P$  is the  $n$ -th cyclotomic polynomial  $\Phi_n$  and  $M$  is a `t_MAT` with `t_INT` or `ZX` coefficients, understood modulo  $P$ .

GEN `ZabM_ker`(GEN `M`, GEN `P`, long `n`) returns an integral (primitive) basis of the kernel of  $M$ .

GEN `ZabM_indexrank`(GEN `M`, GEN `P`, long `n`) return a vector with two `t_VECSMALL` components giving the rank profile of  $M$ . Inefficient (but correct) when  $M$  does not have almost full column rank.

GEN `ZabM_inv`(GEN `M`, GEN `P`, long `n`, GEN `*pden`) assume that  $M$  is invertible; return  $N$  and sets the algebraic integer `*pden` (an integer or a ZX, implicitly modulo  $P$ ) such that  $MN = \text{den} \cdot \text{Id}$ .

GEN `ZabM_pseudoinv`(GEN `M`, GEN `P`, long `n`, GEN `*pv`, GEN `*pden`) analog of `ZM_pseudoinv`. Not gerepile-safe.

GEN `ZabM_inv_ratlift`(GEN `M`, GEN `P`, long `n`, GEN `*pden`) return a primitive matrix  $H$  such that  $MH$  is  $d$  times the identity and set `*pden` to  $d$ . Uses a multimodular algorithm, attempting rational reconstruction along the way. To be used when you expect that the denominator of  $M^{-1}$  is much smaller than  $\det M$  else use `ZabM_inv`.

### 13.1.11 Elements in factored form.

Computational algebraic theory performs extensively linear algebra on  $\mathbf{Z}$ -modules with a natural multiplicative structure ( $K^*$ , fractional ideals in  $K$ ,  $\mathbf{Z}_K^*$ , ideal class group), thereby raising elements to horrendously large powers. A seemingly innocuous elementary linear algebra operation like  $C_i \leftarrow C_i - 10000C_1$  involves raising entries in  $C_1$  to the 10000-th power. Understandably, it is often more efficient to keep elements in factored form rather than expand every such expression. A *factorization matrix* (or *famat*) is a two column matrix, the first column containing *elements* (arbitrary objects which may be repeated in the column), and the second one contains *exponents* (`t_INTs`, allowed to be 0). By abuse of notation, the empty matrix `cgetg(1, t_MAT)` is recognized as the trivial factorization (no element, no exponent).

Even though we think of a *famat* with columns  $g$  and  $e$  as one meaningful object when fully expanded as  $\prod g[i]^{e[i]}$ , *famats* are basically about concatenating information to keep track of linear algebra: the objects stored in a *famat* need not be operation-compatible, they will not even be compared to each other (with one exception: `famat_reduce`). Multiplying two *famats* just concatenates their elements and exponents columns. In a context where a *famat* is expected, an object  $x$  which is not of type `t_MAT` will be treated as the factorization  $x^1$ . The following functions all return *famats*:

GEN `famat_mul`(GEN `f`, GEN `g`)  $f, g$  are *famat*, or objects whose type is *not* `t_MAT` (understood as  $f^1$  or  $g^1$ ). Returns  $fg$ . The empty factorization is the neutral element for *famat* multiplication.

GEN `famat_mul_shallow`(GEN `f`, GEN `g`) shallow version of `famat_mul`.

GEN `famat_pow`(GEN `f`, GEN `n`)  $n$  is a `t_INT`. If  $f$  is a `t_MAT`, assume it is a *famat* and return  $f^n$  (multiplies the exponent column by  $n$ ). Otherwise, understand it as an element and returns the 1-line *famat*  $f^n$ .

GEN `famat_pow_shallow`(GEN `f`, GEN `n`) shallow version of `famat_pow`.

GEN `famat_pows_shallow`(GEN `f`, long `n`) shallow version of `famat_pow` where  $n$  is a small integer.

GEN `famat_mulpow_shallow`(GEN `f`, GEN `g`, GEN `e`) *famat* corresponding to  $f \cdot g^e$ . Shallow function.

GEN `famat_mulpows_shallow`(GEN `f`, GEN `g`, long `e`) *famat* shallow version of `famat_mulpow` where  $e$  is a small integer.

GEN `famat_sqr`(GEN `f`) returns  $f^2$ .

GEN `famat_inv`(GEN `f`) returns  $f^{-1}$ .

GEN `famat_inv_shallow`(GEN `f`) shallow version of `famat_inv`.

GEN `famat_div_shallow`(GEN `f`, GEN `g`) return  $f/g$ ; shallow.

GEN `famat_Z_gcd`(GEN `M`, GEN `n`) restrict the *famat*  $M$  to the prime power dividing  $n$ .

GEN `to_famat`(GEN `x`, GEN `k`) given an element  $x$  and an exponent  $k$ , returns the *famat*  $x^k$ .

GEN `to_famat_shallow`(GEN `x`, GEN `k`) same, as a shallow function.

Note that it is trivial to break up a *famat* into its two constituent columns: `gel(f,1)` and `gel(f,2)` are the elements and exponents respectively. Conversely, `mkmat2` builds a (shallow) *famat* from two `t_COLS` of the same length.

The last two functions makes an assumption about the elements: they must be regular algebraic numbers (not *famats*) over a given number field:

GEN `famat_reduce`(GEN `f`) given a *famat*  $f$ , returns a *famat*  $g$  without repeated elements or 0 exponents, such that the expanded forms of  $f$  and  $g$  would be equal. Shallow function.

GEN `ZM_famat_limit`(GEN `f`, GEN `limit`) given a *famat*  $f$  with `t_INT` entries, returns a *famat*  $g$  with all factors larger than `limit` multiplied out as the last entry (with exponent 1).

GEN `famat_to_nf`(GEN `nf`, GEN `f`) You normally never want to do this! This is a simplified form of `nfactorback`, where we do not check the user input for consistency.

The description of `famat_to_nf` says that you do not want to use this function. Then how do we recover genuine number field elements? Well, in most cases, we do not need to: most of the functions useful in this context accept *famats* as inputs, for instance `nfsign`, `nfsign_arch`, `ideallog` and `bnfisunit`. Otherwise, we can generally make good use of a quotient operation (modulo a fixed conductor, modulo  $\ell$ -th powers); see the end of Section 13.1.22.

**Caveat.** Receiving a *famat* input, `bnfisunit` assumes that it is an algebraic integer, since this is expensive to check, and normally easy to ensure from the user's side; do not feed it ridiculous inputs.

GEN `famatsmall_reduce`(GEN `f`) as `famat_reduce`, but for exponents given by a `t_VECSMALL`.

### 13.1.12 Ideal arithmetic.

#### Conversion to HNF.

GEN `idealhnf`(GEN `nf`, GEN `x`) returns the HNF of the ideal defined by  $x$ :  $x$  may be an algebraic number (defining a principal ideal), a maximal ideal (as given by `idealprimedec` or `idealfactor`), or a matrix whose columns give generators for the ideal. This last format is complicated, but useful to reduce general modules to the canonical form once in a while:

- if strictly less than  $N = [K : Q]$  generators are given,  $x$  is the  $\mathbf{Z}_K$ -module they generate,
- if  $N$  or more are given, it is assumed that they form a  $\mathbf{Z}$ -basis (that the matrix has maximal rank  $N$ ). This acts as `mathnf` since the  $\mathbf{Z}_K$ -module structure is (taken for granted hence) not taken into account in this case.

Extended ideals are also accepted, their principal part being discarded.

GEN `idealhnf0`(GEN `nf`, GEN `x`, GEN `y`) returns the HNF of the ideal generated by the two algebraic numbers  $x$  and  $y$ .

The following low-level functions underlie the above two: they all assume that `nf` is a true *nf* and perform no type checks:

`GEN idealhnf_principal(GEN nf, GEN x)` returns the ideal generated by the algebraic number  $x$ .

`GEN idealhnf_shallow(GEN nf, GEN x)` is `idealhnf` except that the result may not be suitable for `gerepile`: if  $x$  is already in HNF, we return  $x$ , not a copy!

`GEN idealhnf_two(GEN nf, GEN v)` assuming  $a = v[1]$  is a non-zero `t_INT` and  $b = v[2]$  is an algebraic integer, possibly given in regular representation by a `t_MAT` (the multiplication table by  $b$ , see `zk_multable`), returns the HNF of  $a\mathbf{Z}_K + b\mathbf{Z}_K$ .

### Operations.

The basic ideal routines accept all `nfs` ( $nf$ ,  $bnf$ ,  $bnr$ ) and ideals in any form, including extended ideals, and return ideals in HNF, or an extended ideal when that makes sense:

`GEN idealadd(GEN nf, GEN x, GEN y)` returns  $x + y$ .

`GEN idealdiv(GEN nf, GEN x, GEN y)` returns  $x/y$ . Returns an extended ideal if  $x$  or  $y$  is an extended ideal.

`GEN idealmul(GEN nf, GEN x, GEN y)` returns  $xy$ . Returns an extended ideal if  $x$  or  $y$  is an extended ideal.

`GEN idealsqr(GEN nf, GEN x)` returns  $x^2$ . Returns an extended ideal if  $x$  is an extended ideal.

`GEN idealinv(GEN nf, GEN x)` returns  $x^{-1}$ . Returns an extended ideal if  $x$  is an extended ideal.

`GEN idealpow(GEN nf, GEN x, GEN n)` returns  $x^n$ . Returns an extended ideal if  $x$  is an extended ideal.

`GEN idealpows(GEN nf, GEN ideal, long n)` returns  $x^n$ . Returns an extended ideal if  $x$  is an extended ideal.

`GEN idealmulred(GEN nf, GEN x, GEN y)` returns an extended ideal equal to  $xy$ .

`GEN idealpowred(GEN nf, GEN x, GEN n)` returns an extended ideal equal to  $x^n$ .

More specialized routines suffer from various restrictions:

`GEN idealdivexact(GEN nf, GEN x, GEN y)` returns  $x/y$ , assuming that the quotient is an integral ideal. Much faster than `idealdiv` when the norm of the quotient is small compared to  $Nx$ . Strips the principal parts if either  $x$  or  $y$  is an extended ideal.

`GEN idealdivpowprime(GEN nf, GEN x, GEN pr, GEN n)` returns  $x\mathfrak{p}^{-n}$ , assuming  $x$  is an ideal in HNF or a rational number, and `pr` a `prid` attached to `p`. Not suitable for `gerepileupto` since it returns  $x$  when  $n = 0$ .

`GEN idealmulpowprime(GEN nf, GEN x, GEN pr, GEN n)` returns  $x\mathfrak{p}^n$ , assuming  $x$  is an ideal in HNF or a rational number, and `pr` a `prid` attached to `p`. Not suitable for `gerepileupto` since it returns  $x$  when  $n = 0$ .

`GEN idealprodprime(GEN nf, GEN v)` given a list  $v$  of prime ideals in `prid` form, return their product. Assume that  $nf$  is a true  $nf$  structure.

`GEN idealprod(GEN nf, GEN v)` given a list  $v$  of ideals, return their product.

`GEN idealprodval(GEN nf, GEN v, GEN pr)` given a list  $v$  of ideals return the valuation of their product at the prime ideal `pr`.

`GEN idealHNF_mul(GEN nf, GEN x, GEN y)` returns  $xy$ , assuming that `nf` is a true  $nf$ ,  $x$  is an integral ideal in HNF and  $y$  is an integral ideal in HNF or precompiled form (see below). For maximal speed, the second ideal  $y$  may be given in precompiled form  $y = [a, b]$ , where  $a$  is a non-zero `t_INT` and  $b$  is an algebraic integer in regular representation (a `t_MAT` giving the multiplication table by the fixed element): very useful when many ideals  $x$  are going to be multiplied by the same ideal  $y$ . This essentially reduces each ideal multiplication to an  $N \times N$  matrix multiplication followed by a  $N \times 2N$  modular HNF reduction (modulo  $xy \cap \mathbf{Z}$ ).

`GEN idealHNF_inv(GEN nf, GEN I)` returns  $I^{-1}$ , assuming that `nf` is a true  $nf$  and  $x$  is a fractional ideal in HNF.

`GEN idealHNF_inv_Z(GEN nf, GEN I)` returns  $(I \cap \mathbf{Z}) \cdot I^{-1}$ , assuming that `nf` is a true  $nf$  and  $x$  is an integral fractional ideal in HNF. The result is an integral ideal in HNF.

### Approximation.

`GEN idealaddtoone(GEN nf, GEN A, GEN B)` given two coprime integer ideals  $A, B$ , returns  $[a, b]$  with  $a \in A, b \in B$ , such that  $a + b = 1$ . The result is reduced mod  $AB$ , so  $a, b$  will be small.

`GEN idealaddtoone_i(GEN nf, GEN A, GEN B)` as `idealaddtoone` except that `nf` must be a true  $nf$ , and only  $a$  is returned.

`GEN idealaddtoone_raw(GEN nf, GEN A, GEN B)` as `idealaddtoone_i` except that the reduction mod  $AB$  is only performed modulo the lcm of  $A \cap \mathbf{Z}$  and  $B \cap \mathbf{Z}$ , which will increase the size of  $a$ .

`GEN zkchineseinit(GEN nf, GEN A, GEN B, GEN AB)` given two coprime integral ideals  $A$  and  $B$  (in any form, preferably HNF) and their product  $AB$  (in HNF form), initialize a solution to the Chinese remainder problem modulo  $AB$ .

`GEN zkchinese(GEN zkc, GEN x, GEN y)` given `zkc` from `zkchineseinit`, and  $x, y$  two integral elements given as `t_INT` or `ZC`, return a  $z$  modulo  $AB$  such that  $z = x \bmod A$  and  $z = y \bmod B$ .

`GEN zkchinese1(GEN zkc, GEN x)` as `zkchinese` for  $y = 1$ ; useful to lift elements in a nice way from  $(\mathbf{Z}_K/A_i)^*$  to  $(\mathbf{Z}_K/\prod_i A_i)^*$ .

`GEN hnfmerge_get_1(GEN A, GEN B)` given two square upper HNF integral matrices  $A, B$  of the same dimension  $n > 0$ , return  $a$  in the image of  $A$  such that  $1 - a$  is in the image of  $B$ . (By abuse of notation we denote  $1$  the column vector  $[1, 0, \dots, 0]$ .) If such an  $a$  does not exist, return `NULL`. This is the function underlying `idealaddtoone`.

`GEN idealaddmultoone(GEN nf, GEN v)` given a list of  $n$  (globally) coprime integer ideals  $(v[i])$  returns an  $n$ -dimensional vector  $a$  such that  $a[i] \in v[i]$  and  $\sum a[i] = 1$ . If  $[K : \mathbf{Q}] = N$ , this routine computes the HNF reduction (with  $Gl_{nN}(\mathbf{Z})$  base change) of an  $N \times nN$  matrix; so it is well worth pruning "useless" ideals from the list (as long as the ideals remain globally coprime).

`GEN idealapprfact(GEN nf, GEN fx)` as `idealappr`, except that  $x$  *must* be given in factored form. (This is unchecked.)

`GEN idealcoprime(GEN nf, GEN x, GEN y)`. Given 2 integral ideals  $x$  and  $y$ , returns an algebraic number  $\alpha$  such that  $\alpha x$  is an integral ideal coprime to  $y$ .

`GEN idealcoprimefact(GEN nf, GEN x, GEN fy)` same as `idealcoprime`, except that  $y$  is given in factored form, as from `idealfactor`.

`GEN idealchinese(GEN nf, GEN x, GEN y)`

`GEN idealchineseinit(GEN nf, GEN x)`



### 13.1.13 Maximal ideals.

The PARI structure attached to maximal ideals is a *prid* (for *prime ideal*), usually produced by `idealprimedec` and `idealfactor`. In this section, we describe the format; other sections will deal with their daily use.

A *prid* attached to a maximal ideal  $\mathfrak{p}$  stores the following data: the underlying rational prime  $p$ , the ramification degree  $e \geq 1$ , the residue field degree  $f \geq 1$ , a  $p$ -uniformizer  $\pi$  with valuation 1 at  $\mathfrak{p}$  and valuation 0 at all other primes dividing  $p$  and a rescaled “anti-uniformizer”  $\tau$  used to compute valuations. This  $\tau$  is an algebraic integer such that  $\tau/p$  has valuation  $-1$  at  $\mathfrak{p}$  and is integral at all other primes; in particular, the valuation of  $x \in \mathbf{Z}_K$  is positive if and only if the algebraic integer  $x\tau$  is divisible by  $p$  (easy to check for elements in `t_COL` form).

`GEN pr_get_p(GEN pr)` returns  $p$ . Shallow function.

`GEN pr_get_gen(GEN pr)` returns  $\pi$ . Shallow function.

`long pr_get_e(GEN pr)` returns  $e$ .

`long pr_get_f(GEN pr)` returns  $f$ .

`GEN pr_get_tau(GEN pr)` returns `zk_scalar_or_multable(nf,  $\tau$ )`, which is the `t_INT` 1 iff  $p$  is inert, and a `ZM` otherwise. Shallow function.

`int pr_is_inert(GEN pr)` returns 1 if  $p$  is inert, 0 otherwise.

`GEN pr_norm(GEN pr)` returns the norm  $p^f$  of the maximal ideal.

`ulong upr_norm(GEN pr)` returns the norm  $p^f$  of the maximal ideal, as an `ulong`. Assume that the result does not overflow.

`GEN pr_hnf(GEN pr)` return the HNF of  $\mathfrak{p}$ .

`GEN pr_inv(GEN pr)` return the fractional ideal  $\mathfrak{p}^{-1}$ , in HNF.

`GEN pr_inv_p(GEN pr)` return the integral ideal  $p\mathfrak{p}^{-1}$ , in HNF.

`GEN idealprimedec(GEN nf, GEN p)` list of maximal ideals dividing the prime  $p$ .

`GEN idealprimedec_limit_f(GEN nf, GEN p, long f)` as `idealprimedec`, limiting the list to primes of residual degree  $\leq f$  if  $f$  is non-zero.

`GEN idealprimedec_limit_norm(GEN nf, GEN p, GEN B)` as `idealprimedec`, limiting the list to primes of norm  $\leq B$ , which must be a positive `t_INT`.

`GEN idealprimedec_galois(GEN nf, GEN p)` return a single prime ideal above  $p$ .

`GEN idealprimedec_degrees(GEN nf, GEN p)` return a (sorted) `t_VECSMALL` containing the residue degrees  $f(\mathfrak{p}/p)$ .

`GEN idealprimedec_kummer(GEN nf, GEN Ti, long ei, GEN p)` let  $nf$  (true  $nf$ ) correspond to  $K = \mathbf{Q}[X]/(T)$  ( $T$  monic `ZX`). Let  $T \equiv \prod_i T_i^{e_i} \pmod{p}$  be the factorization of  $T$  and let  $(f, g, h)$  be as in Dedekind criterion for prime  $p$ :  $f \equiv \prod T_i$ ,  $g \equiv \prod T_i^{e_i-1}$ ,  $h = (T - fg)/p$ , and let  $D$  be the gcd of  $(f, g, h)$  in  $\mathbf{F}_p[X]$ . Let  $T_i$  (`FpX`) be one irreducible factor  $T_i$  not dividing  $D$ , with  $ei = e_i$ . This function returns the prime ideal attached to  $T_i$  by Kummer / Dedekind criterion, namely  $p\mathbf{Z}_K + T_i(\bar{X})\mathbf{Z}_K$ , which has ramification index  $e_i$  over  $p$ . Shallow function.

`GEN idealhnf_Z_factor(GEN x, GEN *pvN, GEN *pvZ)` given an integral (non-0) ideal  $x$  in HNF, compute both the factorization of  $Nx$  and of  $x \cap \mathbf{Z}$ . This returns the vector of prime divisors of both

and sets `*pvN` and `*pvZ` to the corresponding `t_VECSMALL` vector of exponents for the factorization for the Norm and intersection with  $\mathbf{Z}$  respectively.

`GEN idealHNF_Z_factor_i(GEN x, GEN fa, GEN *pvN, GEN *pvZ)` internal variant of `idealHNF_Z_factor` where `fa` is either a partial factorization of  $x \cap \mathbf{Z}$  ( $= x[1, 1]$ ) or `NULL`. Returns the prime divisors of  $x$  above the rational primes in `fa` and attached `vN` and `vZ`. If `fa` is `NULL`, use the full factorization, i.e. identical to `idealHNF_Z_factor`.

`GEN nf_pV_to_prV(GEN nf, GEN P)` given a vector of rational primes  $P$ , return the vector of all prime ideals above the  $P[i]$ .

`GEN nf_deg1_prime(GEN nf)` let `nf` be a true `nf`. This function returns a degree 1 (unramified) prime ideal not dividing `nf.index`. In fact it returns an ideal above the smallest prime  $p \geq [K : \mathbf{Q}]$  satisfying those conditions.

`GEN prV_lcm_capZ(GEN L)` given a vector  $L$  of `prid` (maximal ideals) return the squarefree positive integer generating their lcm intersected with  $\mathbf{Z}$ . Not `gerepile`-safe.

`GEN pr_uniformizer(GEN pr, GEN F)` given a `prid` attached to  $\mathfrak{p}/p$  and  $F$  in  $\mathbf{Z}$  divisible exactly by  $p$ , return an  $F$ -uniformizer for  $\mathfrak{p}$ , i.e. a  $t$  in  $\mathbf{Z}_K$  such that  $v_{\mathfrak{p}}(t) = 1$  and  $(t, F/\mathfrak{p}) = 1$ . Not `gerepile`-safe.

### 13.1.14 Decomposition group.

`GEN idealramfrobenius(GEN nf, GEN gal, GEN pr, GEN ram)` Let  $K$  be the number field defined by `nf` and assume  $K/\mathbf{Q}$  be a Galois extension with Galois group given `gal=galoisinit(nf)`, and that `pr` is the prime ideal  $\mathfrak{P}$  in `prid` format, and that  $\mathfrak{P}$  is ramified, and `ram` is its list of ramification groups as output by `idealramgroups`. This function returns a permutation of `gal.group` which defines an automorphism  $\sigma$  in the decomposition group of  $\mathfrak{P}$  such that if  $p$  is the unique prime number in  $\mathfrak{P}$ , then  $\sigma(x) \equiv x^p \pmod{\mathbf{P}}$  for all  $x \in \mathbf{Z}_K$ .

`GEN idealramfrobenius_aut(GEN nf, GEN gal, GEN pr, GEN ram, GEN aut)` as `idealramfrobenius(nf, gal, pr, ram)`.

`GEN idealramgroups_aut(GEN nf, GEN gal, GEN pr, GEN aut)` as `idealramgroups(nf, gal, pr)`.

`GEN idealfrobenius_aut(GEN nf, GEN gal, GEN pr, GEN aut)` faster version of `idealfrobenius(nf, gal, pr)` where `aut` must be equal to `nfgaloispermtobasis(nf, gal)`.

### 13.1.15 Reducing modulo maximal ideals.

`GEN nfmodprinit(GEN nf, GEN pr)` returns an abstract `modpr` structure, attached to reduction modulo the maximal ideal `pr`, in `idealprimedec` format. From this data we can quickly project any `pr`-integral number field element to the residue field.

`GEN modpr_get_pr(GEN x)` return the `pr` component from a `modpr` structure.

`GEN modpr_get_p(GEN x)` return the  $p$  component from a `modpr` structure (underlying rational prime).

`GEN modpr_get_T(GEN x)` return the `T` component from a `modpr` structure: either `NULL` (prime of degree 1) or an irreducible `FpX` defining the residue field over  $\mathbf{F}_p$ .

In library mode, it is often easier to use directly

`GEN nf_to_Fq_init(GEN nf, GEN *ppr, GEN *pT, GEN *pp)` concrete version of `nfmodprinit`: `nf` and `*ppr` are the inputs, the return value is a `modpr` and `*ppr`, `*pT` and `*pp` are set as side effects.

The input `*ppr` is either a maximal ideal or already a `modpr` (in which case it is replaced by the underlying maximal ideal). The residue field is realized as  $\mathbf{F}_p[X]/(T)$  for some monic  $T \in \mathbf{F}_p[X]$ , and we set `*pT` to  $T$  and `*pp` to  $p$ . Set  $T = \text{NULL}$  if the prime has degree 1 and the residue field is  $\mathbf{F}_p$ .

In short, this receives (or initializes) a `modpr` structure, and extracts from it  $T$ ,  $p$  and  $\mathfrak{p}$ .

`GEN nf_to_Fq(GEN nf, GEN x, GEN modpr)` returns an `Fq` congruent to  $x$  modulo the maximal ideal attached to `modpr`. The output is canonical: all elements in a given residue class are represented by the same `Fq`.

`GEN Fq_to_nf(GEN x, GEN modpr)` returns an `nf` element lifting the residue field element  $x$ , either a `t_INT` or an algebraic integer in `algtobasis` format.

`GEN modpr_genFq(GEN modpr)` Returns an `nf` element whose image by `nf_to_Fq` is  $X \pmod{T}$ , if  $\deg T > 1$ , else 1.

`GEN zkmodprinit(GEN nf, GEN pr)` as `nfmodprinit`, but we assume we will only reduce algebraic integers, hence do not initialize data allowing to remove denominators. More precisely, we can in fact still handle an  $x$  whose rational denominator is not 0 in the residue field (i.e. if the valuation of  $x$  is non-negative at all primes dividing  $p$ ).

`GEN zk_to_Fq_init(GEN nf, GEN *pr, GEN *T, GEN *p)` as `nf_to_Fq_init`, able to reduce only  $p$ -integral elements.

`GEN zk_to_Fq(GEN x, GEN modpr)` as `nf_to_Fq`, for a  $p$ -integral  $x$ .

`GEN nfM_to_FqM(GEN M, GEN nf, GEN modpr)` reduces a matrix of `nf` elements to the residue field; returns an `FqM`.

`GEN FqM_to_nfM(GEN M, GEN modpr)` lifts an `FqM` to a matrix of `nf` elements.

`GEN nfV_to_FqV(GEN A, GEN nf, GEN modpr)` reduces a vector of `nf` elements to the residue field; returns an `FqV` with the same type as  $A$  (`t_VEC` or `t_COL`).

`GEN FqV_to_nfV(GEN A, GEN modpr)` lifts an `FqV` to a vector of `nf` elements (same type as  $A$ ).

`GEN nfX_to_FqX(GEN Q, GEN nf, GEN modpr)` reduces a polynomial with `nf` coefficients to the residue field; returns an `FqX`.

`GEN FqX_to_nfX(GEN Q, GEN modpr)` lifts an `FqX` to a polynomial with coefficients in `nf`.

The following functions are technical and avoid computing a true `nfmodpr`:

`GEN pr_basis_perm(GEN nf, GEN pr)` given a true `nf` structure and a prime ideal `pr` above  $p$ , return as a `t_VECSMALL` the  $f(\mathfrak{p}/p)$  indices  $i$  such that the `nf.zk[i] mod p` form an  $\mathbf{F}_p$ -basis of the residue field.

`GEN QXQV_to_FpM(GEN v, GEN T, GEN p)` let  $p$  be a positive integer,  $v$  be a vector of  $n$  polynomials with rational coefficients whose denominators are coprime to  $p$ , and  $T$  be a `ZX` (preferably monic) of degree  $d$  whose leading coefficient is coprime to  $p$ . Return the  $d \times n$  `FpM` whose columns are the  $v[i] \bmod T, p$  in the canonical basis  $1, X, \dots, X^{d-1}$ , see `RgX_to_RgC`. This is for instance useful when  $v$  contains a  $\mathbf{Z}$ -basis of the maximal order of a number field  $\mathbf{Q}[X]/(P)$ ,  $p$  is a prime not

dividing the index of  $P$  and  $T$  is an irreducible factor of  $P \bmod p$ , attached to a maximal ideal  $\mathfrak{p}$ : left-multiplication by the matrix maps number field elements (in basis form) to the residue field of  $\mathfrak{p}$ .

### 13.1.16 Valuations.

`long nfval(GEN nf, GEN x, GEN P)` return  $v_P(x)$

**Unsafe functions.** assume that  $P, Q$  are `prid`.

`long ZC_nfval(GEN x, GEN P)` returns  $v_P(x)$ , assuming  $x$  is a `ZC`, representing a non-zero algebraic integer.

`long ZC_nfvalrem(GEN x, GEN P, GEN *newx)` returns  $v = v_P(x)$ , assuming  $x$  is a `ZC`, representing a non-zero algebraic integer, and sets `*newx` to  $x\tau^v$  which is an algebraic integer coprime to  $p$ .

`int ZC_prdvd(GEN x, GEN P)` returns 1 if  $P$  divides  $x$  and 0 otherwise. Assumes that  $x$  is a `ZC`, representing an algebraic integer. Faster than computing  $v_P(x)$ .

`int pr_equal(GEN P, GEN Q)` returns 1 if  $P$  and  $Q$  represent the same maximal ideal: they must lie above the same  $p$  and share the same  $e, f$  invariants, but the  $p$ -uniformizer and  $\tau$  element may differ. Returns 0 otherwise.

### 13.1.17 Signatures.

“Signs” of the real embeddings of number field element are represented in additive notation, using the standard identification  $(\mathbf{Z}/2\mathbf{Z}, +) \rightarrow (\{-1, 1\}, \times)$ ,  $s \mapsto (-1)^s$ .

With respect to a fixed `nf` structure, a selection of real places (a divisor at infinity) is normally given as a `t_VECSMALL` of indices of the roots `nf.roots` of the defining polynomial for the number field. For compatibility reasons, in particular under GP, the (obsolete) `vec01` form is also accepted: a `t_VEC` with `gen_0` or `gen_1` entries.

The following internal functions go back and forth between the two representations for the Archimedean part of divisors (GP: 0/1 vectors, library: list of indices):

`GEN vec01_to_indices(GEN v)` given a `t_VEC`  $v$  with `t_INT` entries return as a `t_VECSMALL` the list of indices  $i$  such that  $v[i] \neq 0$ . (Typically used with 0,1-vectors but not necessarily so.) If  $v$  is already a `t_VECSMALL`, return it: not suitable for `gerepile` in this case.

`GEN vecsmall01_to_indices(GEN v)` as

`vec01_to_indices(zv_to_ZV(v));`

`GEN indices_to_vec01(GEN p, long n)` return the 0/1 vector of length  $n$  with ones exactly at the positions  $p[1], p[2], \dots$

`GEN nfembed(GEN nf, GEN x, long k)` returns a floating point approximation of the  $k$ -th embedding of  $x$  (attached to the  $k$ -th complex root in `nf.roots`).

`GEN nfsign(GEN nf, GEN x)`  $x$  being a number field element and `nf` any form of number field, return the 0 – 1-vector giving the signs of the  $r_1$  real embeddings of  $x$ , as a `t_VECSMALL`. Linear algebra functions like `Flv_add_inplace` then allow keeping track of signs in series of multiplications.

If  $x$  is a `t_VEC` of number field elements, return the matrix whose columns are the signs of the  $x[i]$ .

**GEN nfsign\_arch**(GEN nf, GEN x, GEN arch) arch being a list of distinct real places, either in **vec01** (**t\_VEC** with **gen\_0** or **gen\_1** entries) or **indices** (**t\_VECSMALL**) form (see **vec01\_to\_indices**), returns the signs of  $x$  at the corresponding places. This is the low-level function underlying **nfsign**.

**int nfchecksigns**(GEN nf, GEN x, GEN pl)  $pl$  is a **t\_VECSMALL** with  $r_1$  components, all of which are in  $\{-1, 0, 1\}$ . Return 1 if  $\sigma_i(x)pl[i] \geq 0$  for all  $i$ , and 0 otherwise.

**GEN nfsign\_units**(GEN bnf, GEN archp, int add\_tu) archp being a divisor at infinity in **indices** form (or **NULL** for the divisor including all real places), return the signs at archp of a system of fundamental units for the field, in the same order as **bnf.tufu** if **add\_tu** is set; and in the same order as **bnf.fu** otherwise.

**GEN nfsign\_from\_logarch**(GEN L, GEN invpi, GEN archp) given  $L$  the vector of the  $\log \sigma(x)$ , where  $\sigma$  runs through the (real or complex) embeddings of some number field, **invpi** being a floating point approximation to  $1/\pi$ , and archp being a divisor at infinity in **indices** form, return the signs of  $x$  at the corresponding places. This is the low-level function underlying **nfsign\_units**; the latter is actually a trivial wrapper **bnf** structures include the  $\log \sigma(x)$  for a system of fundamental units of the field.

**GEN set\_sign\_mod\_divisor**(GEN nf, GEN x, GEN y, GEN sarch) let  $f = f_0 f_\infty$  be a divisor, let **sarch** be the output of **nfarchstar**(nf, f0, finf), let  $x$  encode a vector of signs at the places of  $f_\infty$  (see below), and let  $y$  be a non-zero number field element. Returns  $z$  congruent to  $y \bmod f_0$  (integral if  $y$  is) such that  $z$  and  $x$  have the same signs at  $f_\infty$ .

The following formats are supported for  $x$ : a  $\{0, 1\}$ -vector of signs as a **t\_VECSMALL** (0 for positive, 1 for negative); **NULL** for a totally positive element (only 0s); a number field element which is replaced by its signature at  $f_\infty$ .

**GEN nfarchstar**(GEN nf, GEN f0, GEN finf) for a divisor  $f = f_0 f_\infty$  represented by the integral ideal **f0** in **HNF** and the **finf** in **indices** form, returns  $(\mathbf{Z}_K/f_\infty)^*$  in a form suitable for computations mod  $f$ . See **set\_sign\_mod\_divisor**.

**GEN idealprincipalunits**(GEN nf, GEN pr, long e) returns the multiplicative group  $(1 + pr)/(1 + pr^e)$  as an abelian group. Faster than **idealstar** when the norm of  $pr$  is large, since it avoids (useless) work in the multiplicative group of the residue field.

### 13.1.18 Maximal order and discriminant, conversion to nf structure.

A number field  $K = \mathbf{Q}[X]/(T)$  is defined by a monic  $T \in \mathbf{Z}[X]$ . The low-level function computing a maximal order is

**void nfmaxord**(nfmaxord\_t \*S, GEN T0, long flag), where the polynomial  $T_0$  is squarefree with integer coefficients. Let  $K$  be the étale algebra  $\mathbf{Q}[X]/(T_0)$  and let  $T = \mathbf{ZX\_Q\_normalize}(T_0)$ , i.e.  $T = CT_0(X/L)$  is monic and integral for some  $C, Q \in \mathbf{Q}$ .

The structure **nfmaxord\_t** is initialized by the call; it has the following fields:

```
GEN T0, T, dT, dK; /* T0, T, discriminants of T and K */
GEN unscale; /* the integer L */
GEN index; /* index of power basis in maximal order */
GEN dTP, dTE; /* factorization of |dT|, primes / exponents */
GEN dKP, dKE; /* factorization of |dK|, primes / exponents */
GEN basis; /* Z-basis for maximal order of  $\mathbf{Q}[X]/(T)$  */
```

The exponent vectors are `t_VECSMALL`. The primes in `dTP` and `dKP` are pseudoprimes, not proven primes. We recommend restricting to  $T = T_0$ , i.e. either to pass the input polynomial through `ZX_Q_normalize` *before* the call, or to forget about  $T_0$  and go on with the polynomial  $T$ ; otherwise `unscale`  $\neq 1$ , all data is expressed in terms of  $T \neq T_0$ , and needs to be converted to  $T_0$ . For instance to convert the basis to  $\mathbf{Q}[X]/(T_0)$ :

```
RgXV_unscale(S.basis, S.unscale)
```

Instead of passing  $T$  (monic `ZX`), one can use the format  $[T, listP]$  as in `nfbasis` or `nfinit`, which computes an order which is maximal at a set of primes, but need not be the maximal order.

The `flag` is an or-ed combination of the binary flags, both of them deprecated:

`nf_PARTIALFACT`: do not try to fully factor `dT` and only look for primes less than `primelimit`. In that case, the elements in `dTP` and `dKP` need not all be primes. But the resulting `dK`, `index` and `basis` are correct provided there exists no prime  $p > \text{primelimit}$  such that  $p^2$  divides the field discriminant `dK`. This flag is *deprecated*: the  $[T, listP]$  format is safer and more flexible.

`nf_ROUND2`: this flag is *deprecated* and now ignored.

`void nfinit_basic(nfmaxord_t *S, GEN T0)` a wrapper around `nfmaxord` (without the deprecated `flag`) that also accepts number field structures (`nf`, `bnf`, ...) for  $T_0$ .

`GEN nfmaxord_to_nf(nfmaxord_t *S, GEN ro, long prec)` convert an `nfmaxord_t` to an `nf` structure at precision `prec`, where `ro` is `NULL`. The argument `ro` may also be set to a vector with  $r_1 + r_2$  components containing the roots of  $S \rightarrow T$  suitably ordered, i.e. first  $r_1$  `t_REAL` roots, then  $r_2$  `t_COMPLEX` representing the conjugate pairs, but this is *strongly discouraged*: the format is error-prone, and it is hard to compute the roots to the right accuracy in order to achieve `prec` accuracy for the `nf`. This function uses the integer basis  $S \rightarrow \text{basis}$  as is, *without* performing LLL-reduction. Unless the basis is already known to be reduced, use rather the following higher-level function:

`GEN nfinit_complete(nfmaxord_t *S, long flag, long prec)` convert an `nfmaxord_t` to an `nf` structure at precision `prec`. The `flag` has the same meaning as in `nfinitall`. If  $S \rightarrow \text{basis}$  is known to be reduced, it will be faster to use `nfmaxord_to_nf`.

`GEN indexpartial(GEN T, GEN dT)`  $T$  a monic separable `ZX`, `dT` is either `NULL` (no information) or a multiple of the discriminant of  $T$ . Let  $K = \mathbf{Q}[X]/(T)$  and  $\mathbf{Z}_K$  its maximal order. Returns a multiple of the exponent of the quotient group  $\mathbf{Z}_K/(\mathbf{Z}[X]/(T))$ . In other word, a *denominator*  $d$  such that  $dx \in \mathbf{Z}[X]/(T)$  for all  $x \in \mathbf{Z}_K$ .

`GEN FpX_gcd_check(GEN x, GEN y, GEN D)` let  $x$  and  $y$  be two coprime polynomials with integer coefficients and let  $D$  be a factor of the resultant of  $x$  and  $y$ ; try to factor  $D$  by running the Euclidean algorithm on  $x$  and  $y$  modulo  $D$ . This returns `NULL` or a non trivial factor of  $D$ . This is the low-level function underlying `poldiscfactors` (applied to  $x$ , `ZX_deriv(x)` and the discriminant of  $x$ ). It succeeds when  $D$  has at least two prime divisors  $p$  and  $q$  such that one sub-resultant of  $x$  and  $y$  is divisible by  $p$  but not by  $q$ .

### 13.1.19 Computing in the class group.

We compute with arbitrary ideal representatives (in any of the various formats seen above), and call

`GEN bnfisprincipal0(GEN bnf, GEN x, long flag)`. The `bnf` structure already contains information about the class group in the form  $\oplus_{i=1}^n (\mathbf{Z}/d_i\mathbf{Z})g_i$  for canonical integers  $d_i$  (with  $d_n \mid \dots \mid d_1$  all  $> 1$ ) and essentially random generators  $g_i$ , which are ideals in HNF. We normally do not need the value of the  $g_i$ , only that they are fixed once and for all and that any (non-zero) fractional ideal  $x$  can be expressed uniquely as  $x = (t) \prod_{i=1}^n g_i^{e_i}$ , where  $0 \leq e_i < d_i$ , and  $(t)$  is some principal ideal. Computing  $e$  is straightforward, but  $t$  may be very expensive to obtain explicitly. The routine returns (possibly partial) information about the pair  $[e, t]$ , depending on `flag`, which is an or-ed combination of the following symbolic flags:

- `nf_GEN` tries to compute  $t$ . Returns  $[e, t]$ , with  $t$  an empty vector if the computation failed. This flag is normally useless in non-trivial situations since the next two serve analogous purposes in more efficient ways.

- `nf_GENMAT` tries to compute  $t$  in factored form, which is much more efficient than `nf_GEN` if the class group is moderately large; imagine a small ideal  $x = (t)g^{10000}$ : the norm of  $t$  has 10000 as many digits as the norm of  $g$ ; do we want to see it as a vector of huge meaningless integers? The idea is to compute  $e$  first, which is easy, then compute  $(t)$  as  $x \prod g_i^{-e_i}$  using successive `idealmulred`, where the ideal reduction extracts small principal ideals along the way, eventually raised to large powers because of the binary exponentiation technique; the point is to keep this principal part in factored *unexpanded* form. Returns  $[e, t]$ , with  $t$  an empty vector if the computation failed; this should be exceedingly rare, unless the initial accuracy to which `bnf` was computed was ridiculously low (and then `bnfinit` should not have succeeded either). Setting/unsetting `nf_GEN` has no effect when this flag is set.

- `nf_GEN_IF_PRINCIPAL` tries to compute  $t$  *only* if the ideal is principal ( $e = 0$ ). Returns `gen_0` if the ideal is not principal. Setting/unsetting `nf_GEN` has no effect when this flag is set, but setting/unsetting `nf_GENMAT` is possible.

- `nf_FORCE` in the above, insist on computing  $t$ , even if it requires recomputing a `bnf` from scratch. This is a last resort, and normally the accuracy of a `bnf` can be increased without trouble, but it may be that some algebraic information simply cannot be recovered from what we have: see `bnfnewprec`. It should be very rare, though.

In simple cases where you do not care about  $t$ , you may use

`GEN isprincipal(GEN bnf, GEN x)`, which is a shortcut for `bnfisprincipal0(bnf, x, 0)`.

The following low-level functions are often more useful:

`GEN isprincipalfact(GEN bnf, GEN C, GEN L, GEN f, long flag)` is about the same as `bnfisprincipal0` applied to  $C \prod L[i]^{f[i]}$ , where the  $L[i]$  are ideals, the  $f[i]$  integers and  $C$  is either an ideal or `NULL` (omitted). Make sure to include `nf_GENMAT` in `flag`!

`GEN isprincipalfact_or_fail(GEN bnf, GEN C, GEN L, GEN f)` is for delicate cases, where we must be more clever than `nf_FORCE` (it is used when trying to increase the accuracy of a *bnf*, for instance). It performs

```
isprincipalfact(bnf,C, L, f, nf_GENMAT);
```

but if it fails to compute  $t$ , it just returns a `t_INT`, which is the estimated precision (in words, as usual) that would have been sufficient to complete the computation. The point is that `nf_FORCE`

does exactly this internally, but goes on increasing the accuracy of the `bnf`, then discarding it, which is a major inefficiency if you intend to compute lots of discrete logs and have selected a precision which is just too low. (It is sometimes not so bad since most of the really expensive data is cached in `bnf` anyway, if all goes well.) With this function, the *caller* may decide to increase the accuracy using `bnfnewprec` (and keep the resulting `bnf`!), or avoid the computation altogether. In any case the decision can be taken at the place where it is most likely to be correct.

`void bnftestprimes(GEN bnf, GEN B)` is an ingredient to certify unconditionnally a `bnf` computed assuming GRH, cf. `bnfcertify`. Running this function successfully proves that the classes of all prime ideals of norm  $\leq B$  belong to the subgroup of the class group generated by the factorbase used to compute the `bnf` (equal to the class group under GRH). If the condition is not true, then (GRH is false and) the function will run forever.

If it is known that primes of norm less than  $B$  generate the class group (through variants of Minkowski's convex body or Zimmert's twin classes theorems), then the true class group is proven to be a quotient of `bnf.clgp`.

### 13.1.20 Floating point embeddings, the $T_2$ quadratic form.

We assume the `nf` is a true `nf` structure, attached to a number field  $K$  of degree  $n$  and signature  $(r_1, r_2)$ . We saw that

`GEN nf_get_M(GEN nf)` returns the  $(r_1 + r_2) \times n$  matrix  $M$  giving the embeddings of  $K$ , so that if  $v$  is an  $n$ -th dimensional `t_COL` representing the element  $\sum_{i=1}^n v[i]w_i$  of  $K$ , then `RgM_RgC_mul(M, v)` represents the embeddings of  $v$ . Its first  $r_1$  components are real numbers (`t_INT`, `t_FRAC` or `t_REAL`, usually the latter), and the last  $r_2$  are complex numbers (usually of `t_COMPLEX`, but not necessarily for embeddings of rational numbers).

`GEN embed_T2(GEN x, long r1)` assuming  $x$  is the vector of floating point embeddings of some algebraic number  $v$ , i.e.

```
x = RgM_RgC_mul(nf_get_M(nf), algtobasis(nf, v));
```

returns  $T_2(v)$ . If the floating point embeddings themselves are not needed, but only the values of  $T_2$ , it is more efficient to restrict to real arithmetic and use

```
gnorml2( RgM_RgC_mul(nf_get_G(nf), algtobasis(nf, v)));
```

`GEN embednorm_T2(GEN x, long r1)` analogous to `embed_T2`, applied to the `gnorm` of the floating point embeddings. Assuming that

```
x = gnorm( RgM_RgC_mul(nf_get_M(nf), algtobasis(nf, v)) );
```

returns  $T_2(v)$ .

`GEN embed_roots(GEN z, long r1)` given a vector  $z$  of  $r_1 + r_2$  complex embeddings of the algebraic number  $v$ , return the  $r_1 + 2r_2$  roots of its characteristic polynomial. Shallow function.

`GEN embed_disc(GEN z, long r1, long prec)` given a vector  $z$  of  $r_1 + r_2$  complex embeddings of the algebraic number  $v$ , return a floating point approximation of the discriminant of its characteristic polynomial as a `t_REAL` of precision `prec`.

`GEN embed_norm(GEN x, long r1)` given a vector  $z$  of  $r_1 + r_2$  complex embeddings of the algebraic number  $v$ , return (a floating point approximation of) the norm of  $v$ .



### 13.1.21 Ideal reduction, low level.

In the following routines  $nf$  is a true **nf**, attached to a number field  $K$  of degree  $n$ :

**GEN nf\_get\_Gtwist**(**GEN nf**, **GEN v**) assuming  $v$  is a **t\_VECSMALL** with  $r_1 + r_2$  entries, let

$$||x||_v^2 = \sum_{i=1}^{r_1+r_2} 2^{v_i \varepsilon_i} |\sigma_i(x)|^2,$$

where as usual the  $\sigma_i$  are the (real and) complex embeddings and  $\varepsilon_i = 1$ , resp. 2, for a real, resp. complex place. This is a twisted variant of the  $T_2$  quadratic form, the standard Euclidean form on  $K \otimes \mathbf{R}$ . In applications, only the relative size of the  $v_i$  will matter.

Let  $G_v \in M_n(\mathbf{R})$  be a square matrix such that if  $x \in K$  is represented by the column vector  $X$  in terms of the fixed  $\mathbf{Z}$ -basis of  $\mathbf{Z}_K$  in  $nf$ , then

$$||x||_v^2 = {}^t(G_v X) \cdot G_v X.$$

(This is a kind of Cholesky decomposition.) This function returns a rescaled copy of  $G_v$ , rounded to nearest integers, specifically **RM\_round\_maxrank**( $G_v$ ). Suitable for **gerepileupto**, but does not collect garbage. For convenience, also allow  $v = \text{NULL}$  (**nf\_get\_roundG**) and  $v$  a **t\_MAT** as output from the function itself: in both these cases, shallow function.

**GEN nf\_get\_Gtwist1**(**GEN nf**, **long i**). Simple special case. Returns the twisted  $G$  matrix attached to the vector  $v$  whose entries are all 0 except the  $i$ -th one, which is equal to 10.

**GEN idealpseudomin**(**GEN x**, **GEN G**). Let  $x, G$  be two ZMs, such that the product  $Gx$  is well-defined. This returns a “small” integral linear combinations of the columns of  $x$ , given by the LLL-algorithm applied to the lattice  $Gx$ . Suitable for **gerepileupto**, but does not collect garbage.

In applications,  $x$  is an integral ideal,  $G$  approximates a Cholesky form for the  $T_2$  quadratic form as returned by **nf\_get\_Gtwist**, and we return a small element  $a$  in the lattice  $(x, T_2)$ . This is used to implement **idealred**.

**GEN idealpseudomin\_nonscalar**(**GEN x**, **GEN G**). As **idealpseudomin**, but we insist of returning a non-scalar  $a$  (**ZV\_isscalar** is false), if the dimension of  $x$  is  $> 1$ .

In the interpretation where  $x$  defines an integral ideal on a fixed  $\mathbf{Z}_K$  basis whose first element is 1, this means that  $a$  is not rational.

**GEN idealpseudored**(**GEN x**, **GEN G**). As **idealpseudomin** but we return the full reduced  $\mathbf{Z}$ -basis of  $x$  as a **t\_MAT** instead of a single vector.

**GEN idealred\_elt**(**GEN nf**, **GEN x**) shortcut for

**idealpseudomin**(**x**, **nf\_get\_roundG**(**nf**))

### 13.1.22 Ideal reduction, high level.

Given an ideal  $x$  this means finding a “simpler” ideal in the same ideal class. The public GP function is of course available

`GEN idealred0(GEN nf, GEN x, GEN v)` finds an  $a \in K^*$  such that  $(a)x$  is integral of small norm and returns it, as an ideal in HNF. What “small” means depends on the parameter  $v$ , see the GP description. More precisely,  $a$  is returned by `idealpseudomin((xZ) $x$  - 1), G)` divided by  $x_{\mathbf{Z}}$ , where  $x_{\mathbf{Z}} = (x \cap \mathbf{Z})$  and where  $G$  is `nf_get_Gtwist(nf, v)` for  $v \neq \text{NULL}$  and `nf_get_roundG(nf)` otherwise.

Usually one sets  $v = \text{NULL}$  to obtain an element of small  $T_2$  norm in  $x$ :

`GEN idealred(GEN nf, GEN x)` is a shortcut for `idealred0(nf, x, NULL)`.

The function `idealred` remains complicated to use: in order not to lose information  $x$  must be an extended ideal, otherwise the value of  $a$  is lost. There is a subtlety here: the principal ideal  $(a)$  is easy to recover, but  $a$  itself is an instance of the principal ideal problem which is very difficult given only an  $nf$  (once a  $bnf$  structure is available, `bnfisprincipal0` will recover it).

`GEN idealmoddivisor(GEN bnr, GEN x)` A proof-of-concept implementation, useless in practice. If `bnr` is attached to some modulus  $f$ , returns a “small” ideal in the same class as  $x$  in the ray class group modulo  $f$ . The reason why this is useless is that using extended ideals with principal part in a computation, there is a simple way to reduce them: simply reduce the generator of the principal part in  $(\mathbf{Z}_K/f)^*$ .

`GEN famat_to_nf_moddivisor(GEN nf, GEN g, GEN e, GEN bid)` given a true  $nf$  attached to a number field  $K$ , a  $bid$  structure attached to a modulus  $f$ , and an algebraic number in factored form  $\prod g[i]^{e[i]}$ , such that  $(g[i], f) = 1$  for all  $i$ , returns a small element in  $\mathbf{Z}_K$  congruent to it mod  $f$ . Note that if  $f$  contains places at infinity, this includes sign conditions at the specified places.

A simpler case when the conductor has no place at infinity:

`GEN famat_to_nf_modideal_coprime(GEN nf, GEN g, GEN e, GEN f, GEN expo)` as above except that the ideal  $f$  is now integral in HNF (no need for a full  $bid$ ), and we pass the exponent of the group  $(\mathbf{Z}_K/f)^*$  as `expo`; any multiple will also do, at the expense of efficiency. Of course if a  $bid$  for  $f$  is available, it is easy to extract  $f$  and the exact value of `expo` from it (the latter is the first elementary divisor in the group structure). A useful trick: if you set `expo` to *any* positive integer, the result is correct up to `expo`-th powers, hence exact if `expo` is a multiple of the exponent; this is useful when trying to decide whether an element is a square in a residue field for instance! (take `expo=2`).

`GEN nf_to_Fp_coprime(GEN nf, GEN x, GEN modpr)` this low-level function is variant of `famat_to_nf_modideal_coprime`:  $nf$  is a true  $nf$  structure, `modpr` is from `zkmodprinit` attached to a prime of degree 1 above the prime number  $p$ , and  $x$  is either a number field element or a `famat` factorization matrix. We finally assume that no component of  $x$  has a denominator  $p$ .

What to do when the  $g[i]$  are not coprime to  $f$ , but only  $\prod g[i]^{e[i]}$  is? Then the situation is more complicated, and we advise to solve it one prime divisor of  $f$  at a time. Let  $v$  the valuation attached to a maximal ideal  $\mathfrak{p}$  and assume  $v(f) = k > 0$ :

`GEN famat_makecoprime(GEN nf, GEN g, GEN e, GEN pr, GEN prk, GEN expo)` returns an element in  $(\mathbf{Z}_K/\mathfrak{p}^k)^*$  congruent to the product  $\prod g[i]^{e[i]}$ , assumed to be globally coprime to  $f$ . As above, `expo` is any positive multiple of the exponent of  $(\mathbf{Z}_K/\mathfrak{p}^k)^*$ , for instance  $(Nv - 1)p^{k-1}$ ,

if  $p$  is the underlying rational prime. You may use other values of `expo` (see the useful trick in `famat_to_nf_modideal_coprime`).

`GEN Idealstarprk(GEN nf, GEN pr, long k, long flag)` same as `Idealstar` for  $I = \mathfrak{p}^k$

### 13.1.23 Class field theory.

Under GP, a class-field theoretic description of a number field is given by a triple  $A, B, C$ , where the defining set  $[A, B, C]$  can have any of the following forms:  $[bnr]$ ,  $[bnr, subgroup]$ ,  $[bnf, modulus]$ ,  $[bnf, modulus, subgroup]$ . You can still use directly all of (`libpari`'s routines implementing) GP's functions as described in Chapter 3, but they are often awkward in the context of `libpari` programming. In particular, it does not make much sense to always input a triple  $A, B, C$  because of the fringe  $[bnf, modulus, subgroup]$ . The first routine to call, is thus

`GEN Buchray(GEN bnf, GEN mod, long flag)` initializes a *bnr* structure from `bnf` and `modulus`. `mod`. `flag` is an or-ed combination of `nf_GEN` (include generators) and `nf_INIT` (if omitted, do not return a *bnr*, only the ray class group as an abelian group). In fact, a single value of `flag` actually makes sense: `nf_GEN | nf_INIT` to initialize a proper *bnr*: removing `nf_GEN` saves very little time, but the corresponding crippled *bnr* structure will raise errors in most class field theoretic functions. Possibly also 0 to quickly compute the ray class group structure; `bnrclassno` is faster if we only need the *order* of the ray class group.

Now we have a proper *bnr* encoding a `bnf` and a modulus, we no longer need the  $[bnf, modulus]$  and  $[bnf, modulus, subgroup]$  forms, which would internally call `Buchray` anyway. Recall that a subgroup  $H$  is given by a matrix in HNF, whose column express generators of  $H$  on the fixed generators of the ray class group that stored in our *bnr*. You may also code the trivial subgroup by `NULL`.

`GEN bnrconductor(GEN bnr, GEN H, long flag)` see the documentation of the GP function.

`GEN bnrconductor_i(GEN bnr, GEN H, long flag)` shallow variant of `bnrconductor`. Useful when `flag = 2` and the conductor is the *bnr* modulus: avoids copying the *bnr* (wasteful).

`long bnriscconductor(GEN bnr, GEN H)` returns 1 if the class field defined by the subgroup  $H$  (of the ray class group mod  $f$  coded in *bnr*) has conductor  $f$ . Returns 0 otherwise.

`GEN bnrcchar_primitive(GEN bnr, GEN chi, GEN bnrc)` Given a normalized character `chi = [d, c]` on `bnr.clgp` (see `char_normalize`) of conductor `bnrc.mod`, compute the primitive character `chic` on `bnrc.clgp` equivalent to `chi`, given as a normalized character  $[D, C] : \text{chic}(\text{bnrc.gen}[i])$  is  $\zeta_D^{C[i]}$ , where  $D$  is minimal. It is easier to use `bnrconductor_i(bnr, chi, 2)`, but the latter recomputes `bnrc` for each new character.

`GEN bnrdisc(GEN bnr, GEN H, long flag)` returns the discriminant and signature of the class field defined by *bnr* and  $H$ . See the description of the GP function for details. `flag` is an or-ed combination of the flags `rnf_REL` (output relative data) and `rnf_COND` (return 0 unless the modulus is the conductor).

`GEN bnrsurjection(GEN BNR, GEN bnr)` `BNR` and *bnr* defined over the same field  $K$ , for moduli  $F$  and  $f$  with  $F \mid f$ , returns the matrix of the canonical surjection  $\text{Cl}_K(F) \rightarrow \text{Cl}_K(f)$  (giving the image of the fixed ray class group generators of `BNR` in terms of the ones in *bnr*).

`GEN ABC_to_bnr(GEN A, GEN B, GEN C, GEN *H, int addgen)` This is a quick conversion function designed to go from the too general (inefficient)  $A, B, C$  form to the preferred *bnr*,  $H$  form for class fields. Given  $A, B, C$  as explained above (omitted entries coded by `NULL`), return the attached *bnr*, and set  $H$  to the attached subgroup. If `addgen` is 1, make sure that if the *bnr* needed to be computed, then it contains generators.

### 13.1.24 Grunwald–Wang theorem.

GEN `nfgrkwummer`(GEN `nf`, GEN `Lpr`, GEN `Ld`, GEN `pl`, long `var`) low-level version of `nfgrunwaldwang`, assuming that `nf` contains suitable roots of unity, and directly using Kummer theory to construct the extension.

GEN `bnfgwgeneric`(GEN `bnf`, GEN `Lpr`, GEN `Ld`, GEN `pl`, long `var`) low-level version of `nfgrunwaldwang`, assuming that `bnf` is a `bnfinit` structure, and calling `rnfkummer` to construct the extension.

### 13.1.25 Relative equations, Galois conjugates.

GEN `nfissquarefree`(GEN `nf`, GEN `P`) given  $P$  a polynomial with coefficients in  $nf$ , return 1 if  $P$  is squarefree, and 0 otherwise. It is allowed (though less efficient) to replace  $nf$  by a monic `ZX` defining the field.

GEN `rnfequationall`(GEN `A`, GEN `B`, long `*pk`, GEN `*pLPRS`)  $A$  is either an  $nf$  type (corresponding to a number field  $K$ ) or an irreducible `ZX` defining a number field  $K$ .  $B$  is an irreducible polynomial in  $K[X]$ . Returns an absolute equation  $C$  (over  $\mathbf{Q}$ ) for the number field  $K[X]/(B)$ .  $C$  is the characteristic polynomial of  $b + ka$  for some roots  $a$  of  $A$  and  $b$  of  $B$ , and  $k$  is a small rational integer. Set `*pk` to  $k$ .

If `pLPRS` is not NULL set it to  $[h_0, h_1]$ ,  $h_i \in \mathbf{Q}[X]$ , where  $h_0 + h_1Y$  is the last non-constant polynomial in the pseudo-Euclidean remainder sequence attached to  $A(Y)$  and  $B(X - kY)$ , leading to  $C = \text{Res}_Y(A(Y), B(X - kY))$ . In particular  $a := -h_0/h_1$  is a root of  $A$  in  $\mathbf{Q}[X]/(C)$ , and  $X - ka$  is a root of  $B$ .

GEN `nf_rnfeq`(GEN `A`, GEN `B`) wrapper around `rnfequationall` to allow mapping  $K \rightarrow L$  (`eltup`) and converting elements of  $L$  between absolute and relative form (`reltoabs`, `abstorel`), *without* computing a full  $rnf$  structure, which is useful if the relative integral basis is not required. In fact, since  $A$  may be a `t_POL` or an  $nf$ , the integral basis of the base field is not needed either. The return value is the same as `rnf_get_map`. Shallow function.

GEN `nf_rnfeqsimple`(GEN `nf`, GEN `relpol`) as `nf_rnfeq` except some fields are omitted, so that only the `abstorel` operation is supported. Shallow function.

GEN `eltabstorel`(GEN `rnfeq`, GEN `x`) `rnfeq` is as given by `rnf_get_map` (but in this case `rnfeltabstorel` is more robust), `nf_rnfeq` or `nf_rnfeqsimple`, return  $x$  as an element of  $L/K$ , i.e. as a `t_POLMOD` with `t_POLMOD` coefficients. Shallow function.

GEN `eltabstorel_lift`(GEN `rnfeq`, GEN `x`) same as `eltabstorel`, except that  $x$  is returned in partially lifted form, i.e. as a `t_POL` with `t_POLMOD` coefficients.

GEN `eltreltoabs`(GEN `rnfeq`, GEN `x`) `rnfeq` is as given by `rnf_get_map` (but in this case `rnfeltreltoabs` is more robust) or `nf_rnfeq`, return  $x$  in absolute form.

GEN `nf_nfzk`(GEN `nf`, GEN `rnfeq`) `rnfeq` as given by `nf_rnfeq`, `nf` a true  $nf$  structure, return a suitable representation of `nf.zk` allowing quick computation of the map  $K \rightarrow L$  by the function `nfeltup`, *without* computing a full  $rnf$  structure, which is useful if the relative integral basis is not required. The computed value is the same as in `rnf_get_nfzk`. Shallow function.

GEN `nfeltup`(GEN `nf`, GEN `x`, GEN `zknf`) `zknf` and is initialized by `nf_nfzk` or `rnf_get_nfzk` (but in this case `rnfeltup` is more robust); `nf` is a true  $nf$  structure for  $K$ , returns  $x \in K$  as a (lifted) element of  $L$ , in absolute form.

`GEN rnfdisc_factored(GEN nf, GEN pol, GEN *pd)` variant of `rnfdisc` returning the relative discriminant ideal *factorization*, and setting `*pd` to the discriminant as an element in  $K^*/(K^*)^2$ . Shallow function.

`GEN Rg_nffix(const char *f, GEN T, GEN c, int lift)` given a ZX  $T$  and a “coefficient”  $c$  supposedly belonging to  $\mathbf{Q}[y]/(T)$ , check whether this is the case and return a cleaned up version of  $c$ . The string  $f$  is the calling function name, used to report errors.

This means that  $c$  must be one of `t_INT`, `t_FRAC`, `t_POL` in the variable  $y$  with rational coefficients, or `t_POLMOD` modulo  $T$  which lift to a rational `t_POL` as above. The cleanup consists in the following improvements:

- `t_POL` coefficients are reduced modulo  $T$ .
- `t_POL` and `t_POLMOD` belonging to  $\mathbf{Q}$  are converted to rationals, `t_INT` or `t_FRAC`.
- if `lift` is non-zero, convert `t_POLMOD` to `t_POL`, and otherwise convert `t_POL` to `t_POLMODs` modulo  $T$ .

`GEN RgX_nffix(const char *f, GEN T, GEN P, int lift)` check whether  $P$  is a polynomial with coefficients in the number field defined by the absolute equation  $T(y) = 0$ , where  $T$  is a ZX and returns a cleaned up version of  $P$ . This checks whether  $P$  is indeed a `t_POL` with variable compatible with coefficients in  $\mathbf{Q}[y]/(T)$ , i.e.

$$\text{varncmp}(\text{varn}(P), \text{varn}(T)) < 0$$

and applies `Rg_nffix` to each coefficient.

`GEN RgV_nffix(const char *f, GEN T, GEN P, int lift)` as `RgX_nffix` for a vector of coefficients.

`GEN polmod_nffix(const char *f, GEN rnf, GEN x, int lift)` given a `t_POLMOD`  $x$  supposedly defining an element of  $rnf$ , check this and perform `Rg_nffix` cleanups.

`GEN polmod_nffix2(const char *f, GEN T, GEN P, GEN x, int lift)` as in `polmod_nffix`, where the relative extension is explicitly defined as  $L = (\mathbf{Q}[y]/(T))[x]/(P)$ , instead of by an `rnf` structure.

`long numberofconjugates(GEN T, long pinit)` returns a quick multiple for the number of  $\mathbf{Q}$ -automorphism of the (integral, monic) `t_POL`  $T$ , from modular factorizations, starting from prime `pinit` (you can set it to 2). This upper bounds often coincides with the actual number of conjugates. Of course, you should use `nfgaloisconj` to be sure.

`GEN nfroots_if_split(GEN *pt, GEN T)` let `*pt` point either to a number field structure or an irreducible ZX, defining a number field  $K$ . Given  $T$  a monic squarefree polynomial with coefficients in  $\mathbf{Z}_K$ , return the list of roots of `pol` in  $K$  if the polynomial splits completely, and `NULL` otherwise. In other words, this checks whether  $K[X]/(T)$  is normal over  $K$  (hence Galois since  $T$  is separable by assumption).

In the case where `*pT` is a ZX, the function has to compute internally a conditional `nf` attached to  $K$ , whose `nf.zk` may not define the maximal order  $\mathbf{Z}_K$  (see `nfroots`); `*pT` is then replaced by the conditional `nf` to avoid losing that information.

### 13.1.26 Cyclotomics units.

GEN `nfrootsof1(GEN nf)` returns a two-component vector  $[w, z]$  where  $w$  is the number of roots of unity in the number field  $nf$ , and  $z$  is a primitive  $w$ -th root of unity.

GEN `nfcyclotomicunits(GEN nf, GEN zu)` where `zu` is as output by `nfrootsof1(nf)`, return the vector of the cyclotomic units in  $nf$  expressed over the integral basis.

### 13.1.27 Obsolete routines.

Still provided for backward compatibility, but should not be used in new programs. They will eventually disappear.

GEN `zidealstar(GEN nf, GEN x)` short for `Idealstar(nf,x,nf_GEN)`

GEN `zidealstarinit(GEN nf, GEN x)` short for `Idealstar(nf,x,nf_INIT)`

GEN `zidealstarinitgen(GEN nf, GEN x)` short for `Idealstar(nf,x,nf_GEN|nf_INIT)`

GEN `buchimag(GEN D, GEN c1, GEN c2, GEN gCO)` short for

`Buchquad(D,gtodouble(c1),gtodouble(c2), /*ignored*/0)`

GEN `buchreal(GEN D, GEN gsens, GEN c1, GEN c2, GEN RELSUP, long prec)` short for

`Buchquad(D,gtodouble(c1),gtodouble(c2), prec)`

The following use a naming scheme which is error-prone and not easily extensible; besides, they compute generators as per `nf_GEN` and not `nf_GENMAT`. Don't use them:

GEN `isprincipalforce(GEN bnf, GEN x)`

GEN `isprincipalgen(GEN bnf, GEN x)`

GEN `isprincipalgenforce(GEN bnf, GEN x)`

GEN `isprincipalraygen(GEN bnr, GEN x)`, use `bnrisprincipal`.

Variants on `polred`: use `polredbest`.

GEN `factoredpolred(GEN x, GEN fa)`

GEN `factoredpolred2(GEN x, GEN fa)`

GEN `smallpolred(GEN x)`

GEN `smallpolred2(GEN x)`, use `Polred`.

GEN `polred0(GEN x, long flag, GEN p)`

GEN `polredabs(GEN x)`

GEN `polredabs2(GEN x)`

GEN `polredabsall(GEN x, long flun)`

Superseded by `bnrdiscclist0`:

GEN `discrayabslist(GEN bnf, GEN listes)`

GEN `discrayabslistarch(GEN bnf, GEN arch, long bound)`

Superseded by `idealappr` (*flag* is ignored)

GEN `idealappr0(GEN nf, GEN x, long flag)`

## 13.2 Galois extensions of $\mathbb{Q}$ .

This section describes the data structure output by the function `galoisinit`. This will be called a `gal` structure in the following.

### 13.2.1 Extracting info from a `gal` structure.

The functions below expect a `gal` structure and are shallow. See the documentation of `galoisinit` for the meaning of the member functions.

`GEN gal_get_pol(GEN gal)` returns `gal.pol`

`GEN gal_get_p(GEN gal)` returns `gal.p`

`GEN gal_get_e(GEN gal)` returns the integer  $e$  such that `gal.mod==gal.pe`.

`GEN gal_get_mod(GEN gal)` returns `gal.mod`.

`GEN gal_get_roots(GEN gal)` returns `gal.roots`.

`GEN gal_get_invvdm(GEN gal)` `gal[4]`.

`GEN gal_get_den(GEN gal)` return `gal[5]`.

`GEN gal_get_group(GEN gal)` returns `gal.group`.

`GEN gal_get_gen(GEN gal)` returns `gal.gen`.

`GEN gal_get_orders(GEN gal)` returns `gal.orders`.

### 13.2.2 Miscellaneous functions.

`GEN nfgaloispermtobasis(GEN nf, GEN gal)` return the images of the field generator by the automorphisms `gal.orders` expressed on the integral basis `nf.zk`.

`GEN nfgaloismatrix(GEN nf, GEN s)` returns the ZM attached to the automorphism  $s$ , seen as a linear operator expressed on the number field integer basis. This allows to use

```
M = nfgaloismatrix(nf, s);
sx = ZM_ZC_mul(M, x);    /* or RgM_RgC_mul(M, x) if x is not integral */
```

instead of

```
sx = nfgaloisapply(nf, s, x);
```

for an algebraic integer  $x$ .

## 13.3 Quadratic number fields and quadratic forms.

### 13.3.1 Checks.

`void check_quaddisc(GEN x, long *s, long *mod4, const char *f)` checks whether the GEN  $x$  is a quadratic discriminant (`t_INT`, not a square, congruent to 0,1 modulo 4), and raise an exception otherwise. Set  $*s$  to the sign of  $x$  and  $*mod4$  to  $x$  modulo 4 (0 or 1).

`void check_quaddisc_real(GEN x, long *mod4, const char *f)` as `check_quaddisc`; check that `signe(x)` is positive.

`void check_quaddisc_imag(GEN x, long *mod4, const char *f)` as `check_quaddisc`; check that `signe(x)` is negative.

### 13.3.2 Class number.

The function `quadclassunit` uses index calculus and runs in subexponential time but it assumes the truth of the GRH. For imaginary quadratic orders, it is comparatively slow for *small* values, say  $|D| \leq 10^{18}$ . Here are some alternatives:

`GEN classno(GEN D)` corresponds to `qfbclassno(D,0)` and is only useful for  $D < 0$ , uses a baby-step giant-step technique and runs in time  $O(D^{1/4})$ . The result is guaranteed correct for  $|D| < 2 \cdot 10^{10}$  and fastest in that range. For larger values of  $|D|$ , the algorithm is no longer rigorous and may give incorrect results (we know no concrete example); it also becomes relatively less interesting compared to `quadclassunit`.

`GEN classno2(GEN D)` corresponds to `qfbclassno(D,1)` and runs in time  $O(D^{1/2})$ ; it is provided for testing purposes only: it is never competitive.

`GEN hclassno(GEN d)` returns the Hurwitz-Kronecker class number  $H(d)$ . These play a central role in trace formulas and are usually needed for many consecutive values of  $d$ . Thus, the function uses a cache so that later calls for *small* consecutive values of  $d$  are instantaneous, see `getcache`. Large values of  $d$  ( $d > 500000$ ) call `quadclassunit` individually and are not memoized.

`GEN hclassno6(GEN d)` assuming  $d > 0$ , returns the integer  $6H(d)$ . This is a low-level function behind `hclassno`.

`ulong hclassno6u(ulong d)` assuming  $d > 0$ , returns the integer  $6H(d)$ .

### 13.3.3 `t_QFI`, `t_QFR`.

`GEN qfi(GEN x, GEN y, GEN z)` creates the `t_QFI`  $(x, y, z)$ .

`GEN qfr(GEN x, GEN y, GEN z, GEN d)` creates the `t_QFR`  $(x, y, z)$  with distance component  $d$ .

`GEN qfr_1(GEN q)` given a `t_QFR`  $q$ , return the unit form  $q^0$ .

`GEN qfi_1(GEN q)` given a `t_QFI`  $q$ , return the unit form  $q^0$ .

`int qfb_equal1(GEN q)` returns 1 if the `t_QFI` or `t_QFR`  $q$  is the unit form.



### 13.3.3.1 Composition.

GEN `qficomp`(GEN `x`, GEN `y`) compose the two `t_QFI`  $x$  and  $y$ , then reduce the result. This is the same as `gmul(x,y)`.

GEN `qfrcomp`(GEN `x`, GEN `y`) compose the two `t_QFR`  $x$  and  $y$ , then reduce the result. This is the same as `gmul(x,y)`.

GEN `qfisqr`(GEN `x`) as `qficomp(x,y)`.

GEN `qfrsqr`(GEN `x`) as `qfrcomp(x,y)`.

Same as above, *without* reducing the result:

GEN `qficompraw`(GEN `x`, GEN `y`)

GEN `qfrcompraw`(GEN `x`, GEN `y`)

GEN `qfisqrraw`(GEN `x`)

GEN `qfrsqrraw`(GEN `x`)

GEN `qfbcompraw`(GEN `x`, GEN `y`) compose two `t_QFIs` or two `t_QFRs`, without reduce the result.

### 13.3.3.2 Powering.

GEN `powgi`(GEN `x`, GEN `n`) computes  $x^n$  (will work for many more types than `t_QFI` and `t_QFR`, of course). Reduce the result.

GEN `qfrpow`(GEN `x`, GEN `n`) computes  $x^n$  for a `t_QFR`  $x$ , reducing along the way. If the distance component is initially 0, leave it alone; otherwise update it.

GEN `qfbpowraw`(GEN `x`, long `n`) compute  $x^n$  (pure composition, no reduction), for a `t_QFI` or `t_QFR`  $x$ .

GEN `qfipowraw`(GEN `x`, long `n`) as `qfbpowraw`, for a `t_QFI`  $x$ .

GEN `qfrpowraw`(GEN `x`, long `n`) as `qfbpowraw`, for a `t_QFR`  $x$ .

### 13.3.3.3 Order, discrete log.

GEN `qfi_order`(GEN `q`, GEN `o`) assuming that the `t_QFI`  $q$  has order dividing  $o$ , compute its order in the class group. The order can be given in all formats allowed by generic discrete log functions, the preferred format being `[ord, fa]` (`t_INT` and its factorization).

GEN `qfi_log`(GEN `a`, GEN `g`, GEN `o`) given a `t_QFI`  $a$  and assuming that the `t_QFI`  $g$  has order  $o$ , compute an integer  $k$  such that  $a^k = g$ . Return `cgetg(1, t_VEC)` if there are no solutions. Uses a generic Pollig-Hellman algorithm, then either Shanks (small  $o$ ) or Pollard rho (large  $o$ ) method. The order can be given in all formats allowed by generic discrete log functions, the preferred format being `[ord, fa]` (`t_INT` and its factorization).

GEN `qfi_Shanks`(GEN `a`, GEN `g`, long `n`) given a `t_QFI`  $a$  and assuming that the `t_QFI`  $g$  has (small) order  $n$ , compute an integer  $k$  such that  $a^k = g$ . Return `cgetg(1, t_VEC)` if there are no solutions. Directly uses Shanks algorithm, which is inefficient when  $n$  is composite.

### 13.3.3.4 Solve, Cornacchia.

The following functions underly `qfbsolve`;  $p$  denotes a prime number.

`GEN qfisolvp(GEN Q, GEN p)` solves  $Q(x, y) = p$  over the integers, for a `t_QFI`  $Q$ . Return `gen_0` if there are no solutions.

`GEN qfrsolvep(GEN Q, GEN p)` solves  $Q(x, y) = p$  over the integers, for a `t_QFR`  $Q$ . Return `gen_0` if there are no solutions.

`long cornacchia(GEN d, GEN p, GEN *px, GEN *py)` solves  $x^2 + dy^2 = p$  over the integers, where  $d > 0$ . Return 1 if there is a solution (and store it in `*x` and `*y`), 0 otherwise.

`long cornacchia2(GEN d, GEN p, GEN *px, GEN *py)` as `cornacchia`, for the equation  $x^2 + dy^2 = 4p$ .

`long cornacchia2_sqrt(GEN d, GEN p, GEN b, GEN *px, GEN *py)` as `cornacchia2`, where  $p > 2$  and  $b$  is the smallest squareroot of  $d$  modulo  $p$ .

### 13.3.3.5 Prime forms.

`GEN primeform_u(GEN x, ulong p)` `t_QFI` whose first coefficient is the prime  $p$ .

`GEN primeform(GEN x, GEN p, long prec)`

**13.3.4 Efficient real quadratic forms.** Unfortunately, `t_QFRs` are very inefficient, and are only provided for backward compatibility.

- they do not contain needed quantities, which are thus constantly recomputed (the discriminant  $D$ ,  $\sqrt{D}$  and its integer part),

- the distance component is stored in logarithmic form, which involves computing one extra logarithm per operation. It is much more efficient to store its exponential, computed from ordinary multiplications and divisions (taking exponent overflow into account), and compute its logarithm at the very end.

Internally, we have two representations for real quadratic forms:

- `qfr3`, a container  $[a, b, c]$  with at least 3 entries: the three coefficients; the idea is to ignore the distance component.

- `qfr5`, a container with at least 5 entries  $[a, b, c, e, d]$ : the three coefficients a `t_REAL`  $d$  and a `t_INT`  $e$  coding the distance component  $2^{Ne}d$ , in exponential form, for some large fixed  $N$ .

It is a feature that `qfr3` and `qfr5` have no specified length or type. It implies that a `qfr5` or `t_QFR` will do whenever a `qfr3` is expected. Routines using these objects all require a global context, provided by a `struct qfr_data *`:

```
struct qfr_data {
    GEN D;          /* discriminant, t_INT */
    GEN sqrtD;      /* sqrt(D), t_REAL */
    GEN isqrtD;     /* floor(sqrt(D)), t_INT */
};
```

`void qfr_data_init(GEN D, long prec, struct qfr_data *S)` given a discriminant  $D > 0$ , initialize  $S$  for computations at precision `prec` ( $\sqrt{D}$  is computed to that initial accuracy).

All functions below are shallow, and not stack clean.

GEN `qfr3_comp`(GEN `x`, GEN `y`, struct `qfr_data *S`) compose two `qfr3`, reducing the result.

GEN `qfr3_pow`(GEN `x`, GEN `n`, struct `qfr_data *S`) compute  $x^n$ , reducing along the way.

GEN `qfr3_red`(GEN `x`, struct `qfr_data *S`) reduce  $x$ .

GEN `qfr3_rho`(GEN `x`, struct `qfr_data *S`) perform one reduction step; `qfr3_red` just performs reduction steps until we hit a reduced form.

GEN `qfr3_to_qfr`(GEN `x`, GEN `d`) recover an ordinary `t_QFR` from the `qfr3`  $x$ , adding distance component  $d$ .

Before we explain `qfr5`, recall that it corresponds to an ideal, that reduction corresponds to multiplying by a principal ideal, and that the distance component is a clever way to keep track of these principal ideals. More precisely, reduction consists in a number of reduction steps, going from the form  $(a, b, c)$  to  $\rho(a, b, c) = (c, -b \bmod 2c, *)$ ; the distance component is multiplied by (a floating point approximation to)  $(b + \sqrt{D})/(b - \sqrt{D})$ .

GEN `qfr5_comp`(GEN `x`, GEN `y`, struct `qfr_data *S`) compose two `qfr5`, reducing the result, and updating the distance component.

GEN `qfr5_pow`(GEN `x`, GEN `n`, struct `qfr_data *S`) compute  $x^n$ , reducing along the way.

GEN `qfr5_red`(GEN `x`, struct `qfr_data *S`) reduce  $x$ .

GEN `qfr5_rho`(GEN `x`, struct `qfr_data *S`) perform one reduction step.

GEN `qfr5_dist`(GEN `e`, GEN `d`, long `prec`) decode the distance component from exponential (`qfr5`-specific) to logarithmic form (as in a `t_QFR`).

GEN `qfr_to_qfr5`(GEN `x`, long `prec`) convert a `t_QFR` to a `qfr5` with initial trivial distance component ( $= 1$ ).

GEN `qfr5_to_qfr`(GEN `x`, GEN `d`), assume  $x$  is a `qfr5` and  $d$  was the original distance component of some `t_QFR` that we converted using `qfr_to_qfr5` to perform efficiently a number of operations. Convert  $x$  to a `t_QFR` with the correct (logarithmic) distance component.

## 13.4 Linear algebra over $\mathbb{Z}$ .

### 13.4.1 Hermite and Smith Normal Forms.

GEN `ZM_hnf`(GEN `x`) returns the upper triangular Hermite Normal Form of the `ZM`  $x$  (removing 0 columns), using the `ZM_hnfall` algorithm. If you want the true HNF, use `ZM_hnfall(x, NULL, 0)`.

GEN `ZM_hnfmod`(GEN `x`, GEN `d`) returns the HNF of the `ZM`  $x$  (removing 0 columns), assuming the `t_INT`  $d$  is a multiple of the determinant of  $x$ . This is usually faster than `ZM_hnf` (and uses less memory) if the dimension is large,  $> 50$  say.

GEN `ZM_hnfmodid`(GEN `x`, GEN `d`) returns the HNF of the matrix  $(x \mid d\text{Id})$  (removing 0 columns), for a `ZM`  $x$  and a `t_INT`  $d$ .

GEN `ZM_hnfmodprime`(GEN `x`, GEN `p`) returns the HNF of the matrix  $(x \mid p\text{Id})$  (removing 0 columns), for a `ZM`  $x$  and a prime number  $p$ . The algorithm involves only  $\mathbb{F}_p$ -linear algebra and is faster than `ZM_hnfmodid` (which will call it when  $d$  is prime).

GEN ZM\_hnfmodall(GEN x, GEN d, long flag) low-level function underlying the ZM\_hnfmod variants. If flag is 0, calls ZM\_hnfmod(x,d); flag is an or-ed combination of:

- **hnf\_MODID** call ZM\_hnfmodid instead of ZM\_hnfmod,
- **hnf\_PART** return as soon as we obtain an upper triangular matrix, saving time. The pivots are non-negative and give the diagonal of the true HNF, but the entries to the right of the pivots need not be reduced, i.e. they may be large or negative.
- **hnf\_CENTER** returns the centered HNF, where the entries to the right of a pivot  $p$  are centered residues in  $[-p/2, p/2[$ , hence smallest possible in absolute value, but possibly negative.

GEN ZM\_hnfmodall\_i(GEN x, GEN d, long flag) as ZM\_hnfmodall without final garbage collection. Not gerepile-safe.

GEN ZM\_hnfall(GEN x, GEN \*U, long remove) returns the upper triangular HNF  $H$  of the ZM  $x$ ; if  $U$  is not NULL, set it to the matrix  $U$  such that  $xU = H$ . If **remove** = 0,  $H$  is the true HNF, including 0 columns; if **remove** = 1, delete the 0 columns from  $H$  but do not update  $U$  accordingly (so that the integer kernel may still be recovered): we no longer have  $xU = H$ ; if **remove** = 2, remove 0 columns from  $H$  and update  $U$  so that  $xU = H$ . The matrix  $U$  is square and invertible unless **remove** = 2.

This routine uses a naive algorithm which is potentially exponential in the dimension (due to coefficient explosion) but is fast in practice, although it may require lots of memory. The base change matrix  $U$  may be very large, when the kernel is large.

GEN ZM\_hnfall\_i(GEN x, GEN \*U, long remove) as ZM\_hnfall without final garbage collection. Not gerepile-safe.

GEN ZM\_hnfperm(GEN A, GEN \*ptU, GEN \*ptperm) returns the hnf  $H = PAU$  of the matrix  $PA$ , where  $P$  is a suitable permutation matrix, and  $U \in \text{Gl}_n(\mathbf{Z})$ .  $P$  is chosen so as to (heuristically) minimize the size of  $U$ ; in this respect it is less efficient than ZM\_hnf111 but usually faster. Set \*ptU to  $U$  and \*ptperm to a t\_VECSMALL representing the row permutation attached to  $P = (\delta_{i, \text{perm}[i]})$ . If ptU is set to NULL,  $U$  is not computed, saving some time; although useless, setting ptperm to NULL is also allowed.

GEN ZM\_hnf\_knapsack(GEN x) given a ZM  $x$ , compute its HNF  $h$ . Return  $h$  if it has the knapsack property: every column contains only zeroes and ones and each row contains a single 1; return NULL otherwise. Not suitable for gerepile.

GEN ZM\_hnf111(GEN x, GEN \*U, int remove) returns the HNF  $H$  of the ZM  $x$ ; if  $U$  is not NULL, set it to the matrix  $U$  such that  $xU = H$ . The meaning of **remove** is the same as in ZM\_hnfall.

This routine uses the LLL variant of Havas, Majewski and Mathews, which is polynomial time, but rather slow in practice because it uses an exact LLL over the integers instead of a floating point variant; it uses polynomial space but lots of memory is needed for large dimensions, say larger than 300. On the other hand, the base change matrix  $U$  is essentially optimally small with respect to the  $L_2$  norm.

GEN ZM\_hnfcenter(GEN M). Given a ZM in HNF  $M$ , update it in place so that non-diagonal entries belong to a system of *centered* residues. Not suitable for gerepile.

Some direct applications: the following routines apply to upper triangular integral matrices; in practice, these come from HNF algorithms.

**GEN hnf\_divscale**(GEN A, GEN B, GEN t)  $A$  an upper triangular ZM,  $B$  a ZM,  $t$  an integer, such that  $C := tA^{-1}B$  is integral. Return  $C$ .

**GEN hnf\_invscale**(GEN A, GEN t)  $A$  an upper triangular ZM,  $t$  an integer such that  $C := tA^{-1}$  is integral. Return  $C$ . Special case of **hnf\_divscale** when  $B$  is the identity matrix.

**GEN hnf\_solve**(GEN A, GEN B)  $A$  a ZM in upper HNF (not necessarily square),  $B$  a ZM or ZC. Return  $A^{-1}B$  if it is integral, and NULL if it is not.

**GEN hnf\_invimage**(GEN A, GEN b)  $A$  a ZM in upper HNF (not necessarily square),  $b$  a ZC. Return  $A^{-1}B$  if it is integral, and NULL if it is not.

**int hnfdivide**(GEN A, GEN B)  $A$  and  $B$  are two upper triangular ZM. Return 1 if  $A^{-1}B$  is integral, and 0 otherwise.

### Smith Normal Form.

**GEN ZM\_snf**(GEN x) returns the Smith Normal Form (vector of elementary divisors) of the ZM  $x$ .

**GEN ZM\_snfall**(GEN x, GEN \*U, GEN \*V) returns **ZM\_snf**(x) and sets  $U$  and  $V$  to unimodular matrices such that  $UxV = D$  (diagonal matrix of elementary divisors). Either (or both)  $U$  or  $V$  may be NULL in which case the corresponding matrix is not computed.

**GEN ZV\_snfall**(GEN d, GEN \*U, GEN \*V) here  $d$  is a ZV; same as **ZM\_snfall** applied to **diagonal**(d), but faster.

**GEN ZM\_snfall\_i**(GEN x, GEN \*U, GEN \*V, int returnvec) same as **ZM\_snfall**, except that, depending on the value of **returnvec**, we either return a diagonal matrix (as in **ZM\_snfall**, **returnvec** is 0) or a vector of elementary divisors (as in **ZM\_snf**, **returnvec** is 1).

**void ZM\_snfclean**(GEN d, GEN U, GEN V) assuming  $d, U, V$  come from  $d = \mathbf{ZM\_snfall}(x, \&U, \&V)$ , where  $U$  or  $V$  may be NULL, cleans up the output in place. This means that elementary divisors equal to 1 are deleted and  $U, V$  are updated. The output is not suitable for **gerepileupto**.

**void ZV\_snf\_trunc**(GEN D) given a vector  $D$  of elementary divisors (i.e. a ZV such that  $d_i \mid d_{i+1}$ ), truncate it *in place* to leave out the trivial divisors (equal to 1).

**GEN ZM\_snf\_group**(GEN H, GEN \*U, GEN \*Uinv) this function computes data to go back and forth between an abelian group (of finite type) given by generators and relations, and its canonical SNF form. Given an abstract abelian group with generators  $g = (g_1, \dots, g_n)$  and a vector  $X = (x_i) \in \mathbf{Z}^n$ , we write  $gX$  for the group element  $\sum_i x_i g_i$ ; analogously if  $M$  is an  $n \times r$  integer matrix  $gM$  is a vector containing  $r$  group elements. The group neutral element is 0; by abuse of notation, we still write 0 for a vector of group elements all equal to the neutral element. The input is a full relation matrix  $H$  among the generators, i.e. a ZM (not necessarily square) such that  $gX = 0$  for some  $X \in \mathbf{Z}^n$  if and only if  $X$  is in the integer image of  $H$ , so that the abelian group is isomorphic to  $\mathbf{Z}^n / \text{Im}H$ . *The routine assumes that  $H$  is in HNF; replace it by its HNF if it is not the case.* (Of course this defines the same group.)

Let  $G$  a minimal system of generators in SNF for our abstract group: if the  $d_i$  are the elementary divisors ( $\dots \mid d_2 \mid d_1$ ), each  $G_i$  has either infinite order ( $d_i = 0$ ) or order  $d_i > 1$ . Let  $D$  the matrix with diagonal  $(d_i)$ , then

$$GD = 0, \quad G = gU_{\text{inv}}, \quad g = GU,$$

for some integer matrices  $U$  and  $U_{\text{inv}}$ . Note that these are not even square in general; even if square, there is no guarantee that these are unimodular: they are chosen to have minimal entries given the known relations in the group and only satisfy  $D \mid (UU_{\text{inv}} - \text{Id})$  and  $H \mid (U_{\text{inv}}U - \text{Id})$ .

The function returns the vector of elementary divisors ( $d_i$ ); if  $U$  is not NULL, it is set to  $U$ ; if  $U_{\text{inv}}$  is not NULL it is set to  $U_{\text{inv}}$ . The function is not memory clean.

GEN ZV\_snf\_group(GEN d, GEN \*newU, GEN \*newUi), here  $d$  is a ZV; same as ZM\_snf\_group applied to diagonal( $d$ ), but faster.

The following routines underly the various `matrixqz` variants. In all case the  $m \times n$  `t_MAT`  $x$  is assumed to have rational (`t_INT` and `t_FRAC`) coefficients

GEN QM\_ImQ\_hnf(GEN x) returns an HNF basis for  $\text{Im}_{\mathbf{Q}}x \cap \mathbf{Z}^n$ .

GEN QM\_ImZ\_hnf(GEN x) returns an HNF basis for  $\text{Im}_{\mathbf{Z}}x \cap \mathbf{Z}^n$ .

GEN QM\_ImQ\_hnfall(GEN A, GEN \*pB, long remove) as QM\_ImQ\_hnf, further returning the transformation matrix as in ZM\_hnfall.

GEN QM\_ImZ\_hnfall(GEN A, GEN \*pB, long remove) as QM\_ImZ\_hnf, further returning the transformation matrix as in ZM\_hnfall.

GEN QM\_minors\_coprime(GEN x, GEN D), assumes  $m \geq n$ , and returns a matrix in  $M_{m,n}(\mathbf{Z})$  with the same  $\mathbf{Q}$ -image as  $x$ , such that the GCD of all  $n \times n$  minors is coprime to  $D$ ; if  $D$  is NULL, we want the GCD to be 1.

The following routines are simple wrappers around the above ones and are normally useless in library mode:

GEN hnf(GEN x) checks whether  $x$  is a ZM, then calls ZM\_hnf. Normally useless in library mode.

GEN hnfmod(GEN x, GEN d) checks whether  $x$  is a ZM, then calls ZM\_hnfmod. Normally useless in library mode.

GEN hnfmodid(GEN x, GEN d) checks whether  $x$  is a ZM, then calls ZM\_hnfmodid. Normally useless in library mode.

GEN hnfall(GEN x) calls ZM\_hnfall( $x$ , &U, 1) and returns  $[H, U]$ . Normally useless in library mode.

GEN hnfl1l(GEN x) calls ZM\_hnfl1l( $x$ , &U, 1) and returns  $[H, U]$ . Normally useless in library mode.

GEN hnfperm(GEN x) calls ZM\_hnfperm( $x$ , &U, &P) and returns  $[H, U, P]$ . Normally useless in library mode.

GEN smith(GEN x) checks whether  $x$  is a ZM, then calls ZM\_snf. Normally useless in library mode.

GEN smithall(GEN x) checks whether  $x$  is a ZM, then calls ZM\_snfall( $x$ , &U, &V) and returns  $[U, V, D]$ . Normally useless in library mode.

Some related functions over  $K[X]$ ,  $K$  a field:

GEN gsmith(GEN A) the input matrix must be square, returns the elementary divisors.

GEN gsmithall(GEN A) the input matrix must be square, returns the  $[U, V, D]$ ,  $D$  diagonal, such that  $UAV = D$ .

GEN RgM\_hnfall(GEN A, GEN \*pB, long remove) analogous to ZM\_hnfall.

GEN smithclean(GEN z) cleanup the output of smithall or gsmithall (delete elementary divisors equal to 1, updating base change matrices).

### 13.4.2 The LLL algorithm.

The basic GP functions and their immediate variants are normally not very useful in library mode. We briefly list them here for completeness, see the documentation of `qflll` and `qflllgram` for details:

```

    • GEN qflll0(GEN x, long flag)
GEN lll(GEN x) flag= 0
GEN lllint(GEN x) flag= 1
GEN lllkerim(GEN x) flag= 4
GEN lllkerimgen(GEN x) flag= 5
GEN lllgen(GEN x) flag= 8

    • GEN qflllgram0(GEN x, long flag)
GEN lllgram(GEN x) flag= 0
GEN lllgramint(GEN x) flag= 1
GEN lllgramkerim(GEN x) flag= 4
GEN lllgramkerimgen(GEN x) flag= 5
GEN lllgramgen(GEN x) flag= 8

```

The basic workhorse underlying all integral and floating point LLLs is

`GEN ZM_lll(GEN x, double D, long flag)`, where  $x$  is a ZM;  $D \in ]1/4, 1[$  is the Lovász constant determining the frequency of swaps during the algorithm: a larger values means better guarantees for the basis (in principle smaller basis vectors) but longer running times (suggested value:  $D = 0.99$ ).

**Important.** This function does not collect garbage and its output is not suitable for either `gerepile` or `gerepileupto`. We expect the caller to do something simple with the output (e.g. matrix multiplication), then collect garbage immediately.

`flag` is an or-ed combination of the following flags:

- **LLL\_GRAM.** If set, the input matrix  $x$  is the Gram matrix  ${}^t v v$  of some lattice vectors  $v$ .
- **LLL\_INPLACE.** If unset, we return the base change matrix  $U$ , otherwise the transformed matrix  $xU$  or  ${}^t U x U$  (**LLL\_GRAM**). Implies **LLL\_IM** (see below).
- **LLL\_KEEP\_FIRST.** The first vector in the output basis is the same one as was originally input. Provided this is a shortest non-zero vector of the lattice, the output basis is still LLL-reduced. This is used to reduce maximal orders of number fields with respect to the  $T_2$  quadratic form, to ensure that the first vector in the output basis corresponds to 1 (which is a shortest vector).
- **LLL\_COMPATIBLE.** This is a no-op on 64-bit kernels; on 32-bit kernels, restrict to 64-bit-compatible accuracies in the course of LLL algorithms. This is very likely to produce identical results on all kernels, but this is not guaranteed.

The last three flags are mutually exclusive, either 0 or a single one must be set:

- **LLL\_KER** If set, only return a kernel basis  $K$  (not LLL-reduced).

- **LLL\_IM** If set, only return an LLL-reduced lattice basis  $T$ . (This is implied by **LLL\_INPLACE**).
- **LLL\_ALL** If set, returns a 2-component vector  $[K, T]$  corresponding to both kernel and image.

**GEN lllfp**(**GEN x**, **double D**, **long flag**) is a variant for matrices with inexact entries:  $x$  is a matrix with real coefficients (types **t\_INT**, **t\_FRAC** and **t\_REAL**),  $D$  and  $flag$  are as in **ZM\_lll**. The matrix is rescaled, rounded to nearest integers, then fed to **ZM\_lll**. The flag **LLL\_INPLACE** is still accepted but less useful (it returns an LLL-reduced basis attached to rounded input, instead of an exact base change matrix).

**GEN ZM\_lll\_norms**(**GEN x**, **double D**, **long flag**, **GEN \*ptB**) slightly more general version of **ZM\_lll**, setting **\*ptB** to a vector containing the squared norms of the Gram-Schmidt vectors  $(b_i^*)$  attached to the output basis  $(b_i)$ ,  $b_i^* = b_i + \sum_{j < i} \mu_{i,j} b_j^*$ .

**GEN lllintpartial\_inplace**(**GEN x**) given a **ZM x** of maximal rank, returns a partially reduced basis  $(b_i)$  for the space spanned by the columns of  $x$ :  $|b_i \pm b_j| \geq |b_i|$  for any two distinct basis vectors  $b_i, b_j$ . This is faster than the LLL algorithm, but produces much larger bases.

**GEN lllintpartial**(**GEN x**) as **lllintpartial\_inplace**, but returns the base change matrix  $U$  from the canonical basis to the  $b_i$ , i.e.  $xU$  is the output of **lllintpartial\_inplace**.

**GEN RM\_round\_maxrank**(**GEN G**) given a matrix  $G$  with real floating point entries and independent columns, let  $G_e$  be the rescaled matrix  $2^e G$  rounded to nearest integers, for  $e \geq 0$ . Finds a small  $e$  such that the rank of  $G_e$  is equal to the rank of  $G$  (its number of columns) and return  $G_e$ . This is useful as a preconditioning step to speed up LLL reductions, see **nf\_get\_Gtwist**. Suitable for **gerepileupto**, but does not collect garbage.

### 13.4.3 Linear dependencies.

The following functions underly the **lindep** GP function:

**GEN lindep**(**GEN v**) real/complex entries, guess that about only the 80% leading bits of the input are correct.

**GEN lindep\_bit**(**GEN v**, **long b**) real/complex entries, explicit form of the above: multiply the input by  $2^b$  and round to nearest integer before looking for a linear dependency. Truncating dubious bits allows to find better relations.

**GEN lindepfull\_bit**(**GEN v**, **long b**) as **lindep\_bit** but return a matrix  $M$  with  $n = \#v$  columns and  $r$  rows, with  $r = n + 1$  (if  $v$  is real) or  $n + 2$  (general case) which is an LLL-reduced basis of the lattice formed by concatenating vertically an identity matrix and the floor of  $2^b \text{real}(v)$  and  $2^b \text{imag}(v)$  if  $r = n + 2$ . The first  $n$  rows of  $M$  potentially correspond to relations: whenever the last  $r - n$  entries of a column are small. The function **lindep\_bit** essentially returns the first column of  $M$  truncated to  $n$  components.

**GEN lindep\_padic**(**GEN v**)  $p$ -adic entries.

**GEN lindep\_Xadic**(**GEN v**) polynomial entries.

**GEN deplin**(**GEN v**) returns a non-zero kernel vector for a **t\_MAT** input.

Deprecated routine:

**GEN lindep2**(**GEN x**, **long dig**) analogous to **lindep\_bit**, with **dig** counting decimal digits.



#### 13.4.4 Reduction modulo matrices.

GEN `ZC_hnfremdiv`(GEN `x`, GEN `y`, GEN `*Q`) assuming  $y$  is an invertible ZM in HNF and  $x$  is a ZC, returns the ZC  $R$  equal to  $x \bmod y$  (whose  $i$ -th entry belongs to  $[-y_{i,i}/2, y_{i,i}/2[$ ). Stack clean *unless*  $x$  is already reduced (in which case, returns  $x$  itself, not a copy). If  $Q$  is not NULL, set it to the ZC such that  $x = yQ + R$ .

GEN `ZM_hnfdivrem`(GEN `x`, GEN `y`, GEN `*Q`) reduce each column of the ZM  $x$  using `ZC_hnfremdiv`. If  $Q$  is not NULL, set it to the ZM such that  $x = yQ + R$ .

GEN `ZC_hnfrem`(GEN `x`, GEN `y`) alias for `ZC_hnfremdiv(x,y,NULL)`.

GEN `ZM_hnfrem`(GEN `x`, GEN `y`) alias for `ZM_hnfremdiv(x,y,NULL)`.

GEN `ZC_reducemodmatrix`(GEN `v`, GEN `y`) Let  $y$  be a ZM, not necessarily square, which is assumed to be LLL-reduced (otherwise, very poor reduction is expected). Size-reduces the ZC  $v$  modulo the  $\mathbf{Z}$ -module  $Y$  spanned by  $y$ : if the columns of  $y$  are denoted by  $(y_1, \dots, y_{n-1})$ , we return  $y_n \equiv v$  modulo  $Y$ , such that the Gram-Schmidt coefficients  $\mu_{n,j}$  are less than  $1/2$  in absolute value for all  $j < n$ . In short,  $y_n$  is almost orthogonal to  $Y$ .

GEN `ZM_reducemodmatrix`(GEN `v`, GEN `y`) Let  $y$  be as in `ZC_reducemodmatrix`, and  $v$  be a ZM. This returns a matrix  $v$  which is congruent to  $v$  modulo the  $\mathbf{Z}$ -module spanned by  $y$ , whose columns are size-reduced. This is faster than repeatedly calling `ZC_reducemodmatrix` on the columns since most of the Gram-Schmidt coefficients can be reused.

GEN `ZC_reducemodlll`(GEN `v`, GEN `y`) Let  $y$  be an arbitrary ZM, LLL-reduce it then call `ZC_reducemodmatrix`.

GEN `ZM_reducemodlll`(GEN `v`, GEN `y`) Let  $y$  be an arbitrary ZM, LLL-reduce it then call `ZM_reducemodmatrix`.

Besides the above functions, which were specific to integral input, we also have:

GEN `reducemodinvertible`(GEN `x`, GEN `y`)  $y$  is an invertible matrix and  $x$  a `t_COL` or `t_MAT` of compatible dimension. Returns  $x - y[y^{-1}x]$ , which has small entries and differs from  $x$  by an integral linear combination of the columns of  $y$ . Suitable for `gerepileupto`, but does not collect garbage.

GEN `closemodinvertible`(GEN `x`, GEN `y`) returns `x-reducemodinvertible(x,y)`, i.e. an integral linear combination of the columns of  $y$ , which is close to  $x$ .

GEN `reducemodlll`(GEN `x`, GEN `y`) LLL-reduce the non-singular ZM  $y$  and call `reducemodinvertible` to find a small representative of  $x \bmod y\mathbf{Z}^n$ . Suitable for `gerepileupto`, but does not collect garbage.

## 13.5 Finite abelian groups and characters.

### 13.5.1 Abstract groups.

A finite abelian group  $G$  in GP format is given by its Smith Normal Form as a pair  $[h, d]$  or triple  $[h, d, g]$ . Here  $h$  is the cardinality of  $G$ ,  $(d_i)$  is the vector of elementary divisors, and  $(g_i)$  is a vector of generators. In short,  $G = \oplus_{i \leq n} (\mathbf{Z}/d_i \mathbf{Z}) g_i$ , with  $d_n \mid \dots \mid d_2 \mid d_1$  and  $\prod d_i = h$ .

Let  $e(x) := \exp(2i\pi x)$ . For ease of exposition, we restrict to complex-valued characters, but everything applies to more general fields  $K$  where  $e$  denotes a morphism  $(\mathbf{Q}, +) \rightarrow (K^*, \times)$  such that  $e(a/b)$  denotes a  $b$ -th root of unity.

A *character* on the abelian group  $\oplus (\mathbf{Z}/d_j \mathbf{Z}) g_j$  is given by a row vector  $\chi = [a_1, \dots, a_n]$  such that  $\chi(\prod g_j^{n_j}) = e(\sum a_j n_j / d_j)$ .

**GEN cyc\_normalize(GEN d)** shallow function. Given a vector  $(d_i)_{i \leq n}$  of elementary divisors for a finite group (no  $d_i$  vanish), returns the vector  $D = [1]$  if  $n = 0$  (trivial group) and  $[d_1, d_1/d_2, \dots, d_1/d_n]$  otherwise. This will allow to define characters as  $\chi(\prod g_j^{x_j}) = e(\sum_j x_j a_j D_j / D_1)$ , see **char\_normalize**.

**GEN char\_normalize(GEN chi, GEN ncyc)** shallow function. Given a character **chi** =  $(a_j)$  and **ncyc** from **cyc\_normalize** above, returns the normalized representation  $[d, (n_j)]$ , such that  $\chi(\prod g_j^{x_j}) = \zeta_d^{\sum_j n_j x_j}$ , where  $\zeta_d = e(1/d)$  and  $d$  is *minimal*. In particular,  $d$  is the order of **chi**. Shallow function.

**GEN char\_simplify(GEN D, GEN N)** given a quasi-normalized character  $[D, (N_j)]$  such that  $\chi(\prod g_j^{x_j}) = \zeta_D^{\sum_j N_j x_j}$ , but where we only assume that  $D$  is a multiple of the character order, return a normalized character  $[d, (n_j)]$  with  $d$  *minimal*. Shallow function.

**GEN char\_denormalize(GEN cyc, GEN d, GEN n)** given a normalized representation  $[d, n]$  (where  $d$  need not be minimal) of a character on the abelian group with abelian divisors **cyc**, return the attached character (where the image of each generator  $g_i$  is given in terms of roots of unity of different orders **cyc**[ $i$ ]).

**GEN charconj(GEN cyc, GEN chi)** return the complex conjugate of **chi**.

**GEN charmul(GEN cyc, GEN a, GEN b)** return the product character  $a \times b$ .

**GEN chardiv(GEN cyc, GEN a, GEN b)** returns the character  $a/b = a \times \bar{b}$ .

**int char\_check(GEN cyc, GEN chi)** return 1 if **chi** is a character compatible with cyclic factors **cyc**, and 0 otherwise.

**GEN cyc2elts(GEN d)** given a **t\_VEC**  $d = (d_1, \dots, d_n)$  of non-negative integers, return the vector of all **t\_VECSMALLS** of length  $n$  whose  $i$ -th entry lies in  $[0, d_i]$ . Assumes that the product of the  $d_i$  fits in a **long**.

### 13.5.2 Dirichlet characters.

The functions in this section are specific to characters on  $(\mathbf{Z}/N\mathbf{Z})^*$ . The argument  $G$  is a special `bid` structure as returned by `znstar0(N, nf_INIT)`. In this case, there are additional ways to input character via Conrey's representation. The character `chi` is either a `t_INT` (Conrey label), a `t_COL` (a Conrey logarithm) or a `t_VEC` (generic character on `bid.gen` as explained in the previous subsection). The following low-level functions are called by GP's generic character functions.

`int zncharcheck(GEN G, GEN chi)` return 1 if `chi` is a valid character and 0 otherwise.

`GEN zncharconj(GEN G, GEN chi)` as `charconj`.

`GEN znchardiv(GEN G, GEN a, GEN b)` as `chardiv`.

`GEN zncharker(GEN G, GEN chi)` as `charker`.

`GEN znchareval(GEN G, GEN chi, GEN n, GEN z)` as `chareval`.

`GEN zncharmul(GEN G, GEN a, GEN b)` as `charm`.

`GEN zncharpow(GEN G, GEN a, GEN n)` as `charpow`.

`GEN zncharorder(GEN G, GEN chi)` as `charorder`.

The following functions handle characters in Conrey notation (attached to Conrey generators, not `G.gen`):

`int znconrey_check(GEN cyc, GEN chi)` return 1 if `chi` is a valid Conrey logarithm and 0 otherwise.

`GEN znconrey_normalized(GEN G, GEN chi)` return normalized character attached to `chi`, as in `char_normalize` but on Conrey generators.

`GEN znconreyfromchar(GEN G, GEN chi)` return Conrey logarithm attached to the generic (`t_VEC`, on `G.gen`)

`GEN znconreyfromchar_normalized(GEN G, GEN chi)` return normalized Conrey character attached to the generic (`t_VEC`, on `G.gen`) character `chi`.

`GEN znconreylog_normalize(GEN G, GEN m)` given a Conrey logarithm  $m$  (`t_COL`), return the attached normalized Conrey character, as in `char_normalize` but on Conrey generators.

`GEN znchar_quad(GEN G, GEN D)` given a non-zero `t_INT`  $D$  congruent to 0, 1 mod 4, return  $(D/.)$  as a character modulo  $N$ , given by a Conrey logarithm (`t_COL`). Assume that  $|D|$  divides  $N$ .

`GEN Zideallog(GEN G, GEN x)` return the `znconreylog` of  $x$  expressed on `G.gen`, i.e. the ordinary discrete logarithm from `ideallog`.

`GEN ncharvecexpo(GEN G, GEN nchi)` given  $nchi = [d, n]$  a quasi-normalized character ( $d$  may be a multiple of the character order), i.e.  $\chi(g_i) = e(n[i]/d)$  for all Conrey or SNF generators  $g_i$  (as usual, we use SNF generators if  $n$  is a `t_VEC` and the Conrey generators otherwise). Return a `t_VECSMALL`  $v$  such that  $v[i] = -1$  if  $(i, N) > 1$  else  $\chi(i) = e(v[i]/d)$ ,  $1 \leq i \leq N$ .

## 13.6 Central simple algebras.

### 13.6.1 Initialization.

Low-level routines underlying `alginit`.

`GEN alg_csa_table(GEN nf, GEN mt, long v, long maxord)` algebra defined by a multiplication table.

`GEN alg_cyclic(GEN rnf, GEN aut, GEN b, long maxord)` cyclic algebra  $(L/K, \sigma, b)$ .

`GEN alg_hasse(GEN nf, long d, GEN hi, GEN hf, long v, long maxord)` algebra defined by local Hasse invariants.

`GEN alg_hilbert(GEN nf, GEN a, GEN b, long v, long maxord)` quaternion algebra.

`GEN alg_matrix(GEN nf, long n, long v, GEN L, long maxord)` matrix algebra.

`GEN alg_complete(GEN rnf, GEN aut, GEN hi, GEN hf, long maxord)` cyclic algebra  $(L/K, \sigma, b)$  with  $b$  computed from the Hasse invariants.

### 13.6.2 Type checks.

`void checkalg(GEN a)` raise an exception if  $a$  was not initialized by `alginit`.

`void checklat(GEN al, GEN lat)` raise an exception if `lat` is not a valid full lattice in the algebra `al`.

`void checkhasse(GEN nf, GEN hi, GEN hf, long n)` raise an exception if  $(hi, hf)$  do not describe valid Hasse invariants of a central simple algebra of degree  $n$  over  $nf$ .

`long alg_type(GEN al)` internal function called by `algtype`: assume `al` was created by `alginit` (thereby saving a call to `checkalg`). Return values are symbolic rather than numeric:

- `al_NULL`: not a valid algebra.
- `al_TABLE`: table algebra output by `algtableinit`.
- `al_CSA`: central simple algebra output by `alginit` and represented by a multiplication table over its center.
- `al_CYCLIC`: central simple algebra output by `alginit` and represented by a cyclic algebra.

`long alg_model(GEN al, GEN x)` given an element  $x$  in algebra  $al$ , check for inconsistencies (raise a type error) and return the representation model used for  $x$ :

- `al_ALGEBRAIC`: `basistoalg` form, algebraic representation.
- `al_BASIS`: `algtobasis` form, column vector on the integral basis.
- `al_MATRIX`: matrix with coefficients in an algebra.
- `al_TRIVIAL`: trivial algebra of degree 1; can be understood as both basis or algebraic form (since  $e_1 = 1$ ).

### 13.6.3 Shallow accessors.

All these routines assume their argument was initialized by `alginit` and provide minor speedups compared to the GP equivalent. The routines returning a GEN are shallow.

`long alg_get_absdim(GEN al)` low-level version of `algabsdim`.

`long alg_get_dim(GEN al)` low-level version of `algdim`.

`long alg_get_degree(GEN al)` low-level version of `algdegree`.

`GEN alg_get_aut(GEN al)` low-level version of `algaut`.

`GEN alg_get_auts(GEN al)`, given a cyclic algebra  $al = (L/K, \sigma, b)$  of degree  $n$ , returns the vector of  $\sigma^i$ ,  $1 \leq i < n$ .

`GEN alg_get_b(GEN al)` low-level version of `algb`.

`GEN alg_get_basis(GEN al)` low-level version of `algbasis`.

`GEN alg_get_center(GEN al)` low-level version of `algcenter`.

`GEN alg_get_char(GEN al)` low-level version of `algchar`.

`GEN alg_get_hasse_f(GEN al)` low-level version of `alghassef`.

`GEN alg_get_hasse_i(GEN al)` low-level version of `alghassei`.

`GEN alg_get_invbasis(GEN al)` low-level version of `alginvbasis`.

`GEN alg_get_multable(GEN al)` low-level version of `algmultable`.

`GEN alg_get_relmultable(GEN al)` low-level version of `algrelmultable`.

`GEN alg_get_splittingfield(GEN al)` low-level version of `algsplittingfield`.

`GEN alg_get_abssplitting(GEN al)` returns the absolute *nf* structure attached to the *rnf* returned by `algsplittingfield`.

`GEN alg_get_splitpol(GEN al)` returns the relative polynomial defining the *rnf* returned by `algsplittingfield`.

`GEN alg_get_splittingdata(GEN al)` low-level version of `algsplittingdata`.

`GEN alg_get_splittingbasis(GEN al)` the matrix *Lbas* from `algsplittingdata`

`GEN alg_get_splittingbasisinv(GEN al)` the matrix *Lbasinv* from `algsplittingdata`.

`GEN alg_get_tracebasis(GEN al)` returns the traces of the basis elements; used by `algtrace`.

`GEN alglat_get_primbasis(GEN lat)` from the description of *lat* as  $\lambda L$  with  $L \subset \mathcal{O}_0$  and  $\lambda \in \mathbf{Q}$ , returns a basis of *L*.

`GEN alglat_get_scalar(GEN lat)` from the description of *lat* as  $\lambda L$  with  $L \subset \mathcal{O}_0$  and  $\lambda \in \mathbf{Q}$ , returns  $\lambda$ .

### 13.6.4 Other low-level functions.

`GEN conjclasses_algcenter(GEN cc, GEN p)` low-level function underlying `alggroupcenter`, where *cc* is the output of `grouperlts_to_conjclasses`, and *p* is either NULL or a prime number. Not stack clean.

`GEN algsimpledec_ss(GEN al, long maps)` assuming that *al* is semisimple, returns the second component of `algsimpledec(al, maps)`.



## Chapter 14:

### Elliptic curves and arithmetic geometry

This chapter is quite short, but is added as a placeholder, since we expect the library to expand in that direction.

#### 14.1 Elliptic curves.

Elliptic curves are represented in the Weierstrass model

$$(E) : y^2z + a_1xyz + a_3yz = x^3 + a_2x^2z + a_4xz^2 + a_6z^3,$$

by the 5-tuple  $[a_1, a_2, a_3, a_4, a_6]$ . Points in the projective plane are represented as follows: the point at infinity  $(0 : 1 : 0)$  is coded as `[0]`, a finite point  $(x : y : 1)$  outside the projective line at infinity  $z = 0$  is coded as  $[x, y]$ . Note that other points at infinity than  $(0 : 1 : 0)$  cannot be represented; this is harmless, since they do not belong to any of the elliptic curves  $E$  above.

*Points on the curve* are just projective points as described above, they are not tied to a curve in any way: the same point may be used in conjunction with different curves, provided it satisfies their equations (if it does not, the result is usually undefined). In particular, the point at infinity belongs to all elliptic curves.

As with `factor` for polynomial factorization, the 5-tuple  $[a_1, a_2, a_3, a_4, a_6]$  implicitly defines a base ring over which the curve is defined. Point coordinates must be operation-compatible with this base ring (`gadd`, `gmul`, `gdiv` involving them should not give errors).

##### 14.1.1 Types of elliptic curves.

We call a 5-tuple as above an `ell5`; most functions require an `ell` structure, as returned by `ellinit`, which contains additional data (usually dynamically computed as needed), depending on the base field.

`GEN ellinit(GEN E, GEN D, long prec)`, returns an `ell` structure, attached to the elliptic curve  $E$ : either an `ell5`, a pair  $[a_4, a_6]$  or a `t_STR` in Cremona's notation, e.g. "11a1". The optional  $D$  (`NULL` to omit) describes the domain over which the curve is defined.

##### 14.1.2 Type checking.

`void checkell(GEN e)` raise an error unless  $e$  is an `ell`.

`int checkell_i(GEN e)` return 1 if  $e$  is an `ell` and 0 otherwise.

`void checkell5(GEN e)` raise an error unless  $e$  is an `ell` or an `ell5`.

`void checkellpt(GEN z)` raise an error unless  $z$  is a point (either finite or at infinity).

`long ell_get_type(GEN e)` returns the domain type over which the curve is defined, one of

`t_ELL_Q` the field of rational numbers;

`t_ELL_NF` a number field;

`t_ELL_Qp` the field of  $p$ -adic numbers, for some prime  $p$ ;

`t_ELL_Fp` a prime finite field, base field elements are represented as  $\mathbb{F}_p$ , i.e. a `t_INT` reduced modulo  $p$ ;

`t_ELL_Fq` a non-prime finite field (a prime finite field can also be represented by this subtype, but this is inefficient), base field elements are represented as `t_FFELT`;

`t_ELL_Rg` none of the above.

`void checkell_Fq(GEN e)` checks whether  $e$  is an `ell`, defined over a finite field (either prime or non-prime). Otherwise the function raises a `pari_err_TYPE` exception.

`void checkell_Q(GEN e)` checks whether  $e$  is an `ell`, defined over  $\mathbb{Q}$ . Otherwise the function raises a `pari_err_TYPE` exception.

`void checkell_Qp(GEN e)` checks whether  $e$  is an `ell`, defined over some  $\mathbb{Q}_p$ . Otherwise the function raises a `pari_err_TYPE` exception.

`void checkellisog(GEN v)` raise an error unless  $v$  is an isogeny, from `ellisogeny`.

### 14.1.3 Extracting info from an `ell` structure.

These functions expect an `ell` argument. If the required data is not part of the structure, it is computed then inserted, and the new value is returned.

#### 14.1.3.1 All domains.

`GEN ell_get_a1(GEN e)`

`GEN ell_get_a2(GEN e)`

`GEN ell_get_a3(GEN e)`

`GEN ell_get_a4(GEN e)`

`GEN ell_get_a6(GEN e)`

`GEN ell_get_b2(GEN e)`

`GEN ell_get_b4(GEN e)`

`GEN ell_get_b6(GEN e)`

`GEN ell_get_b8(GEN e)`

`GEN ell_get_c4(GEN e)`

`GEN ell_get_c6(GEN e)`

`GEN ell_get_disc(GEN e)`

`GEN ell_get_j(GEN e)`



### 14.1.3.2 Curves over $\mathbf{Q}$ .

`GEN ellQ_get_N(GEN e)` returns the curve conductor

`void ellQ_get_Nfa(GEN e, GEN *N, GEN *faN)` sets  $N$  to the conductor and  $faN$  to its factorization

`int ell_is_integral(GEN e)` return 1 if  $e$  is given by an integral model, and 0 otherwise.

`long ellQ_get_CM(GEN e)` if  $e$  has CM by a principal imaginary quadratic order, return its discriminant. Else return 0.

`long ellap_CM_fast(GEN e, ulong p, long CM)` assuming that  $p$  does not divide the discriminant of  $E$  (in particular,  $E$  has good reduction at  $p$ ), and that  $CM$  is as given by `ellQ_get_CM`, return the trace of Frobenius for  $E/\mathbf{F}_p$ . This is meant to quickly compute lots of  $a_p$ , esp. when  $e$  has CM by a principal quadratic order.

`long ellrootno_global(GEN e)` returns the global root number  $c \in \{-1, 1\}$ .

`GEN ellheightoo(GEN E, GEN P, long prec)` given  $P = [x, y]$  an affine point on  $E$ , return

$$\lambda_\infty(P) + \frac{1}{12} \log |\text{disc} E| = \frac{1}{2} \text{real}(z\eta(z)) - \log |\sigma(E, z)| \in \mathbf{R},$$

where  $\lambda_\infty(P)$  is the canonical local height at infinity and  $z$  is `ellpointtoz`( $E, P$ ). This is computed using Mestre's (quadratically convergent) AGM algorithm.

`long ellorder_Q(GEN E, GEN P)` return the order of  $P \in E(\mathbf{Q})$ , using the impossible value 0 for a point of infinite order. Ultimately called by the generic `ellorder` function.

`GEN point_to_a4a6(GEN E, GEN P, GEN p, GEN *a4)` given  $E/\mathbf{Q}$ ,  $p \neq 2, 3$  not dividing the discriminant of  $E$  and  $P \in E(\mathbf{Q})$  outside the kernel of reduction, return the image of  $P$  on the short Weierstrass model  $y^2 = x^3 + a_4x + a_6$  isomorphic to the reduction  $E_p$  of  $E$  at  $p$ . Also set `a4` to the  $a_4$  coefficient in the above model. This function allows quick computations modulo varying primes  $p$ , avoiding the overhead of `ellinit`( $E, p$ ), followed by a change of coordinates. It produces data suitable for `FpE` routines.

`GEN point_to_a4a6_Fl(GEN E, GEN P, ulong p, ulong *pa4)` as `point_to_a4a6`, returning a `Fle`.

`GEN elldatagenerators(GEN E)` returns generators for  $E(\mathbf{Q})$  extracted from Cremona's table.

`GEN ellanal_globalred(GEN e, GEN *v)` takes an *ell* over  $\mathbf{Q}$  and returns a global minimal model  $E$  (in `ellinit` form, over  $\mathbf{Q}$ ) for  $e$  suitable for analytic computations related to the curve  $L$  series: it contains `ellglobalred` data, as well as global and local root numbers. If  $v$  is not `NULL`, set `*v` to the needed change of variable: `NULL` if  $e$  was already the standard minimal model, such that  $E = \text{ellchangecurve}(e, v)$  otherwise. Compared to the direct use of `ellchangecurve` followed by `ellrootno`, this function avoids converting unneeded dynamic data and avoids potential memory leaks (the changed curve would have had to be deleted using `obj_free`). The original curve  $e$  is updated as well with the same information.

`GEN ellanal_globalred_all(GEN e, GEN *v, GEN *N, GEN *tam)` as `ellanal_globalred`; further set `*N` to the curve conductor and `*tam` to the product of the local Tamagawa numbers, including the factor at infinity (multiply by the number of connected components of  $e(\mathbf{R})$ ).

`GEN ellintegralmodel(GEN e, GEN *pv)` return an integral model for  $e$  (in `ellinit` form, over  $\mathbf{Q}$ ). Set  $v = \text{NULL}$  (already integral, we returned  $e$  itself), else to the variable change  $[u, 0, 0, 0]$  making  $e$  integral. We have  $u = 1/t$ ,  $t > 1$ .

`GEN ellintegralmodel_i(GEN e, GEN *pv)` shallow version of `ellintegralmodel`.

## Deprecated routines.

`GEN elltors0(GEN e, long flag)` this function is deprecated; use `elltors`

### 14.1.3.3 Curves over a number field $nf$ .

Let  $K$  be the number field over which  $E$  is defined, given by a  $nf$  or  $bnf$  structure.

`GEN ellnf_get_nf(GEN E)` returns the underlying  $nf$ .

`GEN ellnf_get_bnf(GEN x)` returns `NULL` if  $K$  does not contain a  $bnf$  structure, else return the  $bnf$ .

`GEN ellnf_vecarea(GEN E)` returns the vector of the period lattices areas of all the complex embeddings of  $E$  in the same order as `E.nf.roots`.

`GEN ellnf_veceta(GEN E)` returns the vector of the quasi-periods of all the complex embeddings of  $E$  in the same order as `E.nf.roots`.

`GEN ellnf_vecomega(GEN E)` returns the vector of the periods of all the complex embeddings of  $E$  in the same order as `E.nf.roots`.

### 14.1.3.4 Curves over $\mathbf{Q}_p$ .

`GEN ellQp_get_p(GEN E)` returns  $p$

`long ellQp_get_prec(GEN E)` returns the default  $p$ -adic accuracy to which we must compute approximate results attached to  $E$ .

`GEN ellQp_get_zero(GEN x)` returns  $O(p^n)$ , where  $n$  is the default  $p$ -adic accuracy as above.

The following functions are only defined when  $E$  has multiplicative reduction (Tate curves):

`GEN ellQp_Tate_uniformization(GEN E, long prec)` returns a `t_VEC` containing  $u^2, u, q, [a, b]$ , at  $p$ -adic precision  $prec$ .

`GEN ellQp_u(GEN E, long prec)` returns  $u$ .

`GEN ellQp_u2(GEN E, long prec)` returns  $u^2$ .

`GEN ellQp_q(GEN E, long prec)` returns the Tate period  $q$ .

`GEN ellQp_ab(GEN E, long prec)` returns  $[a, b]$ .

`GEN ellQp_AGM(GEN E, long prec)` returns  $[a, b, R, v]$ , where  $v$  is an integer,  $a, b, R$  are vectors describing the sequence of 2-isogenous curves  $E_i : y^2 = x(x + A_i)(x + A_i - B_i)$ ,  $i \geq 1$  converging to the singular curve  $E_\infty : y^2 = x^2(x + M)$ . We have  $a[i] = A[i]p^v$ ,  $b[i] = B[i]p^v$ ,  $R[i] = A_i - B_i$ . These are used in `ellpointtoz` and `ellztopoint`.

`GEN ellQp_L(GEN E, long prec)` returns the  $\mathcal{L}$ -invariant  $L$ .

`GEN ellQp_root(GEN E, long prec)` returns  $e_1$ .

#### 14.1.3.5 Curves over a finite field $\mathbf{F}_q$ .

`GEN ellff_get_p(GEN E)` returns the characteristic

`GEN ellff_get_field(GEN E)` returns  $p$  if  $\mathbf{F}_q$  is a prime field, and a `t_FFELT` belonging to  $\mathbf{F}_q$  otherwise.

`GEN ellff_get_card(GEN E)` returns  $\#E(\mathbf{F}_q)$

`GEN ellff_get_gens(GEN E)` returns a minimal set of generators for  $E(\mathbf{F}_q)$ .

`GEN ellff_get_group(GEN E)` returns `ellgroup(E)`.

`GEN ellff_get_m(GEN E)` returns the `t_INT`  $m$  as needed by the `gen_ellgroup` function (the order of the pairing required to verify a generating set).

`GEN ellff_get_o(GEN E)` returns  $[d, \text{factord}]$ , where  $d$  is the exponent of  $E(\mathbf{F}_q)$ .

`GEN ellff_get_D(GEN E)` returns the elementary divisors for  $E(\mathbf{F}_q)$  in a form suitable for `gen_ellgens`: either  $[d_1]$  or  $[d_1, d_2]$ , where  $d_1$  is in `ellff_get_o` format.

$[d, \text{factord}]$ , where  $d$  is the exponent of  $E(\mathbf{F}_q)$ .

`GEN ellff_get_a4a6(GEN E)` returns a canonical “short model” for  $E$ , and the corresponding change of variable  $[u, r, s, t]$ . For  $p \neq 2, 3$ , this is  $[A_4, A_6, [u, r, s, t]]$ , corresponding to  $y^2 = x^3 + A_4x + A_6$ , where  $A_4 = -27c_4$ ,  $A_6 = -54c_6$ ,  $[u, r, s, t] = [6, 3b_2, 3a_1, 108a_3]$ .

- If  $p = 3$  and the curve is ordinary ( $b_2 \neq 0$ ), this is  $[[b_2], A_6, [1, v, -a_1, -a_3]]$ , corresponding to

$$y^2 = x^3 + b_2x^2 + A_6,$$

where  $v = b_4/b_2$ ,  $A_6 = b_6 - v(b_4 + v^2)$ .

- If  $p = 3$  and the curve is supersingular ( $b_2 = 0$ ), this is  $[-b_4, b_6, [1, 0, -a_1, -a_3]]$ , corresponding to

$$y^2 = x^3 + 2b_4x + b_6.$$

- If  $p = 2$  and the curve is ordinary ( $a_1 \neq 0$ ), return  $[A_2, A_6, [a_1^{-1}, da_1^{-2}, 0, (a_4 + d^2)a_1^{-1}]]$ , corresponding to

$$y^2 + xy = x^3 + A_2x^2 + A_6,$$

where  $d = a_3/a_1$ ,  $a_1^2A_2 = (a_2 + d)$  and

$$a_1^6A_6 = d^3 + a_2d^2 + a_4d + a_6 + (a_4^2 + d^4)a_1^{-2}.$$

- If  $p = 2$  and the curve is supersingular ( $a_1 = 0$ ,  $a_3 \neq 0$ ), return  $[[a_3, A_4, 1/a_3], A_6, [1, a_2, 0, 0]]$ , corresponding to

$$y^2 + a_3y = x^3 + A_4x + A_6,$$

where  $A_4 = a_2^2 + a_4$ ,  $A_6 = a_2a_4 + a_6$ . The value  $1/a_3$  is included in the vector since it is frequently needed in computations.

#### 14.1.3.6 Curves over $\mathbf{C}$ . (This includes curves over $\mathbf{Q}$ !)

`long ellR_get_prec(GEN E)` return the default accuracy to which we must compute approximate results attached to  $E$ .

`GEN ellR_ab(GEN E, long prec)` return  $[a, b]$

`GEN ellR_omega(GEN x, long prec)` return periods  $[\omega_1, \omega_2]$ .

`GEN ellR_eta(GEN E, long prec)` return quasi-periods  $[\eta_1, \eta_2]$ .

`GEN ellR_area(GEN x, long prec)` return the area  $(\Im(\omega_1 \overline{\omega_2}))$ .

`GEN ellR_roots(GEN E, long prec)` return  $[e_1, e_2, e_3]$ . If  $E$  is defined over  $\mathbf{R}$ , then  $e_1$  is real. If furthermore  $\text{disc} E > 0$ , then  $e_1 > e_2 > e_3$ .

`long ellR_get_sign(GEN E)` if  $E$  is defined over  $\mathbf{R}$  returns the signe of its discriminant, otherwise return 0.

#### 14.1.4 Points.

`int ell_is_inf(GEN z)` tests whether the point  $z$  is the point at infinity.

`GEN ellinf()` returns the point at infinity  $[0]$ .

#### 14.1.5 Change of variables.

`GEN ellchangeinvert(GEN w)` given a change of variables  $w = [u, r, s, t]$ , returns the inverse change of variables  $w'$ , such that if  $E' = \text{ellchangecurve}(E, w)$ , then  $E = \text{ellchangecurve}(E, w')$ .

#### 14.1.6 Generic helper functions.

The naming scheme assumes an affine equation  $F(x, y) = f(x) - (y^2 + h(x)y) = 0$  in standard Weierstrass form:  $f = x^3 + a_2x^2 + a_4x + a_6$ ,  $h = a_1x + a_3$ . Unless mentionned otherwise, these routine assume that all arguments are compatible with generic functions of `gadd` or `gmul` type. In particular they do not handle elements in number field in `nfalgtobasis` format.

`GEN ellbasechar(GEN E)` returns the characteristic of the base ring over which  $E$  is defined.

`GEN ec_bmodel(GEN E)` returns the polynomial  $4x^3 + b_2x^2 + 2b_4x + b_6$ .

`GEN ec_f_evalx(GEN E, GEN x)` returns  $f(x)$ .

`GEN ec_h_evalx(GEN E, GEN x)` returns  $h(x)$ .

`GEN ec_dFdx_evalQ(GEN E, GEN Q)` returns  $3x^2 + 2a_2x + a_4 - a_1y$ , where  $Q = [x, y]$ .

`GEN ec_dFdy_evalQ(GEN E, GEN Q)` returns  $-(2y + a_1x + a_3)$ , where  $Q = [x, y]$ .

`GEN ec_dmFdy_evalQ(GEN e, GEN Q)` returns  $2y + a_1x + a_3$ , where  $Q = [x, y]$ .

`GEN ec_2divpol_evalx(GEN E, GEN x)` returns  $4x^3 + b_2x^2 + 2b_4x + b_6$ . This function supports inputs in `nfalgtobasis` format.

`GEN ec_half_deriv_2divpol_evalx(GEN E, GEN x)` returns  $6x^2 + b_2x + b_4$ .

`GEN ec_3divpol_evalx(GEN E, GEN x)` returns  $3x^4 + b_2x^2 + 3b_4x^2 + 3b_6x + b_8$ .

### 14.1.7 Functions to handle elliptic curves over finite fields.

#### 14.1.7.1 Tolerant routines.

`GEN ellap(GEN E, GEN p)` given a prime number  $p$  and an elliptic curve defined over  $\mathbf{Q}$  or  $\mathbf{Q}_p$  (assumed integral and minimal at  $p$ ), computes the trace of Frobenius  $a_p = p + 1 - \#E(\mathbf{F}_p)$ . If  $E$  is defined over a non-prime finite field  $\mathbf{F}_q$ , ignore  $p$  and return  $q + 1 - \#E(\mathbf{F}_q)$ . When  $p$  is implied ( $E$  defined over  $\mathbf{Q}_p$  or a finite field),  $p$  can be omitted (set to `NULL`).

**14.1.7.2 Curves defined a non-prime finite field.** In this subsection, we assume that `ell_get_type( $E$ )` is `t_ELL_Fq`. (As noted above, a curve defined over  $\mathbf{Z}/p\mathbf{Z}$  can be represented as a `t_ELL_Fq`.)

`GEN FF_elltwist(GEN E)` returns the coefficients  $[a_1, a_2, a_3, a_4, a_6]$  of the quadratic twist of  $E$ .

`GEN FF_ellmul(GEN E, GEN P, GEN n)` returns  $[n]P$  where  $n$  is an integer and  $P$  is a point on the curve  $E$ .

`GEN FF_ellrandom(GEN E)` returns a random point in  $E(\mathbf{F}_q)$ . This function never returns the point at infinity, unless this is the only point on the curve.

`GEN FF_ellorder(GEN E, GEN P, GEN o)` returns the order of the point  $P$ , where  $o$  is a multiple of the order of  $P$ , or its factorization.

`GEN FF_ellcard(GEN E)` returns  $\#E(\mathbf{F}_q)$ .

`GEN FF_ellcard_SEA(GEN E, long s)` This function returns  $\#E(\mathbf{F}_q)$ , using the Schoof-Elkies-Atkin algorithm. Assume  $p \neq 2, 3$ . The parameter  $s$  has the same meaning as in `Fp_ellcard_SEA`.

`GEN FF_ellgens(GEN E)` returns the generators of the group  $E(\mathbf{F}_q)$ .

`GEN FF_elllog(GEN E, GEN P, GEN G, GEN o)` Let  $G$  be a point of order  $o$ , return  $e$  such that  $[e]P = G$ . If  $e$  does not exists, the result is undefined.

`GEN FF_ellgroup(GEN E, GEN *pm)` returns the structure of the Abelian group  $E(\mathbf{F}_q)$  and set `*pm` to  $m$  (see `gen_ellgens`).

`GEN FF_ellweilpairing(GEN E, GEN P, GEN Q, GEN m)` returns the Weil pairing of the points of  $m$ -torsion  $P$  and  $Q$ .

`GEN FF_elltatepairing(GEN E, GEN P, GEN Q, GEN m)` returns the Tate pairing of  $P$  and  $Q$ , where  $[m]P = 0$ .

## 14.2 Arithmetic on elliptic curve over a finite field in simple form.

The functions in this section no longer operate on elliptic curve structures, as seen up to now. They are used to implement those higher-level functions without using cached information and thus require suitable explicitly enumerated data.

### 14.2.1 Helper functions.

`GEN elltrace_extension(GEN t, long n, GEN q)` Let  $E$  some elliptic curve over  $\mathbf{F}_q$  such that the trace of the Frobenius is  $t$ , returns the trace of the Frobenius over  $\mathbf{F}_q^n$ .

### 14.2.2 Elliptic curves over $\mathbf{F}_p$ , $p > 3$ .

Let  $p$  a prime number and  $E$  the elliptic curve given by the equation  $E : y^2 = x^3 + a_4x + a_6$ , with  $a_4$  and  $a_6$  in  $\mathbf{F}_p$ . A **FpE** is a point of  $E(\mathbf{F}_p)$ . Since an affine point and  $a_4$  determine an unique  $a_6$ , most functions do not take  $a_6$  as an argument. A **FpE** is either the point at infinity (**ellinf()**) or a  $FpV$  with two components. The parameters  $a_4$  and  $a_6$  are given as **t\_INTs** when required.

**GEN Fp\_ellj**(**GEN a4**, **GEN a6**, **GEN p**) returns the  $j$ -invariant of the curve  $E$ .

**int Fp\_elljissupersingular**(**GEN j**, **GEN p**) returns 1 if  $j$  is the  $j$ -invariant of a supersingular curve over  $\mathbf{F}_p$ , 0 otherwise.

**GEN Fp\_ellcard**(**GEN a4**, **GEN a6**, **GEN p**) returns the cardinality of the group  $E(\mathbf{F}_p)$ .

**GEN Fp\_ellcard\_SEA**(**GEN a4**, **GEN a6**, **GEN p**, **long s**) This function returns  $\#E(\mathbf{F}_p)$ , using the Schoof-Elkies-Atkin algorithm. If the **seadata** package is installed, the function will be faster.

The extra flag **s**, if set to a non-zero value, causes the computation to return **gen\_0** (an impossible cardinality) if one of the small primes  $\ell$  divides the curve order but does not divide  $s$ . For cryptographic applications, where one is usually interested in curves of prime order, setting  $s = 1$  efficiently weeds out most uninteresting curves; if curves of order a power of 2 times a prime are acceptable, set  $s = 2$ . If moreover **s** is negative, similar checks are performed for the twist of the curve.

**GEN Fp\_ffellcard**(**GEN a4**, **GEN a6**, **GEN q**, **long n**, **GEN p**) returns the cardinality of the group  $E(\mathbf{F}_q)$  where  $q = p^n$ .

**GEN Fp\_ellgroup**(**GEN a4**, **GEN a6**, **GEN N**, **GEN p**, **GEN \*pm**) returns the group structure  $D$  of the group  $E(\mathbf{F}_p)$ , which is assumed to be of order  $N$  and set **\*pm** to  $m$ .

**GEN Fp\_ellgens**(**GEN a4**, **GEN a6**, **GEN ch**, **GEN D**, **GEN m**, **GEN p**) returns generators of the group  $E(\mathbf{F}_p)$  with the base change **ch** (see **FpE\_changepoint**), where  $D$  and  $m$  are as returned by **Fp\_ellgroup**.

**GEN Fp\_elldivpol**(**GEN a4**, **GEN a6**, **long n**, **GEN p**) returns the  $n$ -division polynomial of the elliptic curve  $E$ .

**void Fp\_elltwist**(**GEN a4**, **GEN a6**, **GEN p**, **GEN \*pA4**, **GEN \*pA6**) sets **\*pA4** and **\*pA6** to the corresponding parameters for the quadratic twist of  $E$ .

### 14.2.3 FpE.

**GEN FpE\_add**(**GEN P**, **GEN Q**, **GEN a4**, **GEN p**) returns the sum  $P + Q$  in the group  $E(\mathbf{F}_p)$ , where  $E$  is defined by  $E : y^2 = x^3 + a_4x + a_6$ , for any value of  $a_6$  compatible with the points given.

**GEN FpE\_sub**(**GEN P**, **GEN Q**, **GEN a4**, **GEN p**) returns  $P - Q$ .

**GEN FpE\_dbl**(**GEN P**, **GEN a4**, **GEN p**) returns  $2P$ .

**GEN FpE\_neg**(**GEN P**, **GEN p**) returns  $-P$ .

**GEN FpE\_mul**(**GEN P**, **GEN n**, **GEN a4**, **GEN p**) return  $nP$ .

**GEN FpE\_changepoint**(**GEN P**, **GEN m**, **GEN a4**, **GEN p**) returns the image  $Q$  of the point  $P$  on the curve  $E : y^2 = x^3 + a_4x + a_6$  by the coordinate change  $m$  (which is a **FpV**).

**GEN FpE\_changepointinv**(**GEN P**, **GEN m**, **GEN a4**, **GEN p**) returns the image  $Q$  on the curve  $E : y^2 = x^3 + a_4x + a_6$  of the point  $P$  by the inverse of the coordinate change  $m$  (which is a **FpV**).

GEN random\_FpE(GEN a4, GEN a6, GEN p) returns a random point on  $E(\mathbf{F}_p)$ , where  $E$  is defined by  $E : y^2 = x^3 + a_4x + a_6$ .

GEN FpE\_order(GEN P, GEN o, GEN a4, GEN p) returns the order of  $P$  in the group  $E(\mathbf{F}_p)$ , where  $o$  is a multiple of the order of  $P$ , or its factorization.

GEN FpE\_log(GEN P, GEN G, GEN o, GEN a4, GEN p) Let  $G$  be a point of order  $o$ , return  $e$  such that  $e.P = G$ . If  $e$  does not exist, the result is currently undefined.

GEN FpE\_tatepairing(GEN P, GEN Q, GEN m, GEN a4, GEN p) returns the Tate pairing of the point of  $m$ -torsion  $P$  and the point  $Q$ .

GEN FpE\_weilpairing(GEN P, GEN Q, GEN m, GEN a4, GEN p) returns the Weil pairing of the points of  $m$ -torsion  $P$  and  $Q$ .

GEN FpE\_to\_mod(GEN P, GEN p) returns  $P$  as a vector of `t_INTMODs`.

GEN RgE\_to\_FpE(GEN P, GEN p) returns the FpE obtained by applying `Rg_to_Fp` coefficientwise.

**14.2.4 Fle.** Let  $p$  be a prime `ulong`, and  $E$  the elliptic curve given by the equation  $E : y^2 = x^3 + a_4x + a_6$ , where  $a_4$  and  $a_6$  are `ulong`. A `Fle` is either the point at infinity (`ellinf()`), or a `Flv` with two components  $[x, y]$ .

`long Fl_elltrace(ulong a4, ulong a6, ulong p)` returns the trace  $t$  of the Frobenius of  $E(\mathbf{F}_p)$ . The cardinality of  $E(\mathbf{F}_p)$  is thus  $p + 1 - t$ , which might not fit in an `ulong`.

`long Fl_elltrace_CM(long CM, ulong a4, ulong a6, ulong p)` as `Fl_elltrace`. If  $CM$  is 0, use the standard algorithm; otherwise assume the curve has  $CM$  by a principal imaginary quadratic order of discriminant  $CM$  and use a faster algorithm. Useful when the curve is the reduction of  $E/\mathbf{Q}$ , which has  $CM$  by a principal order, and we need the trace of Frobenius for many distinct  $p$ , see `ellQ_get_CM`.

`ulong Fl_elldisc(ulong a4, ulong a6, ulong p)` returns the discriminant of the curve  $E$ .

`ulong Fl_elldisc_pre(ulong a4, ulong a6, ulong p, ulong pi)` returns the discriminant of the curve  $E$ , assuming  $pi$  is the pseudo inverse of  $p$ .

`ulong Fl_ellj(ulong a4, ulong a6, ulong p)` returns the  $j$ -invariant of the curve  $E$ .

`ulong Fl_ellj_pre(ulong a4, ulong a6, ulong p, ulong pi)` returns the  $j$ -invariant of the curve  $E$ , assuming  $pi$  is the pseudo inverse of  $p$ .

`void Fl_ellj_to_a4a6(ulong j, ulong p, ulong *pa4, ulong *pa6)` sets  $*pa4$  to  $a_4$  and  $*pa6$  to  $a_6$  where  $a_4$  and  $a_6$  define a fixed elliptic curve with  $j$ -invariant  $j$ .

`void Fl_elltwist(ulong a4, ulong a6, ulong p, ulong *pA4, ulong *pA6)` set  $*pA4$  to  $A_4$  and  $*pA6$  to  $A_6$  where  $A_4$  and  $A_6$  define the twist of  $E$ .

`void Fl_elltwist_disc(ulong a4, ulong a6, ulong D, ulong p, ulong *pA4, ulong *pA6)` sets  $*pA4$  to  $A_4$  and  $*pA6$  to  $A_6$  where  $A_4$  and  $A_6$  define the twist of  $E$  by the discriminant  $D$ .

GEN Fle\_add(GEN P, GEN Q, ulong a4, ulong p)

GEN Fle\_dbl(GEN P, ulong a4, ulong p)

GEN Fle\_sub(GEN P, GEN Q, ulong a4, ulong p)

GEN Fle\_mul(GEN P, GEN n, ulong a4, ulong p)

GEN Fle\_mulu(GEN P, ulong n, ulong a4, ulong p)  
 GEN Fle\_order(GEN P, GEN o, ulong a4, ulong p)  
 GEN Fle\_log(GEN P, GEN G, GEN o, ulong a4, ulong p)  
 GEN random\_Fle(ulong a4, ulong a6, ulong p)  
 GEN random\_Fle\_pre(ulong a4, ulong a6, ulong p, ulong pi)  
 GEN Fle\_changepoint(GEN x, GEN ch, ulong p), ch is assumed to give the change of coordinates  $[u, r, s, t]$  as a `t_VECSMALL`.  
 GEN Fle\_changepointinv(GEN x, GEN ch, ulong p), as `Fle_changepoint`

#### 14.2.5 FpJ.

Let  $p$  be a prime `t_INT`, and  $E$  the elliptic curve given by the equation  $E : y^2 = x^3 + a_4x + a_6$ , where  $a_4$  and  $a_6$  are `t_INT`. A `FpJ` is a `FpV` with three components  $[x, y, z]$ , representing the affine point  $[x/z^2, y/z^3]$  in Jacobian coordinates, the point at infinity being represented by  $[1, 1, 0]$ . The following must holds:  $y^2 = x^3 + a_4xz^4 + a_6z^6$ . For all non-zero  $u$ , the points  $[u^2x, u^3y, uz]$  and  $[x, y, z]$  are representing the same affine point.

GEN FpJ\_add(GEN P, GEN Q, GEN a4, GEN p)  
 GEN FpJ\_dbl(GEN P, GEN a4, GEN p)  
 GEN FpJ\_mul(GEN P, GEN n, GEN a4, GEN p);  
 GEN FpJ\_neg(GEN P, GEN p) return  $-P$ .  
 GEN FpJ\_to\_FpE(GEN P, GEN p) return the corresponding `FpE`.  
 GEN FpE\_to\_FpJ(GEN P) return the corresponding `FpJ`.

#### 14.2.6 Flj.

Below, `pi` is assumed to be the precomputed inverse of  $p$ .

GEN Fle\_to\_Flj(GEN P) convert a `Fle` to an equivalent `Flj`.  
 GEN Flj\_to\_Fle\_pre(GEN P) convert a `Flj` to the equivalent `Fle`.  
 GEN Flj\_add\_pre(GEN P, GEN Q, ulong a4, ulong p, ulong pi)  
 GEN Flj\_dbl\_pre(GEN P, ulong a4, ulong p, ulong pi)  
 GEN Flj\_neg(GEN P, ulong p) return  $-P$ .  
 GEN Flj\_mulu\_pre(GEN P, ulong n, ulong a4, ulong p, ulong pi)  
 GEN random\_Flj\_pre(ulong a4, ulong a6, ulong p, ulong pi)



**14.2.7 Elliptic curves over  $\mathbf{F}_{2^n}$ .** Let  $T$  be an irreducible  $\mathbf{F}_2[x]$  and  $E$  the elliptic curve given by either the equation  $E : y^2 + x * y = x^3 + a_2 x^2 + a_6$ , where  $a_2, a_6$  are  $\mathbf{F}_2[x]$  in  $\mathbf{F}_2[X]/(T)$  (ordinary case) or  $E : y^2 + a_3 * y = x^3 + a_4 x + a_6$ , where  $a_3, a_4, a_6$  are  $\mathbf{F}_2[x]$  in  $\mathbf{F}_2[X]/(T)$  (supersingular case).

A  $\mathbf{F}_2[x]E$  is a point of  $E(\mathbf{F}_2[X]/(T))$ . In the supersingular case, the parameter  $a_2$  is actually the  $t\_VEC [a_3, a_4, a_3^{-1}]$ .

`GEN F2xq_ellcard(GEN a2, GEN a6, GEN T)` Return the order of the group  $E(\mathbf{F}_2[X]/(T))$ .

`GEN F2xq_ellgroup(GEN a2, GEN a6, GEN N, GEN T, GEN *pm)` Return the group structure  $D$  of the group  $E(\mathbf{F}_2[X]/(T))$ , which is assumed to be of order  $N$  and set  $*pm$  to  $m$ .

`GEN F2xq_ellgens(GEN a2, GEN a6, GEN ch, GEN D, GEN m, GEN T)` Returns generators of the group  $E(\mathbf{F}_2[X]/(T))$  with the base change  $ch$  (see `F2xqE_changepoint`), where  $D$  and  $m$  are as returned by `F2xq_ellgroup`.

`void F2xq_elltwist(GEN a4, GEN a6, GEN T, GEN *a4t, GEN *a6t)` sets  $*a4t$  and  $*a6t$  to the parameters of the quadratic twist of  $E$ .

#### 14.2.8 $\mathbf{F}_2[x]E$ .

`GEN F2xqE_changepoint(GEN P, GEN m, GEN a2, GEN T)` returns the image  $Q$  of the point  $P$  on the curve  $E : y^2 + x * y = x^3 + a_2 x^2 + a_6$  by the coordinate change  $m$  (which is a  $\mathbf{F}_2[x]V$ ).

`GEN F2xqE_changepointinv(GEN P, GEN m, GEN a2, GEN T)` returns the image  $Q$  on the curve  $E : y^2 = x^3 + a_4 x + a_6$  of the point  $P$  by the inverse of the coordinate change  $m$  (which is a  $\mathbf{F}_2[x]V$ ).

`GEN F2xqE_add(GEN P, GEN Q, GEN a2, GEN T)`

`GEN F2xqE_sub(GEN P, GEN Q, GEN a2, GEN T)`

`GEN F2xqE_dbl(GEN P, GEN a2, GEN T)`

`GEN F2xqE_neg(GEN P, GEN a2, GEN T)`

`GEN F2xqE_mul(GEN P, GEN n, GEN a2, GEN T)`

`GEN random_F2xqE(GEN a2, GEN a6, GEN T)`

`GEN F2xqE_order(GEN P, GEN o, GEN a2, GEN T)` returns the order of  $P$  in the group  $E(\mathbf{F}_2[X]/(T))$ , where  $o$  is a multiple of the order of  $P$ , or its factorization.

`GEN F2xqE_log(GEN P, GEN G, GEN o, GEN a2, GEN T)` Let  $G$  be a point of order  $o$ , return  $e$  such that  $e.P = G$ . If  $e$  does not exist, the result is currently undefined.

`GEN F2xqE_tatepairing(GEN P, GEN Q, GEN m, GEN a2, GEN T)` returns the Tate pairing of the point of  $m$ -torsion  $P$  and the point  $Q$ .

`GEN F2xqE_weilpairing(GEN P, GEN Q, GEN m, GEN a2, GEN T)` returns the Weil pairing of the points of  $m$ -torsion  $P$  and  $Q$ .

`GEN RgE_to_F2xqE(GEN P, GEN T)` returns the  $\mathbf{F}_2[x]E$  obtained by applying `Rg.to_F2xq` coefficient-wise.

**14.2.9 Elliptic curves over  $\mathbf{F}_q$ , small characteristic  $p > 2$ .** Let  $p > 2$  be a prime `ulong`,  $T$  an irreducible `Flx` mod  $p$ , and  $E$  the elliptic curve given by the equation  $E : y^2 = x^3 + a_4x + a_6$ , where  $a_4$  and  $a_6$  are `Flx` in  $\mathbf{F}_p[X]/(T)$ . A `FlxqE` is a point of  $E(\mathbf{F}_p[X]/(T))$ .

In the special case  $p = 3$ , ordinary elliptic curves ( $j(E) \neq 0$ ) cannot be represented as above, but admit a model  $E : y^2 = x^3 + a_2x^2 + a_6$  with  $a_2$  and  $a_6$  being `Flx` in  $\mathbf{F}_3[X]/(T)$ . In that case, the parameter `a2` is actually stored as a `t_VEC`,  $[a_2]$ , to avoid ambiguities.

`GEN Flxq_ellj(GEN a4, GEN a6, GEN T, ulong p)` returns the  $j$ -invariant of the curve  $E$ .

`void Flxq_ellj_to_a4a6(GEN j, GEN T, ulong p, GEN *pa4, GEN *pa6)` sets `*pa4` to  $a_4$  and `*pa6` to  $a_6$  where  $a_4$  and  $a_6$  define a fixed elliptic curve with  $j$ -invariant  $j$ .

`GEN Flxq_ellcard(GEN a4, GEN a6, GEN T, ulong p)` returns the order of  $E(\mathbf{F}_p[X]/(T))$ .

`GEN Flxq_ellgroup(GEN a4, GEN a6, GEN N, GEN T, ulong p, GEN *pm)` returns the group structure  $D$  of the group  $E(\mathbf{F}_p[X]/(T))$ , which is assumed to be of order  $N$  and sets `*pm` to  $m$ .

`GEN Flxq_ellgens(GEN a4, GEN a6, GEN ch, GEN D, GEN m, GEN T, ulong p)` returns generators of the group  $E(\mathbf{F}_p[X]/(T))$  with the base change `ch` (see `FlxqE_changepoint`), where  $D$  and  $m$  are as returned by `Flxq_ellgroup`.

`void Flxq_elltwist(GEN a4, GEN a6, GEN T, ulong p, GEN *pA4, GEN *pA6)` sets `*pA4` and `*pA6` to the corresponding parameters for the quadratic twist of  $E$ .

#### 14.2.10 `FlxqE`.

Let  $p > 2$  be a prime number.

`GEN FlxqE_changepoint(GEN P, GEN m, GEN a4, GEN T, ulong p)` returns the image  $Q$  of the point  $P$  on the curve  $E : y^2 = x^3 + a_4x + a_6$  by the coordinate change  $m$  (which is a `FlxqV`).

`GEN FlxqE_changepointinv(GEN P, GEN m, GEN a4, GEN T, ulong p)` returns the image  $Q$  on the curve  $E : y^2 = x^3 + a_4x + a_6$  of the point  $P$  by the inverse of the coordinate change  $m$  (which is a `FlxqV`).

`GEN FlxqE_add(GEN P, GEN Q, GEN a4, GEN T, ulong p)`

`GEN FlxqE_sub(GEN P, GEN Q, GEN a4, GEN T, ulong p)`

`GEN FlxqE_dbl(GEN P, GEN a4, GEN T, ulong p)`

`GEN FlxqE_neg(GEN P, GEN T, ulong p)`

`GEN FlxqE_mul(GEN P, GEN n, GEN a4, GEN T, ulong p)`

`GEN random_FlxqE(GEN a4, GEN a6, GEN T, ulong p)`

`GEN FlxqE_order(GEN P, GEN o, GEN a4, GEN T, ulong p)` returns the order of  $P$  in the group  $E(\mathbf{F}_p[X]/(T))$ , where  $o$  is a multiple of the order of  $P$ , or its factorization.

`GEN FlxqE_log(GEN P, GEN G, GEN o, GEN a4, GEN T, ulong p)` Let  $G$  be a point of order  $o$ , return  $e$  such that  $e.P = G$ . If  $e$  does not exist, the result is currently undefined.

`GEN FlxqE_tatepairing(GEN P, GEN Q, GEN m, GEN a4, GEN T, ulong p)` returns the Tate pairing of the point of  $m$ -torsion  $P$  and the point  $Q$ .

`GEN FlxqE_weilpairing(GEN P, GEN Q, GEN m, GEN a4, GEN T, ulong p)` returns the Weil pairing of the points of  $m$ -torsion  $P$  and  $Q$ .

`GEN RgE_to_FlxqE(GEN P, GEN T, ulong p)` returns the `FlxqE` obtained by applying `Rg_to_Flxq` coefficientwise.

#### 14.2.11 Elliptic curves over $\mathbf{F}_q$ , large characteristic .

Let  $p > 3$  be a prime number,  $T$  an irreducible polynomial mod  $p$ , and  $E$  the elliptic curve given by the equation  $E : y^2 = x^3 + a_4x + a_6$  with  $a_4$  and  $a_6$  in  $\mathbf{F}_p[X]/(T)$ . A **FpXQE** is a point of  $E(\mathbf{F}_p[X]/(T))$ .

**GEN FpXQ\_ellj**(**GEN a4**, **GEN a6**, **GEN T**, **GEN p**) returns the  $j$ -invariant of the curve  $E$ .

**int FpXQ\_elljissupersingular**(**GEN j**, **GEN T**, **GEN p**) returns 1 if  $j$  is the  $j$ -invariant of a supersingular curve over  $\mathbf{F}_p[X]/(T)$ , 0 otherwise.

**GEN FpXQ\_ellcard**(**GEN a4**, **GEN a6**, **GEN T**, **GEN p**) returns the order of  $E(\mathbf{F}_p[X]/(T))$ .

**GEN Fq\_ellcard\_SEA**(**GEN a4**, **GEN a6**, **GEN q**, **GEN T**, **GEN p**, **long s**) This function returns  $\#E(\mathbf{F}_p[X]/(T))$ , using the Schoof-Elkies-Atkin algorithm. Assume  $p \neq 2, 3$ , and  $q$  is the cardinality of  $\mathbf{F}_p[X]/(T)$ . The parameter  $s$  has the same meaning as in **Fp\_ellcard\_SEA**. If the **seadata** package is installed, the function will be faster.

**GEN FpXQ\_ellgroup**(**GEN a4**, **GEN a6**, **GEN N**, **GEN T**, **GEN p**, **GEN \*pm**) Return the group structure  $D$  of the group  $E(\mathbf{F}_p[X]/(T))$ , which is assumed to be of order  $N$  and set **\*pm** to  $m$ .

**GEN FpXQ\_ellgens**(**GEN a4**, **GEN a6**, **GEN ch**, **GEN D**, **GEN m**, **GEN T**, **GEN p**) Returns generators of the group  $E(\mathbf{F}_p[X]/(T))$  with the base change **ch** (see **FpXQE\_changepoint**), where  $D$  and  $m$  are as returned by **FpXQ\_ellgroup**.

**GEN FpXQ\_elldivpol**(**GEN a4**, **GEN a6**, **long n**, **GEN T**, **GEN p**) returns the  $n$ -division polynomial of the elliptic curve  $E$ .

**GEN Fq\_elldivpolmod**(**GEN a4**, **GEN a6**, **long n**, **GEN h**, **GEN T**, **GEN p**) returns the  $n$ -division polynomial of the elliptic curve  $E$  modulo the polynomial  $h$ .

**void FpXQ\_elltwist**(**GEN a4**, **GEN a6**, **GEN T**, **GEN p**, **GEN \*pA4**, **GEN \*pA6**) sets **\*pA4** and **\*pA6** to the corresponding parameters for the quadratic twist of  $E$ .

#### 14.2.12 FpXQE.

**GEN FpXQE\_changepoint**(**GEN P**, **GEN m**, **GEN a4**, **GEN T**, **GEN p**) returns the image  $Q$  of the point  $P$  on the curve  $E : y^2 = x^3 + a_4x + a_6$  by the coordinate change  $m$  (which is a **FpXQV**).

**GEN FpXQE\_changepointinv**(**GEN P**, **GEN m**, **GEN a4**, **GEN T**, **GEN p**) returns the image  $Q$  on the curve  $E : y^2 = x^3 + a_4x + a_6$  of the point  $P$  by the inverse of the coordinate change  $m$  (which is a **FpXQV**).

**GEN FpXQE\_add**(**GEN P**, **GEN Q**, **GEN a4**, **GEN T**, **GEN p**)

**GEN FpXQE\_sub**(**GEN P**, **GEN Q**, **GEN a4**, **GEN T**, **GEN p**)

**GEN FpXQE\_dbl**(**GEN P**, **GEN a4**, **GEN T**, **GEN p**)

**GEN FpXQE\_neg**(**GEN P**, **GEN T**, **GEN p**)

**GEN FpXQE\_mul**(**GEN P**, **GEN n**, **GEN a4**, **GEN T**, **GEN p**)

**GEN random\_FpXQE**(**GEN a4**, **GEN a6**, **GEN T**, **GEN p**)

**GEN FpXQE\_log**(**GEN P**, **GEN G**, **GEN o**, **GEN a4**, **GEN T**, **GEN p**) Let  $G$  be a point of order  $o$ , return  $e$  such that  $e.P = G$ . If  $e$  does not exists, the result is currently undefined.

**GEN FpXQE\_order**(**GEN P**, **GEN o**, **GEN a4**, **GEN T**, **GEN p**) returns the order of  $P$  in the group  $E(\mathbf{F}_p[X]/(T))$ , where  $o$  is a multiple of the order of  $P$ , or its factorization.

`GEN FpXQE_tatepairing(GEN P, GEN Q, GEN m, GEN a4, GEN T, GEN p)` returns the Tate pairing of the point of  $m$ -torsion  $P$  and the point  $Q$ .

`GEN FpXQE_weilpairing(GEN P, GEN Q, GEN m, GEN a4, GEN T, GEN p)` returns the Weil pairing of the points of  $m$ -torsion  $P$  and  $Q$ .

`GEN RgE_to_FpXQE(GEN P, GEN T, GEN p)` returns the FpXQE obtained by applying `Rg_to_FpXQ` coefficientwise.

### 14.3 Functions related to modular polynomials.

Variants of `polmodular`, returning the modular polynomial of prime level  $L$  for the invariant coded by `inv` (0:  $j$ , 1: Weber- $f$ , see `polclass` for the full list).

`GEN polmodular_ZXX(long L, long inv, long xvar, long yvar)` returns a bivariate polynomial in variables `xvar` and `yvar`.

`GEN polmodular_ZM(long L, long inv)` returns a matrix of (integral) coefficients.

`GEN Fp_polmodular_evalx(long L, long inv, GEN J, GEN p, long v, int derivs)` returns the modular polynomial evaluated at  $J$  modulo the prime  $p$  in the variable  $v$  (if `derivs` is non-zero, returns a vector containing the modular polynomial and its first and second derivatives, all evaluated at  $J$  modulo  $p$ ).

#### 14.3.1 Functions related to modular invariants.

`void check_modinv(long inv)` report an error if `inv` is not a valid code for a modular invariant.

`int modinv_good_disc(long inv, long D)` test whether the invariant `inv` is defined for the discriminant  $D$ .

`int modinv_good_prime(long inv, long D)` test whether the invariant `inv` is defined for the prime  $p$ .

`long modinv_height_factor(long inv)` return the height factor of the modular invariant `inv` with respect to the  $j$ -invariant. This is an integer  $n$  such that the  $j$ -invariant is asymptotically of the order of the  $n$ -th power of the invariant `inv`.

`long modinv_is_Weber(long inv)` test whether the invariant `inv` is a power of Weber  $f$ .

`long modinv_is_double_eta(long inv)` test whether the invariant `inv` is a double  $\eta$  quotient.

`long disc_best_modinv(long D)` the integer  $D$  being a negative discriminant, return the modular invariant compatible with  $D$  with the highest height factor.

`GEN Fp_modinv_to_j(GEN x, long inv, GEN p)` Let  $\Phi$  the modular equation between  $j$  and the modular invariant `inv`, return  $y$  such that  $\Phi(y, x) = 0 \pmod{p}$ .

## 14.4 Other curves.

The following functions deal with hyperelliptic curves in weighted projective space  $\mathbf{P}_{(1,d,1)}$ , with coordinates  $(x, y, z)$  and a model of the form  $y^2 = T(x, z)$ , where  $T$  is homogeneous of degree  $2d$ , and squarefree. Thus the curve is nonsingular of genus  $d - 1$ .

`long hyperell_locally_soluble(GEN T, GEN p)` assumes that  $T \in \mathbf{Z}[X]$  is integral. Returns 1 if the curve is locally soluble over  $\mathbf{Q}_p$ , 0 otherwise.

`long nf_hyperell_locally_soluble(GEN nf, GEN T, GEN pr)` let  $K$  be a number field, attached to `nf`, `pr` a *prid* attached to some maximal ideal  $\mathfrak{p}$ ; assumes that  $T \in \mathbf{Z}_K[X]$  is integral. Returns 1 if the curve is locally soluble over  $K_{\mathfrak{p}}$ .



## Chapter 15: *L*-functions

### 15.1 Accessors.

```
long is_linit(GEN data)
GEN ldata_get_an(GEN ldata)
GEN ldata_get_dual(GEN ldata)
long ldata_isreal(GEN ldata)
GEN ldata_get_gammavec(GEN ldata)
long ldata_get_degree(GEN ldata)
long ldata_get_k(GEN ldata)
GEN ldata_get_conductor(GEN ldata)
GEN ldata_get_rootno(GEN ldata)
GEN ldata_get_residue(GEN ldata)
GEN ldata_vecan(GEN ldata, long L, long prec)
long ldata_get_type(GEN ldata)
long linit_get_type(GEN linit)
GEN linit_get_ldata(GEN linit)
GEN linit_get_tech(GEN linit)
GEN lfun_get_domain(GEN tech)
GEN lfun_get_dom(GEN tech)
long lfun_get_bitprec(GEN tech)
GEN lfun_get_factgammavec(GEN tech)
GEN lfun_get_step(GEN tech)
GEN lfun_get_pol(GEN tech)
GEN lfun_get_Residue(GEN tech)
GEN lfun_get_k2(GEN tech)
GEN lfun_get_w2(GEN tech)
GEN lfun_get_expot(GEN tech)
long lfun_get_bitprec(GEN tech)
GEN lfunprod_get_fact(GEN tech)
```

```

GEN theta_get_an(GEN tdata)
GEN theta_get_K(GEN tdata)
GEN theta_get_R(GEN tdata)
long theta_get_bitprec(GEN tdata)
long theta_get_m(GEN tdata)
GEN theta_get_tdom(GEN tdata)
GEN theta_get_sqrtN(GEN tdata)

```

## 15.2 Conversions and constructors.

```

GEN lfunmisc_to_ldata(GEN ldata)
GEN lfunmisc_to_ldata_shallow(GEN ldata)
GEN lfunrttopoles(GEN r)
int sdomain_isincl(GEN dom, GEN dom0)

```

## 15.3 Variants of GP functions.

```

GEN lfun(GEN ldata, GEN s, long bitprec)
GEN lfuninit(GEN ldata, GEN dom, long der, long bitprec)
GEN lfuninit_make(long t, GEN ldata, GEN molin, GEN domain)
GEN lfunlambda(GEN ldata, GEN s, long bitprec)
long lfunthetacost(GEN ldata, GEN tdom, long m, long bitprec): lfunthetacost0 when
the first argument is known to be an Ldata.
GEN lfunthetacheckinit(GEN data, GEN tinf, long m, long bitprec)
GEN lfunrootno(GEN data, long bitprec)
GEN lfunzetakinit(GEN pol, GEN dom, long der, long flag, long bitprec)
GEN lfunellmfpeters(GEN E, long bitprec)
GEN ellanalyticrank(GEN E, long prec) DEPRECATED.
GEN ellL1(GEN E, long prec) DEPRECATED.

```

## 15.4 Inverse Mellin transforms of Gamma products.

```

GEN gammamellininv(GEN Vga, GEN s, long m, long bitprec)
GEN gammamellinininit(GEN Vga, long m, long bitprec)
GEN gammamellininvrt(GEN K, GEN s, long bitprec)
double dbllambertW0(double a)
double dbllambertW_1(double a)
double dblllemma526(double a, double b, double c, long B)
double dblcoro526(double a, double c, long B)

```



## Chapter 16: Modular symbols

`void checkms(GEN W)` raise an exception if  $W$  is not an *ms* structure from `msinit`.

`void checkmspadic(GEN W)` raise an exception if  $W$  is not an *mspadic* structure from `mspadicinit`.

Variants of `mfnumcusps` :

`ulong mfnumcuspsu(ulong n)`

`GEN mfnumcusps_fact(GEN fa)` where `fa` is `factor(n)`.

`ulong mfnumcuspsu_fact(GEN fa)` where `fa` is `factoru(n)`.

## Chapter 17: Modular forms

### 17.1 Implementation of public data structures.

`void checkMF(GEN mf)` raise an exception if the argument is not a modular form space.

`GEN checkMF_i(GEN mf)` return the underlying modular form space if `mf` is either directly a modular form space from `mfinit` or a symbol from `mfsymbol`. Return `NULL` otherwise.

`int checkmf_i(GEN mf)` return 1 if the argument is a modular form and 0 otherwise.

#### 17.1.1 Accessors for modular form spaces.

Shallow functions; assume that their argument is a modular form space is created by `mfinit` and checked using `checkMF`.

`GEN MF_get_gN(GEN mf)` return the level  $N$  as a `t_INT`.

`long MF_get_N(GEN mf)` return the level  $N$  as a `long`.

`GEN MF_get_gk(GEN mf)` return the level  $k$  as a `t_INT`.

`long MF_get_k(GEN mf)` return the level  $k$  as a `long`.

`long MF_get_r(GEN mf)` assuming the level is a half-integer, return the integer  $r = k - (1/2)$ .

`GEN MF_get_CHI(GEN mf)` return the nebentypus  $\chi$ , which is a special form of character structure attached to Dirichlet characters (see next section). Its values are given as algebraic numbers: either  $\pm 1$  or `t_POLMOD` in  $t$ .

`long MF_get_space(GEN mf)` returns the space type, corresponding to `mfinit`'s `space` flag. The current list is

`mf_NEW, mf_CUSP, mf_OLD, mf_EISEN, mf_FULL`

`GEN MF_get_basis(GEN mf)` return the  $\mathbf{Q}$ -basis of the space, concatenation of `MF_get_E` and `MF_get_S`, in this order; the forms have coefficients in  $\mathbf{Q}(\chi)$ . Low-level version of `mfbasis`.

`long MF_get_dim(GEN mf)` returns the dimension  $d$  of the space. It is the cardinality of `MF_get_basis`.

`GEN MF_get_E(GEN mf)` returns a  $\mathbf{Q}$ -basis for the subspace spanned by Eisenstein series in the space; the forms have coefficients in  $\mathbf{Q}(\chi)$ .

`GEN MF_get_S(GEN mf)` returns a  $\mathbf{Q}$ -basis for the cuspidal subspace in the space; the forms have coefficients in  $\mathbf{Q}(\chi)$ .

`GEN MF_get_fields(GEN mf)` returns the vector of polynomials defining each Galois orbit of newforms over  $\mathbf{Q}(\chi)$ . Uses memoization: a first call splits the space and may be costly; subsequent calls return the cached result.

`GEN MF_get_newforms(GEN mf)` returns a vector `vF` containing the coordinates of the eigenforms on `MF_get_basis` (`mftobasis` form). Low-level version of `mfeigenbasis`, whose elements are recovered as `mflinear(mf, gel(vF,i))`. Uses memoization, sharing the same data as `MF_get_fields`. Note that it is much more efficient to use `mfcoefs(mf,)` then multiply by this vector than to compute the coefficients of eigenforms from `mfeigenbasis` individually.

The following accessors are technical,

`GEN MF_get_M(GEN mf)` the  $(1+m) \times d$  matrix whose  $j$ -th column contain the coefficients of the  $j$ -th entry in `MF_get_basis`,  $m$  is the optimal “Sturm bound” for the space: the maximum of the  $v_\infty(f)$  over non-zero forms. It has entries in  $\mathbf{Q}(\chi)$ .

`GEN MF_get_Mindex(GEN mf)` is a `t_VECSMALL` containing  $d$  row indices, the corresponding rows of  $M$  form an invertible matrix  $M_0$ .

`GEN MF_get_Minv(GEN mf)` the inverse of  $M_0$  in a form suitable for fast multiplication.

`GEN MF_cusp_get_vMjd(GEN mf)` valid only for a full *cuspidal* space. Then the functions in `MF_get_S` are of the form  $B_d T_j Tr_M^{new}$ . This returns the vector of triples (`t_VECSMALL`)  $[M, j, d]$ , in the same order.

`GEN MF_new_get_vj(GEN mf)` valid only for a *new* space. Then the functions in `MF_get_S` are of the form  $T_j Tr_N^{new}$ . This returns a `t_VECSMALL` of the Hecke indices  $j$ , in the same order.

### 17.1.2 Accessors for individual modular forms.

`GEN mf_get_gN(GEN F)` return the level of  $F$ , which may be a multiple of the conductor, as a `t_INT`

`long mf_get_N(GEN F)` return the level as a `long`.

`GEN mf_get_gk(GEN F)` return the weight of  $F$  as a `t_INT` or a `t_FRAC` with denominator 2 (half-integral weight).

`long mf_get_k(GEN F)` return the weight as a `long`; if the weight is not integral, this raises an exception.

`long mf_get_r(GEN F)` assuming  $F$  is a modular form of half-integral weight  $k = (2r+1)/2$ , return  $r = k - (1/2)$ .

`GEN mf_get_CHI(GEN F)` return the nebentypus, which is a special form of character structure attached to Dirichlet characters (see next section). Its values are given as algebraic numbers: either  $\pm 1$  or `t_POLMOD` in  $t$ .

`GEN mf_get_field(GEN F)` return the polynomial (in variable  $y$ ) defining  $\mathbf{Q}(f)$  over  $\mathbf{Q}(\chi)$ .

`GEN mf_get_NK(GEN F)` return the tag attached to  $F$ : a vector containing `gN`, `gk`, `CHI`, `field`. Never use its component directly, use individual accessors as above.

`long mf_get_type(GEN F)` returns a symbolic name for the constructor used to create the form, e.g. `t_MF_EISEN` for a general Eisenstein series. A form has a recursive structure represented by a tree: its definition may involve other forms, e.g. the tree attached to  $T_n f$  contains  $f$  as a subtree. Such trees have *leaves*, forms which do not contain a strict subtree, e.g. `t_MF_DELTA` is a leaf, attached to Ramanujan's  $\Delta$ .

Here is the current list of types; since the names are liable to change, they are not documented at this point. Use `mfdescribe` to visualize their mathematical structure.

```
/*leaves*/
  t_MF_CONST, t_MF_EISEN, t_MF_Ek, t_MF_DELTA, t_MF_ETAQUO, t_MF_ELL,
  t_MF_DIHEDRAL, t_MF_THETA, t_MF_TRACE, t_MF_NEWTRACE,
/*recursive*/
  t_MF_MUL, t_MF_POW, t_MF_DIV, t_MF_BRACKET, t_MF_LINEAR, t_MF_LINEAR_BHN,
  t_MF_SHIFT, t_MF_DERIV, t_MF_DERIVE2, t_MF_TWIST, t_MF_HECKE,
  t_MF_BD,
```

**17.1.3 Nebentypus.** The characters stored in modular forms and modular form spaces have a special structure. One can recover the parameters of an ordinary Dirichlet character by `G = gel(CHI,1)` (the underlying `znstar`) and `chi = gel(CHI,2)` (the underlying character in `znconreylog` form).

`long mfcharmodulus(GEN CHI)` the modulus of  $\chi$ .

`long mfcharorder(GEN CHI)` the order of  $\chi$ .

`GEN mfcharpol(GEN CHI)` the cyclotomic polynomial  $\Phi_n$  defining  $\mathbf{Q}(\chi)$ , always normalized so that  $n$  is not 2 mod 4.

#### 17.1.4 Miscellaneous functions.

`long mfnewdim(long N, long k, GEN CHI)` dimension of the new part of the cuspidal space.

`long mfcuspdim(long N, long k, GEN CHI)` dimension of the cuspidal space.

`long mfolddim(long N, long k, GEN CHI)` dimension of the old part of the cuspidal space.

`long mfeisensteindim(long N, long k, GEN CHI)` dimension of the Eisenstein subspace.

`long mffulldim(long N, long k, GEN CHI)` dimension of the full space.

`GEN mfeisensteinspaceinit(GEN NK)`

`GEN mfdiv_val(GEN F, GEN G, long vG)`

`GEN mfembed(GEN E, GEN v)`

`GEN mfmatembed(GEN E, GEN v)`

`GEN mfvecembed(GEN E, GEN v)`

`long mfsturmNgk(long N, GEN k)`

`long mfsturmNk(long N, long k)`

```
long mfsturm_mf(GEN mf)
long mfiscuspidal(GEN mf, GEN F)
GEN mftobasisES(GEN mf, GEN F)
GEN mftocol(GEN F, long lim, long d)
GEN mfvectomat(GEN vF, long lim, long d)
```

## Chapter 18: Plots

A `PARI_plot` canvas is a record of dimensions, with the following fields:

```
long width; /* window width */
long height; /* window height */
long hunit; /* length of horizontal 'ticks' */
long vunit; /* length of vertical 'ticks' */
long fwidth; /* font width */
long fheight; /* font height */
void (*draw)(PARI_plot *T, GEN w, GEN x, GEN y);
```

The `draw` method performs the actual drawing of a `t_VECSMALL` `w` (rectwindow indices);  $x$  and  $y$  are `t_VECSMALL`s of the same length and rectwindow  $w[i]$  is drawn with its upper left corner at offset  $(x[i], y[i])$ . No plot engine is available in `libpari` by default, since this would introduce a dependency on extra graphical libraries. See the files `src/graph/plot*` for basic implementations of various plot engines: `plotsvg` is particularly simple (`draw` is a 1-liner).

`void pari_set_plot_engine(void (*T)(PARI_plot *))` installs the graphical engine  $T$  and initializes the graphical subsystem. No routine in this chapter will work without this initialization.

`void pari_kill_plot_engine(void)` closes the graphical subsystem and frees the resources it occupies.

**18.0.5 Highlevel function.** Those functions plot  $f(E, x)$  for  $x \in [a, b]$ , using  $n$  regularly spaced points (by default).

`GEN ploth(void *E, GEN(*f)(void*, GEN), GEN a, GEN b, long flags, long n, long prec)` draw physically.

`GEN plotrecth(void *E, GEN(*f)(void*, GEN), long w, GEN a, GEN b, ulong flags, long n, long prec)` draw in rectwindow  $w$ .

### 18.0.6 Function .

```
void plotbox(long ne, GEN gx2, GEN gy2)
void plotclip(long rect)
void plotcolor(long ne, long color)
void plotcopy(long source, long dest, GEN xoff, GEN yoff, long flag)
GEN plotcursor(long ne)
void plotdraw(GEN list, long flag)
GEN plothraw(GEN listx, GEN listy, long flag)
GEN plothsizes(long flag)
void plotinit(long ne, GEN x, GEN y, long flag)
void plotkill(long ne)
```

```

void plotline(long ne, GEN x2, GEN y2)
void plotlines(long ne, GEN listx, GEN listy, long flag)
void plotlinetype(long ne, long t)
void plotmove(long ne, GEN x, GEN y)
void plotpoints(long ne, GEN listx, GEN listy)
void plotpointsize(long ne, GEN size)
void plotpointtype(long ne, long t)
void plotrbox(long ne, GEN x2, GEN y2)
GEN plotrecthraw(long ne, GEN data, long flags)
void plotrline(long ne, GEN x2, GEN y2)
void plotrmove(long ne, GEN x, GEN y)
void plotrpoint(long ne, GEN x, GEN y)
void plotscale(long ne, GEN x1, GEN x2, GEN y1, GEN y2)
void plotstring(long ne, char *x, long dir)

```

**18.0.7 Obsolete functions.** These draw directly to a PostScript file specified by a global variable and should no longer be used. Use `plotexport` and friends instead.

```

void psdraw(GEN list, long flag)
GEN psplthraw(GEN listx, GEN listy, long flag)
GEN psplth(void *E, GEN(*f)(void*, GEN), GEN a, GEN b, long flags, long n, long
prec) draw to a PostScript file.

```

#### 18.0.8 Dump rectwindows to a PostScript or SVG file.

$w, x, y$  are three `t_VECSMALLs` indicating the rectwindows to dump, at which offsets. If  $T$  is `NULL`, rescale with respect to the installed graphic engine dimensions; else with respect to  $T$ .

```

char* rect2ps(GEN w, GEN x, GEN y, PARI_plot *T)
char* rect2ps_i(GEN w, GEN x, GEN y, PARI_plot *T, int plotps) if plotps is 0, as above;
else private version used to implement the plotps graphic engine (do not rescale, rotate to portrait
orientation).
char* rect2svg(GEN w, GEN x, GEN y, PARI_plot *T)

```

#### 18.0.9 Technical functions exported for convenience.

```

void pari_plot_by_file(const char *env, const char *suf, const char *img) backend
used by the plotps and plotsvg graphic engines.
void colorname_to_rgb(const char *s, int *r, int *g, int *b) convert an X11 colorname
to RGB values.
void color_to_rgb(GEN c, int *r, int *g, int *b) convert a pari color (t_VECSMALL RGB
triple or t_STR name) to RGB values.
void long_to_rgb(long c, int *r, int *g, int *b) split a standard hexadecimal color value
0xfdf5e6 to its rgb components (0xfd, 0xf5, 0xe6).

```

## Appendix A: A Sample program and Makefile

We assume that you have installed the PARI library and include files as explained in Appendix A or in the installation guide. If you chose differently any of the directory names, change them accordingly in the Makefiles.

If the program example that we have given is in the file `extgcd.c`, then a sample Makefile might look as follows. Note that the actual file `examples/Makefile` is more elaborate and you should have a look at it if you intend to use `install()` on custom made functions.

```
CC = cc
INCDIR = /usr/include
LIBDIR = /usr/lib
CFLAGS = -O -I$(INCDIR) -L$(LIBDIR)

all: extgcd

extgcd: extgcd.c
        $(CC) $(CFLAGS) -o extgcd extgcd.c -lpari -lm
```

We then give the listing of the program `examples/extgcd.c` seen in detail in Section 4.10.

```
#include <pari/pari.h>
/*
GP;install("extgcd", "GG&&", "gcdex", "./libextgcd.so");
*/

/* return d = gcd(a,b), sets u, v such that au + bv = gcd(a,b) */
GEN
extgcd(GEN A, GEN B, GEN *U, GEN *V)
{
    pari_sp av = avma;
    GEN ux = gen_1, vx = gen_0, a = A, b = B;

    if (typ(a) != t_INT) pari_err_TYPE("extgcd",a);
    if (typ(b) != t_INT) pari_err_TYPE("extgcd",b);
    if (signe(a) < 0) { a = negi(a); ux = negi(ux); }
    while (!gequal0(b))
    {
        GEN r, q = dvmdii(a, b, &r), v = vx;
        vx = subii(ux, mulii(q, vx));
        ux = v; a = b; b = r;
    }
    *U = ux;
    *V = diviiexact( subii(a, mulii(A,ux)), B );
    gerepileall(av, 3, &a, U, V); return a;
}

int
main()
```

```

{
    GEN x, y, d, u, v;
    pari_init(1000000,2);
    printf("x = "); x = gp_read_stream(stdin);
    printf("y = "); y = gp_read_stream(stdin);
    d = extgcd(x, y, &u, &v);
    pari_printf("gcd = %Ps\nu = %Ps\nv = %Ps\n", d, u, v);
    pari_close();
    return 0;
}

```



## Appendix B:

### PARI and threads

To use PARI in multi-threaded programs, you must configure it using `Configure --enable-tls`. Your system must implement the `__thread` storage class. As a major side effect, this breaks the `libpari` ABI: the resulting library is not compatible with the old one, and `-tls` is appended to the PARI library `soname`. On the other hand, this library is now thread-safe.

PARI provides some functions to set up PARI subthreads. In our model, each concurrent thread needs its own PARI stack. The following scheme is used:

Child thread:

```
void *child_thread(void *arg)
{
    GEN data = pari_thread_start((struct pari_thread*)arg);
    GEN result = ...; /* Compute result from data */
    pari_thread_close();
    return (void*)result;
}
```

Parent thread:

```
pthread_t th;
struct pari_thread pth;
GEN data, result;

pari_thread_alloc(&pth, s, data);
pari_thread_sync();
pthread_create(&th, NULL, &child_thread, (void*)&pth); /* start child */
... /* do stuff in parent */
pthread_join(th, (void*)&result); /* wait until child terminates */
result = gcopy(result); /* copy result from thread stack to main stack */
pari_thread_free(&pth); /* ... and clean up */
```

`void pari_thread_valloc(struct pari_thread *pth, size_t s, size_t v, GEN arg)` Allocate a PARI stack of size `s` which can grow to at most `v` (as with `parisize` and `parisizemax`) and associate it, together with the argument `arg`, with the PARI thread data `pth`.

`void pari_thread_alloc(struct pari_thread *pth, size_t s, GEN arg)` As above but the stack cannot grow beyond `s`.

`void pari_thread_free(struct pari_thread *pth)` Free the PARI stack attached to the PARI thread data `pth`. This is called after the child thread terminates, i.e. after `pthread_join` in the parent. Any `GEN` objects returned by the child in the thread stack need to be saved before running this command.

`void pari_thread_sync(void)` Record states from the main thread so that they are available to `pari_thread_start()`. Must be called in the main thread before the subthreads starts.

`void pari_thread_init(void)` Initialize the thread-local PARI data structures. This function is called by `pari_thread_start`.

`GEN pari_thread_start(struct pari_thread *t)` Initialize the thread-local PARI data structures and set up the thread stack using the PARI thread data `pth`. This function returns the thread argument `arg` that was given to `pari_thread_alloc`.

`void pari_thread_close(void)` Free the thread-local PARI data structures, but keeping the thread stack, so that a GEN returned by the thread remains valid.

Under this model, some PARI states are reset in new threads. In particular

- the random number generator is reset to the starting seed;
- the system stack exhaustion checking code, meant to catch infinite recursions, is disabled (use `pari_stackcheck_init()` to reenale it);
- cached real constants (returned by `mppi`, `mpeuler` and `mplog2`) are not shared between threads and will be recomputed as needed;

The following sample program can be compiled using

```
cc thread.c -o thread.o -lpari -lpthread
```

(Add `-I/-L` paths as necessary.)

```
#include <pari/pari.h> /* Include PARI headers */
#include <pthread.h> /* Include POSIX threads headers */

void *
mydet(void *arg)
{
    GEN F, M;
    /* Set up thread stack and get thread parameter */
    M = pari_thread_start((struct pari_thread*) arg);
    F = det(M);
    /* Free memory used by the thread */
    pari_thread_close();
    return (void*)F;
}

void *
myfactor(void *arg) /* same principle */
{
    GEN F, N;
    N = pari_thread_start((struct pari_thread*) arg);
    F = factor(N);
    pari_thread_close();
    return (void*)F;
}

int
main(void)
{
    GEN M,N1,N2, F1,F2,D;
    pthread_t th1, th2, th3; /* POSIX-thread variables */
    struct pari_thread pth1, pth2, pth3; /* pari thread variables */
    /* Initialise the main PARI stack and global objects (gen_0, etc.) */
```

```

pari_init(4000000,500000);
/* Compute in the main PARI stack */
N1 = addis(int2n(256), 1); /*  $2^{256} + 1$  */
N2 = subis(int2n(193), 1); /*  $2^{193} - 1$  */
M = mathilbert(80);
/* Sync with main thread */
pari_thread_sync();
/* Allocate pari thread structures */
pari_thread_alloc(&pth1,4000000,N1);
pari_thread_alloc(&pth2,4000000,N2);
pari_thread_alloc(&pth3,4000000,M);
/* pthread_create() and pthread_join() are standard POSIX-thread
   * functions to start and get the result of threads. */
pthread_create(&th1,NULL, &myfactor, (void*)&pth1);
pthread_create(&th2,NULL, &myfactor, (void*)&pth2);
pthread_create(&th3,NULL, &mydet, (void*)&pth3); /* Start 3 threads */
pthread_join(th1,(void*)&F1);
pthread_join(th2,(void*)&F2);
pthread_join(th3,(void*)&D); /* Wait for termination, get the results */
pari_printf("F1=%Ps\nF2=%Ps\nlog(D)=%Ps\n", F1, F2, glog(D,3));
pari_thread_free(&pth1);
pari_thread_free(&pth2);
pari_thread_free(&pth3); /* clean up */
return 0;
}

```

## Index

*SomeWord* refers to PARI-GP concepts.  
*SomeWord* is a PARI-GP keyword.  
*SomeWord* is a generic index entry.

### A

ABC_to_bnr . . . . .	291	addumului . . . . .	91
abelian_group . . . . .	235	adduu . . . . .	90
abgrp_get_cyc . . . . .	268	affc_fixlg . . . . .	232
abgrp_get_gen . . . . .	268	affects sign . . . . .	60
abgrp_get_no . . . . .	268	affects sign_safe . . . . .	60
abscmpii . . . . .	89	affgr . . . . .	82
abscmpiu . . . . .	89	affii . . . . .	82
abscmprr . . . . .	89	affir . . . . .	82
abscmpui . . . . .	89	affiz . . . . .	82
absdiviu_rem . . . . .	93	affrr . . . . .	82
absequalii . . . . .	89	affrr_fixlg . . . . .	82, 232
absequaliu . . . . .	89	affsi . . . . .	82
absequalui . . . . .	89	affsr . . . . .	82
absfrac . . . . .	223	affsz . . . . .	82
absfrac_shallow . . . . .	223	affui . . . . .	82
absi . . . . .	87	affur . . . . .	82
absi_shallow . . . . .	87	alarm . . . . .	247
absr . . . . .	87	algin it . . . . .	308
absrnz_equal1 . . . . .	89	alglat_get_primbasis . . . . .	309
absrnz_equal2n . . . . .	89	alglat_get_scalar . . . . .	309
abstorel . . . . .	292	algsimpledec_ss . . . . .	309
absZ_factor . . . . .	158	algtype . . . . .	308
absZ_factor_limit . . . . .	158	alg_complete . . . . .	308
addhelp . . . . .	74	alg_csa_table . . . . .	307
addii . . . . .	15	alg_cyclic . . . . .	307
addii_sign . . . . .	91	alg_get_absdim . . . . .	308
addir . . . . .	15	alg_get_abssplitting . . . . .	309
addir_sign . . . . .	91	alg_get_aut . . . . .	308
addis . . . . .	15	alg_get_auts . . . . .	308
addiu . . . . .	90	alg_get_b . . . . .	309
addll . . . . .	77	alg_get_basis . . . . .	309
addllx . . . . .	77	alg_get_center . . . . .	309
addmul . . . . .	77	alg_get_char . . . . .	309
addmulii . . . . .	91	alg_get_degree . . . . .	308
addmulii_inplace . . . . .	91	alg_get_dim . . . . .	308
addmuliu . . . . .	91	alg_get_hasse_f . . . . .	309
addmuliu_inplace . . . . .	91	alg_get_hasse_i . . . . .	309
addri . . . . .	15	alg_get_invbasis . . . . .	309
addr . . . . .	15	alg_get_multable . . . . .	309
addr_sign . . . . .	92	alg_get_relmultable . . . . .	309
addsi_sign . . . . .	92	alg_get_splitpol . . . . .	309
addui . . . . .	90	alg_get_splittingbasis . . . . .	309
addui_sign . . . . .	92	alg_get_splittingbasisinv . . . . .	309
		alg_get_splittingdata . . . . .	309
		alg_get_splittingfield . . . . .	309
		alg_get_tracebasis . . . . .	309
		alg_hasse . . . . .	307
		alg_hilbert . . . . .	308
		alg_matrix . . . . .	308

alg_model . . . . .	308
alg_type . . . . .	308
assignment . . . . .	26
avma . . . . .	17, 26

## B

bb_algebra . . . . .	195
bb_field . . . . .	193
bb_group . . . . .	191
bb_ring . . . . .	196
bernfrac . . . . .	233, 234
Bernoulli . . . . .	233, 234
bernreal . . . . .	233, 234
bezout . . . . .	46, 96
bfffo . . . . .	77
bid_get_arch . . . . .	271
bid_get_archp . . . . .	271
bid_get_cyc . . . . .	272
bid_get_fact . . . . .	271
bid_get_gen . . . . .	272
bid_get_gen_nocheck . . . . .	272
bid_get_grp . . . . .	271
bid_get_ideal . . . . .	271
bid_get_mod . . . . .	271
bid_get_no . . . . .	271
bid_get_sarch . . . . .	272
bid_get_sprk . . . . .	272
bid_get_U . . . . .	272
BIGDEFAULTPREC . . . . .	16, 62
bigomegau . . . . .	99
BIL . . . . .	49
binary_2k . . . . .	85, 86
binary_2k_nv . . . . .	86
binary_zv . . . . .	85
bincopy_relink . . . . .	65
binomial . . . . .	223
binomialuu . . . . .	223
bin_copy . . . . .	65
BITS_IN_HALFULONG . . . . .	62
BITS_IN_LONG . . . . .	16, 49, 62, 86
bits_to_int . . . . .	86
bits_to_u . . . . .	86
bit_accuracy . . . . .	16, 57
bit_accuracy_mul . . . . .	57
bit_prec . . . . .	57
bl_base . . . . .	69
bl_next . . . . .	69
bl_num . . . . .	69

bl_prev . . . . .	69
bl_refc . . . . .	69
bnfgwgeneric . . . . .	292
bnfisprincipal0 . . . . .	273, 286, 289
bnfisunit . . . . .	278
bnfnewprec . . . . .	273, 287
bnfnewprec_shallow . . . . .	273
bnfttestprimes . . . . .	288
bnf_build_cycgen . . . . .	272
bnf_build_matalpha . . . . .	272
bnf_build_units . . . . .	273
bnf_get_clgp . . . . .	270
bnf_get_cyc . . . . .	270
bnf_get_fu . . . . .	270
bnf_get_fu_nocheck . . . . .	270
bnf_get_gen . . . . .	270
bnf_get_logfu . . . . .	270
bnf_get_nf . . . . .	269
bnf_get_no . . . . .	270
bnf_get_reg . . . . .	270
bnf_get_tuN . . . . .	270
bnf_get_tuU . . . . .	270
bnrchar_primitive . . . . .	291
bnrclassno . . . . .	291
bnrconductor . . . . .	291
bnrconductor_i . . . . .	291
bnrdisc . . . . .	291
bnrdiscclist0 . . . . .	294
bnrisconductor . . . . .	291
bnrisprincipal . . . . .	294
bnrnewprec . . . . .	273
bnrnewprec_shallow . . . . .	273
bnrsurjection . . . . .	291
bnr_get_bid . . . . .	270
bnr_get_bnf . . . . .	270
bnr_get_clgp . . . . .	270
bnr_get_cyc . . . . .	270
bnr_get_gen . . . . .	270
bnr_get_gen_nocheck . . . . .	270
bnr_get_mod . . . . .	270
bnr_get_nf . . . . .	270
bnr_get_no . . . . .	270
both_odd . . . . .	78
boundfact . . . . .	158
BPSW_isprime . . . . .	163
BPSW_psp . . . . .	163
brent_kung_optpow . . . . .	195
brute . . . . .	242
buchimag . . . . .	294

Buchray . . . . .	291
buchreal . . . . .	294

## C

CATCH_ALL . . . . .	44
cbezout . . . . .	96
cbrtr . . . . .	231
cbrtr_abs . . . . .	231
cb_pari_ask_confirm . . . . .	53, 54
cb_pari_break_loop . . . . .	53
cb_pari_err_handle . . . . .	53
cb_pari_err_recover . . . . .	53
cb_pari_handle_exception . . . . .	53
cb_pari_init_histfile . . . . .	53
cb_pari_is_interactive . . . . .	53, 55
cb_pari_pre_recover . . . . .	53
cb_pari_quit . . . . .	53
cb_pari_sigint . . . . .	53
cb_pari_start_output . . . . .	53
cb_pari_whatnow . . . . .	53
ceildivuu . . . . .	93
ceilr . . . . .	84
ceil_safe . . . . .	84
centerlift . . . . .	201
centerlift0 . . . . .	201
centermod . . . . .	215
centermodii . . . . .	91
centermod_i . . . . .	215
cgcd . . . . .	95
cgetalloc . . . . .	65
cgetc . . . . .	24, 55, 64, 81, 232
cgetg . . . . .	24, 25, 55, 64
cgetg_block . . . . .	69
cgetg_copy . . . . .	55
cgeti . . . . .	24, 55, 64, 81
cgetineg . . . . .	81
cgetipos . . . . .	81
cgetp . . . . .	64
cgetr . . . . .	24, 55, 64, 81
cgetr_block . . . . .	69
cgiv . . . . .	18, 65
character string . . . . .	34
<i>character</i> . . . . .	305
characteristic . . . . .	225
charconj . . . . .	306
chardiv . . . . .	306
charmul . . . . .	306
chartoGENstr . . . . .	239

char_check . . . . .	306
char_denormalize . . . . .	306
char_normalize . . . . .	291, 306
char_simplify . . . . .	306
checkabgrp . . . . .	268
checkalg . . . . .	308
checkbid . . . . .	267
checkbid_i . . . . .	267
checkbnf . . . . .	267
checkbnf_i . . . . .	267
checkbnr . . . . .	267
checkbnrgen . . . . .	267
checkell . . . . .	311
checkell5 . . . . .	311
checkellisog . . . . .	312
checkellpt . . . . .	311
checkell_Fq . . . . .	312
checkell_i . . . . .	311
checkell_Q . . . . .	312
checkell_Qp . . . . .	312
checkgal . . . . .	267
checkgroup . . . . .	235
checkgroupelts . . . . .	235
checkhasse . . . . .	308
checklat . . . . .	308
checkMF . . . . .	329
checkMF_i . . . . .	329
checkmf_i . . . . .	329
checkmodpr . . . . .	268
checkms . . . . .	329
checkmspadi . . . . .	329
checknf . . . . .	267
checknfelt_mod . . . . .	268
checknf_i . . . . .	267
checkprid . . . . .	268
checkprid_i . . . . .	268
checkrnf . . . . .	267
checkrnf_i . . . . .	267
checksqmat . . . . .	268
checkznstar_i . . . . .	267
check_arith_all . . . . .	161
check_arith_non0 . . . . .	161
check_arith_pos . . . . .	161
check_modinv . . . . .	324
check_quaddisc . . . . .	295
check_quaddisc_imag . . . . .	295
check_quaddisc_real . . . . .	295
check_ZKmodule . . . . .	268
chinese1 . . . . .	146

chinese1_coprime_Z	146	cmp_universal	174, 210, 214
chk_gerepileupto	68	colorname_to_rgb	334
classno	296	colors	241, 242
classno2	296	color_to_rgb	334
clcm	96	coltrunc_init	56
cleanroots	181, 225	column vector	33
clean_Z_factor	161	col_ei	204
clone	68	compile_str	54, 260
clone	14, 27	complex number	31
CLONEBIT	62	compo	61
closemodinvertible	305	conjclasses_algcenter	309
closure	72	conjclasses_repr	236
closure	34	conjvec	225, 232
closure_arity	34	conj_i	223
closure_callgen1	72, 259	constant_coeff	32, 61
closure_callgen1prec	259	constcatalan	233
closure_callgen2	259	consteuler	233
closure_callgenall	259	constlog2	233
closure_callgenvec	259	constpi	233
closure_callgenvecprec	259	const_col	255
closure_callvoid1	259	const_F2v	113
closure_context	260	const_vec	255
closure_deriv	260	const_vecsmall	255
closure_disassemble	259	content	216
closure_err	261	conversions	27
closure_evalbrk	259	coprimes_zv	96
closure_evalgen	72, 259	copy	27
closure_evalnobrk	259	copybin_unlink	65
closure_evalres	259	copyifstack	68
closure_evalvoid	72, 259	copy_bin	65
closure_func_err	53	copy_bin_canon	65
closure_is_variadic	34	core	162
closure_trapgen	260	core2	162
cmpii	88	core2partial	162
cmpir	88	corediscs	100
cmpis	88	corepartial	162
cmpiu	88	coreu	100
cmpri	88	coreu_fact	100
cmprr	88	cornacchia	297
cmprs	88	cornacchia2	298
cmpsi	88	cornacchia2_sqrt	298
cmpsr	88	creation	24
cmpui	88	cvstop2	200
cmp_Flx	214	cvtop	200
cmp_nodata	214	cvtop2	200
cmp_padic	214	cxcompotor	199
cmp_prime_ideal	214	cxEk	232
cmp_prime_over_p	214	cxexpm1	223
cmp_RgX	214	cxnorm	223

cxredsl2 . . . . .	232
cxredsl2_i . . . . .	232
cxtofp . . . . .	199
cxtoreal . . . . .	200
cx_approx_equal . . . . .	223
cyc2elts . . . . .	306
cyclicgroup . . . . .	235
cyclic_perm . . . . .	234
cyc_normalize . . . . .	305
cyc_pow . . . . .	234
cyc_pow_perm . . . . .	234
c_ERR . . . . .	241
c_HELP . . . . .	242
c_HIST . . . . .	241
C_INCLUDE_PATH . . . . .	13
c_INPUT . . . . .	242
c_NONE . . . . .	242
c_OUTPUT . . . . .	242
c_PROMPT . . . . .	242
c_TIME . . . . .	242

## D

datadir . . . . .	76
dbgGEN . . . . .	40
dbg_gerepile . . . . .	68
dbg_gerepileupto . . . . .	68
dbg_pari_heap . . . . .	41
dblcoro526 . . . . .	328
dbllexpo . . . . .	199
dbllambertW0 . . . . .	328
dbllambertW_1 . . . . .	328
dbllemma526 . . . . .	328
dbllog2r . . . . .	199
dblmantissa . . . . .	199
dblmodulus . . . . .	199
dbltor . . . . .	27, 199
debug . . . . .	40
debugging . . . . .	40
DEBUGLEVEL . . . . .	40
DEBUGMEM . . . . .	40
debugmem . . . . .	40
default0 . . . . .	52
DEFAULTPREC . . . . .	16, 62
definite binary quadratic form . . . . .	33
deg1pol . . . . .	203
deg1pol_shallow . . . . .	208
deg1_from_roots . . . . .	209
deg2pol_shallow . . . . .	208

degpol . . . . .	32, 58, 225
degree . . . . .	32, 224
delete_var . . . . .	36, 70
denom_i . . . . .	216
deplin . . . . .	304
derivser . . . . .	226
destruction . . . . .	18
detint . . . . .	167
diagonal_shallow . . . . .	208
dicyclicgroup . . . . .	235
diffptr . . . . .	14
discrayabslist . . . . .	294
discrayabslistarch . . . . .	294
disc_best_modinv . . . . .	324
diviexact . . . . .	92
diviiround . . . . .	84
divisors . . . . .	42
divisorsu . . . . .	100
divisorsu_fact . . . . .	100
divis_rem . . . . .	93
diviuexact . . . . .	92
diviuuexact . . . . .	92
divll . . . . .	77
divll_pre . . . . .	79
divsBIL . . . . .	62
divsi_rem . . . . .	93
divss_rem . . . . .	93
dvdi . . . . .	92
dvdiiz . . . . .	92
dvdis . . . . .	92
dvdisz . . . . .	92
dvdiu . . . . .	92
dvdiuz . . . . .	92
dvdsi . . . . .	92
dvdui . . . . .	92
dvmdii . . . . .	92
dvmdiiz . . . . .	93
dvmdis . . . . .	93
dvmdsBIL . . . . .	62
dvmdsi . . . . .	93
dvmdss . . . . .	93
dvmduBIL . . . . .	62
dynamic array . . . . .	251
d_ACKNOWLEDGE . . . . .	261, 262, 264
d_INITRC . . . . .	261, 263
d_RETURN . . . . .	261, 263, 264
d_SILENT . . . . .	262

## E



echo . . . . .	55	ellQp_AGM . . . . .	314
ecpp . . . . .	163	ellQp_Ei . . . . .	233
ecppexport . . . . .	163	ellQp_get_p . . . . .	314
ecppisvalid . . . . .	163	ellQp_get_prec . . . . .	314
ec_2divpol_evalx . . . . .	316	ellQp_get_zero . . . . .	314
ec_3divpol_evalx . . . . .	316	ellQp_L . . . . .	314
ec_bmodel . . . . .	316	ellQp_q . . . . .	314
ec_dFdx_evalQ . . . . .	316	ellQp_root . . . . .	314
ec_dFdy_evalQ . . . . .	316	ellQp_Tate_uniformization . . . . .	314
ec_dmFdy_evalQ . . . . .	316	ellQp_u . . . . .	314
ec_f_evalx . . . . .	316	ellQp_u2 . . . . .	314
ec_half_deriv_2divpol_evalx . . . . .	316	ellQ_get_CM . . . . .	312, 319
ec_h_evalx . . . . .	316	ellQ_get_N . . . . .	312
effective length . . . . .	29	ellQ_get_Nfa . . . . .	312
ei_multable . . . . .	276	ellrootno_global . . . . .	313
ellff_get_o . . . . .	315	ellR_ab . . . . .	315
ell . . . . .	311	ellR_area . . . . .	315
ellanalyticrank . . . . .	328	ellR_eta . . . . .	315
ellanal_globalred . . . . .	313	ellR_get_prec . . . . .	315
ellanal_globalred_all . . . . .	313	ellR_get_sign . . . . .	316
ellap . . . . .	316	ellR_omega . . . . .	315
ellap_CM_fast . . . . .	312	ellR_roots . . . . .	315
ellbasechar . . . . .	316	elltors . . . . .	314
ellchangeinvert . . . . .	316	elltors0 . . . . .	313
elldatagenerators . . . . .	313	elltrace_extension . . . . .	317
ellff_get_a4a6 . . . . .	315	ell_get_a1 . . . . .	312
ellff_get_card . . . . .	314	ell_get_a2 . . . . .	312
ellff_get_D . . . . .	315	ell_get_a3 . . . . .	312
ellff_get_field . . . . .	314	ell_get_a4 . . . . .	312
ellff_get_gens . . . . .	314	ell_get_a6 . . . . .	312
ellff_get_group . . . . .	314	ell_get_b2 . . . . .	312
ellff_get_m . . . . .	314	ell_get_b4 . . . . .	312
ellff_get_o . . . . .	315	ell_get_b6 . . . . .	312
ellff_get_p . . . . .	314	ell_get_b8 . . . . .	312
ellheightoo . . . . .	313	ell_get_c4 . . . . .	312
ellinf . . . . .	316	ell_get_c6 . . . . .	312
ellinit . . . . .	311	ell_get_disc . . . . .	312
ellintegralmodel . . . . .	313	ell_get_j . . . . .	312
ellintegralmodel_i . . . . .	313	ell_get_type . . . . .	311, 317
ellisogeny . . . . .	312	ell_is_inf . . . . .	316
ellL1 . . . . .	328	ell_is_integral . . . . .	312
ellnf_get_bnf . . . . .	314	eltabstorel . . . . .	292
ellnf_get_nf . . . . .	314	eltabstorel_lift . . . . .	292
ellnf_vecarea . . . . .	314	eltreltoabs . . . . .	292
ellnf_veceta . . . . .	314	embednorm_T2 . . . . .	288
ellnf_vecomega . . . . .	314	embed_disc . . . . .	288
ellorder . . . . .	313	embed_norm . . . . .	288
ellorder_Q . . . . .	313	embed_roots . . . . .	288
ellQp_ab . . . . .	314	embed_T2 . . . . .	288



F2xqM_F2xC_gauss . . . . .	141	F2xq_ellgroup . . . . .	320
F2xqM_F2xC_invimage . . . . .	141	F2xq_elltwist . . . . .	321
F2xqM_F2xC_mul . . . . .	141	F2xq_inv . . . . .	140
F2xqM_gauss . . . . .	141	F2xq_invsafe . . . . .	140
F2xqM_image . . . . .	141	F2xq_log . . . . .	141
F2xqM_indexrank . . . . .	141	F2xq_matrix_pow . . . . .	141
F2xqM_inv . . . . .	141	F2xq_mul . . . . .	140
F2xqM_invimage . . . . .	141	F2xq_order . . . . .	141
F2xqM_ker . . . . .	142	F2xq_pow . . . . .	140
F2xqM_mul . . . . .	142	F2xq_powers . . . . .	141
F2xqM_rank . . . . .	142	F2xq_powu . . . . .	140
F2xqM_suppl . . . . .	142	F2xq_pow_init . . . . .	140
F2xqXQV_red . . . . .	144	F2xq_pow_table . . . . .	140
F2xqXQ_autpow . . . . .	144	F2xq_sqr . . . . .	140
F2xqXQ_auttrace . . . . .	144	F2xq_sqrt . . . . .	141
F2xqXQ_inv . . . . .	143	F2xq_sqrtn . . . . .	141
F2xqXQ_invsafe . . . . .	143	F2xq_sqrt_fast . . . . .	141
F2xqXQ_mul . . . . .	144	F2xq_trace . . . . .	140
F2xqXQ_pow . . . . .	144	F2xV_to_F2m . . . . .	157
F2xqXQ_powers . . . . .	144	F2xV_to_FlxV_inplace . . . . .	155
F2xqXQ_sqr . . . . .	144	F2xV_to_ZXV_inplace . . . . .	155
F2xqX_ddf . . . . .	144	F2xXC_to_ZXXC . . . . .	142
F2xqX_degfact . . . . .	144	F2xX_add . . . . .	142
F2xqX_div . . . . .	143	F2xX_deriv . . . . .	142
F2xqX_divrem . . . . .	143	F2xX_F2x_add . . . . .	142
F2xqX_extgcd . . . . .	143	F2xX_F2x_mul . . . . .	142
F2xqX_F2xXQV_eval . . . . .	143	F2xX_renormalize . . . . .	142
F2xqX_F2xXQ_eval . . . . .	143	F2xX_to_Kronecker . . . . .	142
F2xqX_F2xq_mul . . . . .	143	F2xX_to_ZXX . . . . .	142
F2xqX_F2xq_mul_to_monic . . . . .	143	F2xY_degreex . . . . .	142
F2xqX_factor . . . . .	143	F2xY_F2xqV_evalx . . . . .	142
F2xqX_factor_squarefree . . . . .	143	F2xY_F2xq_evalx . . . . .	142
F2xqX_gcd . . . . .	143	F2x_1_add . . . . .	139
F2xqX_get_red . . . . .	143	F2x_add . . . . .	139
F2xqX_invBarrett . . . . .	143	F2x_clear . . . . .	139
F2xqX_isplayer . . . . .	143	F2x_coeff . . . . .	139
F2xqX_mul . . . . .	143	F2x_copy . . . . .	139
F2xqX_normalize . . . . .	143	F2x_ddf . . . . .	140
F2xqX_powu . . . . .	143	F2x_deflate . . . . .	140
F2xqX_red . . . . .	143	F2x_degfact . . . . .	140
F2xqX_rem . . . . .	143	F2x_degree . . . . .	139
F2xqX_roots . . . . .	143	F2x_deriv . . . . .	140
F2xqX_sqr . . . . .	143	F2x_div . . . . .	140
F2xq_Artin_Schreier . . . . .	141	F2x_divrem . . . . .	139
F2xq_autpow . . . . .	141	F2x_equal . . . . .	139
F2xq_conjvec . . . . .	141	F2x_equal1 . . . . .	139
F2xq_div . . . . .	140	F2x_eval . . . . .	140
F2xq_ellcard . . . . .	320	F2x_even_odd . . . . .	140
F2xq_ellgens . . . . .	320	F2x_extgcd . . . . .	140

F2x_F2xqV_eval . . . . .	141	famat_pow . . . . .	277
F2x_F2xq_eval . . . . .	141	famat_pows_shallow . . . . .	277
F2x_factor . . . . .	140	famat_pow_shallow . . . . .	277
F2x_factor_squarefree . . . . .	140	famat_reduce . . . . .	277, 278
F2x_flip . . . . .	139	famat_sqr . . . . .	277
F2x_Frobenius . . . . .	140	famat_to_nf . . . . .	278
F2x_gcd . . . . .	140	famat_to_nf_moddivisor . . . . .	290
F2x_get_red . . . . .	138	famat_to_nf_modideal_coprime . . . . .	290
F2x_halfgcd . . . . .	140	famat_Z_gcd . . . . .	277
F2x_issquare . . . . .	140	fetch_user_var . . . . .	35, 70
F2x_is_irred . . . . .	140	fetch_var . . . . .	36, 70
F2x_matFrobenius . . . . .	140	fetch_var_higher . . . . .	36
F2x_mul . . . . .	139	fetch_var_value . . . . .	36, 70
F2x_rem . . . . .	140	FFM_deplin . . . . .	230
F2x_renormalize . . . . .	140	FFM_det . . . . .	230
F2x_set . . . . .	139	FFM_FFC_gauss . . . . .	229
F2x_shift . . . . .	140	FFM_FFC_invimage . . . . .	229
F2x_sqr . . . . .	139	FFM_FFC_mul . . . . .	229
F2x_sqrt . . . . .	140	FFM_gauss . . . . .	230
F2x_to_F2v . . . . .	157	FFM_image . . . . .	230
F2x_to_F2xX . . . . .	139	FFM_indexrank . . . . .	230
F2x_to_Flx . . . . .	139	FFM_inv . . . . .	230
F2x_to_ZX . . . . .	139	FFM_invimage . . . . .	230
F2x_valrem . . . . .	140	FFM_ker . . . . .	230
factmod . . . . .	145	FFM_mul . . . . .	230
factor . . . . .	311	FFM_rank . . . . .	230
factorback . . . . .	220	FFM_suppl . . . . .	230
factoredpolred . . . . .	294	FFXQ_inv . . . . .	230
factoredpolred2 . . . . .	294	FFXQ_mul . . . . .	230
factorial_lval . . . . .	87	FFXQ_sqr . . . . .	230
factorint . . . . .	160	FFX_ddf . . . . .	229
factoru . . . . .	159	FFX_degfact . . . . .	229
factoru_pow . . . . .	159	FFX_factor . . . . .	229
factor_Aurifeuille . . . . .	159	FFX_factor_squarefree . . . . .	229
factor_Aurifeuille_prime . . . . .	159	FFX_isplayer . . . . .	229
factor_pn_1 . . . . .	158	FFX_mul . . . . .	229
factor_pn_1_limit . . . . .	159	FFX_preimage . . . . .	229
factor_proven . . . . .	162	FFX_rem . . . . .	229
<i>famat</i> . . . . .	277	FFX_roots . . . . .	229
famat_small_reduce . . . . .	278	FFX_sqr . . . . .	229
famat_div_shallow . . . . .	277	FF_1 . . . . .	227
famat_inv . . . . .	277	FF_add . . . . .	227
famat_inv_shallow . . . . .	277	FF_charpoly . . . . .	228
famat_makecoprime . . . . .	290	FF_conjvec . . . . .	228
famat_mul . . . . .	277	FF_div . . . . .	227
famat_mulpow . . . . .	277	FF_ellcard . . . . .	317
famat_mulpows_shallow . . . . .	277	FF_ellcard_SEA . . . . .	317
famat_mulpow_shallow . . . . .	277	FF_ellgens . . . . .	317
famat_mul_shallow . . . . .	277	FF_ellgroup . . . . .	317

FF_elllog . . . . .	317	FF_Z_Z_muldiv . . . . .	228
FF_ellmul . . . . .	317	file_is_binary . . . . .	242
FF_ellorder . . . . .	317	finite field element . . . . .	31
FF_ellrandom . . . . .	317	fixlg . . . . .	67, 83
FF_elltatepairing . . . . .	317	Flc_Flv_mul . . . . .	110
FF_elltwist . . . . .	317	Flc_FpV_mul . . . . .	111
FF_ellweilpairing . . . . .	317	Flc_lincomb1_inplace . . . . .	110
FF_equal . . . . .	227	Flc_to_mod . . . . .	144
FF_equal0 . . . . .	227	Flc_to_ZC . . . . .	155
FF_equal1 . . . . .	227	Flc_to_ZC_inplace . . . . .	155
FF_equalitym1 . . . . .	227	Fle_add . . . . .	319
FF_f . . . . .	226	Fle_changepoint . . . . .	320
FF_Frobenius . . . . .	228	Fle_changepointinv . . . . .	320
FF_gen . . . . .	226	Fle_dbl . . . . .	319
FF_inv . . . . .	228	Fle_log . . . . .	319
FF_ispower . . . . .	228	Fle_mul . . . . .	319
FF_issquare . . . . .	228	Fle_mulu . . . . .	319
FF_issquareall . . . . .	228	Fle_order . . . . .	319
FF_log . . . . .	228	Fle_sub . . . . .	319
FF_map . . . . .	228	Fle_to_Flj . . . . .	320
FF_minpoly . . . . .	228	Flj_add_pre . . . . .	320
FF_mod . . . . .	226	Flj_dbl_pre . . . . .	320
FF_mul . . . . .	227	Flj_mulu_pre . . . . .	320
FF_mul2n . . . . .	228	Flj_neg . . . . .	320
FF_neg . . . . .	228	Flj_to_Fle_pre . . . . .	320
FF_neg_i . . . . .	228	Flm_add . . . . .	111
FF_norm . . . . .	228	Flm_adjoint . . . . .	111
FF_order . . . . .	228	Flm_center . . . . .	109
FF_p . . . . .	226	Flm_charpoly . . . . .	111
FF_pow . . . . .	228	Flm_copy . . . . .	109
FF_primroot . . . . .	228	Flm_deplin . . . . .	111
FF_p_i . . . . .	226	Flm_det . . . . .	111
FF_q . . . . .	226	Flm_det_sp . . . . .	111
FF_Q_add . . . . .	227	Flm_Flc_gauss . . . . .	111
FF_samefield . . . . .	227	Flm_Flc_invimage . . . . .	111
FF_sqr . . . . .	228	Flm_Flc_mul . . . . .	110
FF_sqrt . . . . .	228	Flm_Flc_mul_pre . . . . .	110
FF_sqrtn . . . . .	228	Flm_Flc_mul_pre_Flx . . . . .	110
FF_sub . . . . .	227	Flm_Fl_add . . . . .	110
FF_to_F2xq . . . . .	226	Flm_Fl_mul . . . . .	110
FF_to_F2xq_i . . . . .	226	Flm_Fl_mul_inplace . . . . .	110
FF_to_Flxq . . . . .	226	Flm_gauss . . . . .	111
FF_to_Flxq_i . . . . .	226	Flm_hess . . . . .	112
FF_to_FpXQ . . . . .	226	Flm_image . . . . .	112
FF_to_FpXQ_i . . . . .	226	Flm_indexrank . . . . .	111
FF_trace . . . . .	228	Flm_intersect . . . . .	112
FF_zero . . . . .	227	Flm_inv . . . . .	111
FF_Z_add . . . . .	227	Flm_invimage . . . . .	111
FF_Z_mul . . . . .	227	Flm_ker . . . . .	112

Flm_ker_sp . . . . .	112	FlxC_neg . . . . .	133
Flm_mul . . . . .	111	FlxC_sub . . . . .	133
Flm_neg . . . . .	110	FlxC_to_F2xC . . . . .	157
Flm_powers . . . . .	111	FlxC_to_ZXC . . . . .	155
Flm_powu . . . . .	111	FlxM_eval_powers_pre . . . . .	133
Flm_rank . . . . .	112	FlxM_Flx_add_shallow . . . . .	113
Flm_row . . . . .	111	FlxM_neg . . . . .	133
Flm_sub . . . . .	111	FlxM_sub . . . . .	133
Flm_suppl . . . . .	112	FlxM_to_ZXM . . . . .	155
Flm_to_F2m . . . . .	113	Flxn_inv . . . . .	130
Flm_to_FlxV . . . . .	156	Flxn_mul . . . . .	130
Flm_to_FlxX . . . . .	156	FlxqC_Flxq_mul . . . . .	114
Flm_to_mod . . . . .	144	FlxqE_add . . . . .	322
Flm_to_ZM . . . . .	155	FlxqE_changepoint . . . . .	322
Flm_to_ZM_inplace . . . . .	155	FlxqE_changepointinv . . . . .	322
Flm_transpose . . . . .	112	FlxqE_dbl . . . . .	322
floorr . . . . .	84	FlxqE_log . . . . .	322
floor_safe . . . . .	84	FlxqE_mul . . . . .	322
flush . . . . .	240	FlxqE_neg . . . . .	322
Flv_add . . . . .	110	FlxqE_order . . . . .	322
Flv_add_inplace . . . . .	110, 284	FlxqE_sub . . . . .	322
Flv_center . . . . .	109	FlxqE_tatepairing . . . . .	322
Flv_copy . . . . .	109	FlxqE_weilpairing . . . . .	322
Flv_dotproduct . . . . .	110	FlxqM_deplin . . . . .	114
Flv_dotproduct_pre . . . . .	110	FlxqM_det . . . . .	114
Flv_Flm_polint . . . . .	133	FlxqM_FlxqC_gauss . . . . .	114
Flv_Fl_div . . . . .	110	FlxqM_FlxqC_invimage . . . . .	114
Flv_Fl_div_inplace . . . . .	110	FlxqM_FlxqC_mul . . . . .	114
Flv_Fl_mul . . . . .	110	FlxqM_Flxq_mul . . . . .	114
Flv_Fl_mul_inplace . . . . .	110	FlxqM_gauss . . . . .	114
Flv_Fl_mul_part_inplace . . . . .	110	FlxqM_image . . . . .	114
Flv_inv . . . . .	111	FlxqM_indexrank . . . . .	114
Flv_invVandermonde . . . . .	133	FlxqM_inv . . . . .	114
Flv_inv_inplace . . . . .	111	FlxqM_invimage . . . . .	114
Flv_inv_pre . . . . .	111	FlxqM_ker . . . . .	114
Flv_inv_pre_inplace . . . . .	111	FlxqM_mul . . . . .	114
Flv_neg . . . . .	110	FlxqM_rank . . . . .	114
Flv_neg_inplace . . . . .	110	FlxqM_suppl . . . . .	114
Flv_polint . . . . .	133	FlxqV_dotproduct . . . . .	113
Flv_prod . . . . .	110	FlxqV_roots_to_pol . . . . .	134
Flv_prod_pre . . . . .	111	FlxqXQ_autpow . . . . .	138
Flv_roots_to_pol . . . . .	133	FlxqXQ_autsum . . . . .	138
Flv_sub . . . . .	110	FlxqXQ_auttrace . . . . .	138
Flv_sub_inplace . . . . .	110	FlxqXQ_div . . . . .	138
Flv_sum . . . . .	110	FlxqXQ_halfFrobenius . . . . .	138
Flv_to_F2v . . . . .	113	FlxqXQ_inv . . . . .	138, 144
Flv_to_Flx . . . . .	156	FlxqXQ_invsafe . . . . .	138, 144
Flv_to_ZV . . . . .	155	FlxqXQ_matrix_pow . . . . .	138
FlxC_eval_powers_pre . . . . .	133	FlxqXQ_minpoly . . . . .	138

FlxqXQ_mul . . . . .	138	Flxq_elltwist . . . . .	322
FlxqXQ_pow . . . . .	138	Flxq_ffisom_inv . . . . .	134
FlxqXQ_powers . . . . .	138	Flxq_inv . . . . .	134
FlxqXQ_powu . . . . .	138	Flxq_invsafe . . . . .	134
FlxqXQ_sqr . . . . .	138	Flxq_is2npower . . . . .	134
FlxqXV_prod . . . . .	137	Flxq_issquare . . . . .	134
FlxqX_ddf . . . . .	138	Flxq_log . . . . .	134
FlxqX_ddf_degree . . . . .	138	Flxq_lroot . . . . .	135
FlxqX_degfact . . . . .	138	Flxq_lroot_fast . . . . .	135
FlxqX_div . . . . .	137	Flxq_matrix_pow . . . . .	134
FlxqX_divrem . . . . .	137	Flxq_minpoly . . . . .	135
FlxqX_dotproduct . . . . .	137	Flxq_mul . . . . .	134
FlxqX_extgcd . . . . .	137	Flxq_norm . . . . .	135
FlxqX_factor . . . . .	137	Flxq_order . . . . .	134
FlxqX_factor_squarefree . . . . .	138	Flxq_pow . . . . .	134
FlxqX_FlxqXQV_eval . . . . .	138	Flxq_powers . . . . .	134
FlxqX_FlxqXQ_eval . . . . .	138	Flxq_powu . . . . .	134
FlxqX_Flxq_mul . . . . .	137	Flxq_pow_init . . . . .	134
FlxqX_Flxq_mul_to_monic . . . . .	137	Flxq_pow_table . . . . .	134
FlxqX_Frobenius . . . . .	137	Flxq_sqr . . . . .	134
FlxqX_gcd . . . . .	137	Flxq_sqrt . . . . .	134
FlxqX_get_red . . . . .	136	Flxq_sqrtn . . . . .	134
FlxqX_halfgcd . . . . .	137	Flxq_sub . . . . .	134
FlxqX_invBarrett . . . . .	137	Flxq_trace . . . . .	135
FlxqX_ispower . . . . .	137	FlxT_red . . . . .	134
FlxqX_is_squarefree . . . . .	137	FlxV_Flc_mul . . . . .	133
FlxqX_mul . . . . .	137	FlxV_Flx_fromdigits . . . . .	130
FlxqX_nbfact . . . . .	138	FlxV_prod . . . . .	133
FlxqX_nbfact_Frobenius . . . . .	138	FlxV_red . . . . .	133
FlxqX_nbroots . . . . .	138	FlxV_to_Flm . . . . .	156
FlxqX_normalize . . . . .	137	FlxV_to_ZXV . . . . .	155
FlxqX_powu . . . . .	137	FlxV_to_ZXV_inplace . . . . .	155
FlxqX_red . . . . .	137	FlxCX_to_F2xCX . . . . .	142
FlxqX_rem . . . . .	137	FlxCX_to_ZXXC . . . . .	155
FlxqX_roots . . . . .	137	FlxXM_to_ZXXM . . . . .	155
FlxqX_safegcd . . . . .	137	FlxXV_to_FlxM . . . . .	157
FlxqX_sqr . . . . .	137	FlxX_add . . . . .	135
Flxq_add . . . . .	134	FlxX_deriv . . . . .	135
Flxq_autpow . . . . .	134	FlxX_double . . . . .	135
Flxq_autsum . . . . .	134	FlxX_Flx_add . . . . .	135
Flxq_auttrace . . . . .	134	FlxX_Flx_mul . . . . .	135
Flxq_charpoly . . . . .	135	FlxX_Flx_sub . . . . .	135
Flxq_conjvec . . . . .	135	FlxX_Fl_mul . . . . .	135
Flxq_div . . . . .	134	FlxX_neg . . . . .	135
Flxq_ellcard . . . . .	322	FlxX_renormalize . . . . .	136
Flxq_ellgens . . . . .	322	FlxX_resultant . . . . .	136
Flxq_ellgroup . . . . .	322	FlxX_shift . . . . .	136
Flxq_ellj . . . . .	321	FlxX_sub . . . . .	135
Flxq_ellj_to_a4a6 . . . . .	321	FlxX_swap . . . . .	136

FlxX_to_F2xX . . . . .	142	Flx_gcd . . . . .	131
FlxX_to_Flm . . . . .	156	Flx_get_red . . . . .	129
FlxX_to_Flx . . . . .	156	Flx_halfgcd . . . . .	131
FlxX_to_FlxC . . . . .	156	Flx_halve . . . . .	130
FlxX_to_ZXX . . . . .	155	Flx_inflate . . . . .	132
FlxX_triple . . . . .	135	Flx_invBarrett . . . . .	132
FlxYqq_pow . . . . .	136	Flx_ispower . . . . .	131
FlxY_degreeex . . . . .	135	Flx_is_irred . . . . .	132
FlxY_evalx . . . . .	135	Flx_is_smooth . . . . .	132
FlxY_evalx_powers_pre . . . . .	135	Flx_is_squarefree . . . . .	132
FlxY_eval_powers_pre . . . . .	136	Flx_lead . . . . .	130
FlxY_FlxqV_evalx . . . . .	136	Flx_matFrobenius . . . . .	131
FlxY_Flxq_evalx . . . . .	136	Flx_mod_Xn1 . . . . .	131
FlxY_Flx_div . . . . .	135	Flx_mod_Xnm1 . . . . .	131
FlxY_Flx_translate . . . . .	135	Flx_mul . . . . .	130
Flx_add . . . . .	130	Flx_mulu . . . . .	130
Flx_constant . . . . .	130	Flx_nbfact . . . . .	133
Flx_copy . . . . .	130	Flx_nbfact_by_degree . . . . .	133
Flx_ddf . . . . .	131	Flx_nbfact_Frobenius . . . . .	133
Flx_deflate . . . . .	132	Flx_nbroots . . . . .	133
Flx_degfact . . . . .	131, 133	Flx_neg . . . . .	130
Flx_deriv . . . . .	130	Flx_neg_inplace . . . . .	130
Flx_diff1 . . . . .	130	Flx_normalize . . . . .	131
Flx_digits . . . . .	130	Flx_oneroot . . . . .	131
Flx_div . . . . .	130	Flx_oneroot_split . . . . .	131
Flx_divrem . . . . .	130	Flx_powu . . . . .	130
Flx_div_by_X_x . . . . .	132	Flx_recip . . . . .	132
Flx_dotproduct . . . . .	132	Flx_red . . . . .	130
Flx_double . . . . .	130	Flx_rem . . . . .	130
Flx_equal . . . . .	130	Flx_renormalize . . . . .	132
Flx_equal1 . . . . .	130	Flx_rescale . . . . .	131
Flx_eval . . . . .	132	Flx_resultant . . . . .	132
Flx_eval_powers_pre . . . . .	132	Flx_roots . . . . .	131
Flx_eval_pre . . . . .	132	Flx_rootsff . . . . .	131
Flx_extgcd . . . . .	131	Flx_shift . . . . .	132
Flx_extresultant . . . . .	132	Flx_splitting . . . . .	132
Flx_factor . . . . .	131	Flx_sqr . . . . .	130
Flx_factorff_irred . . . . .	131	Flx_sub . . . . .	130
Flx_factor_squarefree . . . . .	131	Flx_to_F2x . . . . .	139
Flx_ffintersect . . . . .	133	Flx_to_Flv . . . . .	156
Flx_ffisom . . . . .	131	Flx_to_FlxX . . . . .	155
Flx_Flv_multieval . . . . .	132	Flx_to_ZX . . . . .	155
Flx_FlxqV_eval . . . . .	134	Flx_to_ZX_inplace . . . . .	155
Flx_Flxq_eval . . . . .	134	Flx_translate1 . . . . .	130
Flx_FlxY_resultant . . . . .	136	Flx_triple . . . . .	130
Flx_Fl_add . . . . .	130	Flx_val . . . . .	132
Flx_Fl_mul . . . . .	130	Flx_valrem . . . . .	132
Flx_Fl_mul_to_monic . . . . .	130	Flx_to_FlxY . . . . .	157
Flx_Frobenius . . . . .	131	Fl_2gener_pre . . . . .	80



Fl_add . . . . .	78	format . . . . .	40
Fl_addmulmul_pre . . . . .	80	forpart . . . . .	42
Fl_addmul_pre . . . . .	79	forpart_init . . . . .	42
Fl_center . . . . .	78	forpart_next . . . . .	42
Fl_div . . . . .	79	forpart_prev . . . . .	42
Fl_double . . . . .	78	forpart_t . . . . .	42
Fl_elldisc . . . . .	319	forperm . . . . .	43
Fl_elldisc_pre . . . . .	319	forperm_init . . . . .	43
Fl_ellj . . . . .	319	forperm_next . . . . .	43
Fl_ellj_pre . . . . .	319	forprime . . . . .	42
Fl_ellj_to_a4a6 . . . . .	319	forprimestep_init . . . . .	163
Fl_elltrace . . . . .	319	forprime_init . . . . .	43, 44, 163, 164
Fl_elltrace_CM . . . . .	319	forprime_next . . . . .	43, 163
Fl_elltwist . . . . .	319	forprime_t . . . . .	43, 44
Fl_elltwist_disc . . . . .	319	forsubgroup . . . . .	42
Fl_half . . . . .	78	forsubgroup(H = G, B,) . . . . .	42
Fl_inv . . . . .	78	forsubset . . . . .	43
Fl_invgen . . . . .	78	forsubset_init . . . . .	43
Fl_invsafe . . . . .	78	forsubset_next . . . . .	43
Fl_log . . . . .	79	forvec . . . . .	42
Fl_log_pre . . . . .	80	forvec_init . . . . .	42
Fl_mul . . . . .	78	forvec_next . . . . .	42
Fl_mul_pre . . . . .	79	FpC_add . . . . .	107
Fl_neg . . . . .	78	FpC_center . . . . .	107
Fl_order . . . . .	79	FpC_center_inplace . . . . .	107
Fl_powers . . . . .	79	FpC_FpV_mul . . . . .	108
Fl_powers_pre . . . . .	80	FpC_Fp_mul . . . . .	108
Fl_powu . . . . .	79	FpC_ratlift . . . . .	149
Fl_powu_pre . . . . .	80	FpC_red . . . . .	107
Fl_sqr . . . . .	78	FpC_sub . . . . .	107
Fl_sqrt . . . . .	79	FpC_to_mod . . . . .	144
Fl_sqrt1 . . . . .	79	FpE_add . . . . .	318
Fl_sqrt1_pre . . . . .	80	FpE_changepoint . . . . .	318
Fl_sqrtn . . . . .	79	FpE_changepointinv . . . . .	318
Fl_sqrtn_pre . . . . .	80	FpE_dbl . . . . .	318
Fl_sqrt_pre . . . . .	80	FpE_log . . . . .	319
Fl_sqrt_pre_i . . . . .	80	FpE_mul . . . . .	318
Fl_sqr_pre . . . . .	79	FpE_neg . . . . .	318
Fl_sub . . . . .	78	FpE_order . . . . .	318
Fl_to_Flx . . . . .	156	FpE_sub . . . . .	318
Fl_triple . . . . .	78	FpE_tatepairing . . . . .	319
forallsubset_init . . . . .	43	FpE_to_FpJ . . . . .	320
forcomposite . . . . .	42	FpE_to_mod . . . . .	319
forcomposite_init . . . . .	42	FpE_weilpairing . . . . .	319
forcomposite_next . . . . .	42	FpJ_add . . . . .	320
fordiv . . . . .	42	FpJ_dbl . . . . .	320
forell . . . . .	42	FpJ_mul . . . . .	320
forell(ell,a,b,) . . . . .	42	FpJ_neg . . . . .	320
forksubset_init . . . . .	43	FpJ_to_FpE . . . . .	320

FpMs_FpCs_solve . . . . .	175	FpXM_FpXQV_eval . . . . .	122
FpMs_FpCs_solve_safe . . . . .	175	FpXn_exp . . . . .	122
FpMs_FpC_mul . . . . .	175	FpXn_inv . . . . .	122
FpMs_leftkernel_elt . . . . .	175	FpXn_mul . . . . .	122
FpM_add . . . . .	107	FpXn_sqr . . . . .	122
FpM_center . . . . .	107	FpXQC_to_mod . . . . .	144
FpM_center_inplace . . . . .	107	FpXQE_add . . . . .	323
FpM_charpoly . . . . .	109	FpXQE_changepoint . . . . .	323
FpM_deplin . . . . .	108	FpXQE_changepointinv . . . . .	323
FpM_det . . . . .	108	FpXQE_dbl . . . . .	323
FpM_FpC_gauss . . . . .	108	FpXQE_log . . . . .	323
FpM_FpC_invimage . . . . .	108	FpXQE_mul . . . . .	323
FpM_FpC_mul . . . . .	108	FpXQE_neg . . . . .	323
FpM_FpC_mul_FpX . . . . .	108	FpXQE_order . . . . .	323
FpM_Fp_mul . . . . .	108	FpXQE_sub . . . . .	323
FpM_gauss . . . . .	108	FpXQE_tatepairing . . . . .	323
FpM_hess . . . . .	109	FpXQE_weilpairing . . . . .	323
FpM_image . . . . .	108	FpXQM_autsum . . . . .	122
FpM_indexrank . . . . .	109	FpXQXn_exp . . . . .	126
FpM_intersect . . . . .	108	FpXQXn_inv . . . . .	126
FpM_inv . . . . .	108	FpXQXn_mul . . . . .	126
FpM_invimage . . . . .	108	FpXQXn_sqr . . . . .	125
FpM_ker . . . . .	108	FpXQXQ_autpow . . . . .	127
FpM_mul . . . . .	108	FpXQXQ_autsum . . . . .	127
FpM_powu . . . . .	108	FpXQXQ_auttrace . . . . .	127
FpM_rank . . . . .	108	FpXQXQ_div . . . . .	126
FpM_ratlift . . . . .	149	FpXQXQ_halfFrobenius . . . . .	127
FpM_red . . . . .	107	FpXQXQ_inv . . . . .	126
FpM_sub . . . . .	107	FpXQXQ_invsafe . . . . .	126
FpM_suppl . . . . .	109	FpXQXQ_matrix_pow . . . . .	127
FpM_to_mod . . . . .	144	FpXQXQ_minpoly . . . . .	127
FpVV_to_mod . . . . .	144	FpXQXQ_mul . . . . .	126
FpV_add . . . . .	107	FpXQXQ_pow . . . . .	126
FpV_dotproduct . . . . .	108	FpXQXQ_powers . . . . .	127
FpV_dotsquare . . . . .	108	FpXQXQ_sqr . . . . .	126
FpV_factorback . . . . .	108	FpXQXV_FpXQX_fromdigits . . . . .	125
FpV_FpC_mul . . . . .	108	FpXQXV_prod . . . . .	125
FpV_FpMs_mul . . . . .	175	FpXQX_ddf . . . . .	128
FpV_FpM_polint . . . . .	117	FpXQX_ddf_degree . . . . .	128
FpV_inv . . . . .	97	FpXQX_degfact . . . . .	128
FpV_invVandermonde . . . . .	117	FpXQX_digits . . . . .	125
FpV_polint . . . . .	117	FpXQX_div . . . . .	125
FpV_red . . . . .	107	FpXQX_divrem . . . . .	125
FpV_roots_to_pol . . . . .	116	FpXQX_div_by_X_x . . . . .	125
FpV_sub . . . . .	107	FpXQX_dotproduct . . . . .	125
FpV_to_mod . . . . .	144	FpXQX_extgcd . . . . .	125
FpXC_center . . . . .	122	FpXQX_factor . . . . .	128
FpXC_FpXQV_eval . . . . .	122	FpXQX_factor_squarefree . . . . .	128
FpXM_center . . . . .	122	FpXQX_FpXQXQV_eval . . . . .	126

FpXQX_FpXQXQ_eval . . . . .	125	FpXQ_red . . . . .	119
FpXQX_FpXQ_mul . . . . .	125	FpXQ_sqr . . . . .	120
FpXQX_Frobenius . . . . .	129	FpXQ_sqrt . . . . .	120
FpXQX_gcd . . . . .	125	FpXQ_sqrtn . . . . .	120, 121, 134
FpXQX_get_red . . . . .	126	FpXQ_sub . . . . .	119
FpXQX_halfgcd . . . . .	125	FpXQ_trace . . . . .	121
FpXQX_invBarrett . . . . .	125	FpXT_red . . . . .	114
FpXQX_ispower . . . . .	128	FpXV_FpC_mul . . . . .	116
FpXQX_mul . . . . .	125	FpXV_FpX_fromdigits . . . . .	115
FpXQX_nbfact . . . . .	129	FpXV_prod . . . . .	116
FpXQX_nbfact_Frobenius . . . . .	129	FpXV_red . . . . .	114
FpXQX_nbroots . . . . .	129	FpXX_add . . . . .	122
FpXQX_normalize . . . . .	124	FpXX_deriv . . . . .	123
FpXQX_powu . . . . .	125	FpXX_FpX_mul . . . . .	122
FpXQX_red . . . . .	125	FpXX_Fp_mul . . . . .	122
FpXQX_rem . . . . .	125	FpXX_half . . . . .	123
FpXQX_renormalize . . . . .	125	FpXX_integ . . . . .	123
FpXQX_roots . . . . .	128	FpXX_mulu . . . . .	122
FpXQX_split_part . . . . .	128	FpXX_neg . . . . .	122
FpXQX_sqr . . . . .	125	FpXX_red . . . . .	122
FpXQX_to_mod . . . . .	125	FpXX_renormalize . . . . .	122
FpXQ_add . . . . .	119	FpXX_sub . . . . .	122
FpXQ_autpow . . . . .	121	FpXYQQ_pow . . . . .	123
FpXQ_autpowers . . . . .	122	FpXY_eval . . . . .	123
FpXQ_autsum . . . . .	121	FpXY_evalx . . . . .	123
FpXQ_auttrace . . . . .	121	FpXY_evaly . . . . .	123
FpXQ_charpoly . . . . .	121	FpXY_FpXQV_evalx . . . . .	123
FpXQ_conjvec . . . . .	121	FpXY_FpXQ_evalx . . . . .	123
FpXQ_div . . . . .	120	FpXY_Fq_evaly . . . . .	123
FpXQ_ellcard . . . . .	323	FpX_add . . . . .	114
FpXQ_elldivpol . . . . .	323	FpX_center . . . . .	115, 116
FpXQ_ellgens . . . . .	323	FpX_center_i . . . . .	116
FpXQ_ellgroup . . . . .	323	FpX_chinese_coprime . . . . .	117
FpXQ_ellj . . . . .	323	FpX_conv . . . . .	115
FpXQ_elljissupersingular . . . . .	323	FpX_ddf . . . . .	117
FpXQ_elltwist . . . . .	323	FpX_ddf_degree . . . . .	117
FpXQ_ffisom_inv . . . . .	129	FpX_degfact . . . . .	117, 131, 140
FpXQ_inv . . . . .	120	FpX_deriv . . . . .	115
FpXQ_invsafe . . . . .	120	FpX_digits . . . . .	115
FpXQ_issquare . . . . .	120	FpX_disc . . . . .	118
FpXQ_log . . . . .	120, 121, 134	FpX_div . . . . .	115
FpXQ_matrix_pow . . . . .	121	FpX_divrem . . . . .	115
FpXQ_minpoly . . . . .	121	FpX_div_by_X_x . . . . .	115
FpXQ_mul . . . . .	120	FpX_dotproduct . . . . .	116
FpXQ_norm . . . . .	121	FpX_eval . . . . .	116
FpXQ_order . . . . .	120, 134	FpX_extgcd . . . . .	115
FpXQ_pow . . . . .	120	FpX_factor . . . . .	117
FpXQ_powers . . . . .	121	FpX_factorff . . . . .	128
FpXQ_powu . . . . .	120	FpX_factorff_irred . . . . .	128, 131

FpX_factor_squarefree . . . . .	117	FpX_resultant . . . . .	118
FpX_ffintersect . . . . .	129	FpX_roots . . . . .	118
FpX_ffisom . . . . .	128, 131	FpX_rootsff . . . . .	128, 131
FpX_FpC_nfpoleval . . . . .	274	FpX_split_part . . . . .	118
FpX_FpV_multieval . . . . .	116	FpX_sqr . . . . .	115
FpX_FpXQV_eval . . . . .	122	FpX_sub . . . . .	115
FpX_FpXQ_eval . . . . .	122	FpX_to_mod . . . . .	144
FpX_FpXY_resultant . . . . .	118	FpX_translate . . . . .	115
FpX_Fp_add . . . . .	116	FpX_valrem . . . . .	115
FpX_Fp_add_shallow . . . . .	116	Fp_2gener . . . . .	98
FpX_Fp_mul . . . . .	116	Fp_add . . . . .	15, 97
FpX_Fp_mulspec . . . . .	116	Fp_addmul . . . . .	97
FpX_Fp_mul_to_monic . . . . .	116	Fp_center . . . . .	97
FpX_Fp_sub . . . . .	116	Fp_center_i . . . . .	97
FpX_Fp_sub_shallow . . . . .	116	Fp_div . . . . .	97
FpX_Frobenius . . . . .	116	Fp_ellcard . . . . .	318
FpX_fromNewton . . . . .	118	Fp_ellcard_SEA . . . . .	318
FpX_gcd . . . . .	115	Fp_elldivpol . . . . .	318
FpX_gcd_check . . . . .	286	Fp_ellgens . . . . .	318
FpX_get_red . . . . .	118	Fp_ellgroup . . . . .	318
FpX_halfgcd . . . . .	115	Fp_ellj . . . . .	317
FpX_half . . . . .	115	Fp_elljissupersingular . . . . .	317
FpX_integ . . . . .	115	Fp_elltwist . . . . .	318
FpX_invBarret . . . . .	118	Fp_factored_order . . . . .	98
FpX_invBarrett . . . . .	116	Fp_ffellcard . . . . .	318
FpX_invLaplace . . . . .	118	Fp_FpXQ_log . . . . .	120
FpX_isplayer . . . . .	117	Fp_FpX_sub . . . . .	116
FpX_is_irred . . . . .	117, 140	Fp_half . . . . .	97
FpX_is_squarefree . . . . .	117	Fp_inv . . . . .	97
FpX_is_totally_split . . . . .	117	Fp_invgen . . . . .	97
FpX_Laplace . . . . .	118	Fp_invsafe . . . . .	97
FpX_matFrobenius . . . . .	116	Fp_isplayer . . . . .	98
FpX_mul . . . . .	115	Fp_issquare . . . . .	98
FpX_mulspec . . . . .	115	Fp_log . . . . .	98, 121
FpX_mulu . . . . .	116	Fp_modinv_to_j . . . . .	324
FpX_nbfact . . . . .	117	Fp_mul . . . . .	97
FpX_nbfact_Frobenius . . . . .	117	Fp_muls . . . . .	97
FpX_nbroots . . . . .	117	Fp_mulu . . . . .	97
FpX_neg . . . . .	114	Fp_neg . . . . .	97
FpX_Newton . . . . .	118	Fp_order . . . . .	98
FpX_normalize . . . . .	116	Fp_polmodular_evalx . . . . .	324
FpX_oneroot . . . . .	118	Fp_pow . . . . .	97
FpX_oneroot_split . . . . .	118	Fp_powers . . . . .	97
FpX_powu . . . . .	115	Fp_pows . . . . .	97
FpX_ratlift . . . . .	149	Fp_powu . . . . .	97
FpX_red . . . . .	114	Fp_pow_init . . . . .	97
FpX_rem . . . . .	115	Fp_pow_table . . . . .	97
FpX_renormalize . . . . .	115	Fp_ratlift . . . . .	149
FpX_rescale . . . . .	116	Fp_red . . . . .	96

Fp_sqr . . . . .	97	FqX_ddf . . . . .	128
Fp_sqrt . . . . .	98	FqX_degfact . . . . .	128
Fp_sqrtn . . . . .	98	FqX_deriv . . . . .	124
Fp_sqrt_i . . . . .	98	FqX_div . . . . .	124
Fp_sub . . . . .	97	FqX_divrem . . . . .	124
Fp_to_mod . . . . .	144	FqX_div_by_X_x . . . . .	124
FqC_add . . . . .	109	FqX_eval . . . . .	124
FqC_Fq_mul . . . . .	109	FqX_extgcd . . . . .	124
FqC_sub . . . . .	109	FqX_factor . . . . .	128
FqC_to_FlxC . . . . .	154	FqX_factor_squarefree . . . . .	128
FqM_deplin . . . . .	109	FqX_Fp_mul . . . . .	124
FqM_det . . . . .	109	FqX_Fq_add . . . . .	123
FqM_FqC_gauss . . . . .	109	FqX_Fq_mul . . . . .	124
FqM_FqC_invimage . . . . .	109	FqX_Fq_mul_to_monic . . . . .	124
FqM_FqC_mul . . . . .	109	FqX_Fq_sub . . . . .	123
FqM_gauss . . . . .	109	FqX_gcd . . . . .	124
FqM_image . . . . .	109	FqX_get_red . . . . .	126
FqM_indexrank . . . . .	109	FqX_halfgcd . . . . .	124
FqM_inv . . . . .	109	FqX_halve . . . . .	124
FqM_invimage . . . . .	109	FqX_integ . . . . .	124
FqM_ker . . . . .	109	FqX_ispower . . . . .	128
FqM_mul . . . . .	109	FqX_is_squarefree . . . . .	128
FqM_rank . . . . .	109	FqX_mul . . . . .	124
FqM_suppl . . . . .	109	FqX_mulu . . . . .	124
FqM_to_FlxM . . . . .	155	FqX_nbfact . . . . .	129
FqM_to_mod . . . . .	144	FqX_nbroots . . . . .	129
FqM_to_nfM . . . . .	283	FqX_neg . . . . .	123
FqV_inv . . . . .	121	FqX_normalize . . . . .	124
FqV_red . . . . .	119	FqX_powu . . . . .	124
FqV_roots_to_pol . . . . .	128	FqX_red . . . . .	119
FqV_to_FlxV . . . . .	155	FqX_rem . . . . .	124
FqV_to_nfV . . . . .	283	FqX_roots . . . . .	128
FqXn_exp . . . . .	126	FqX_sqr . . . . .	124
FqXn_inv . . . . .	126	FqX_sub . . . . .	123
FqXn_mul . . . . .	126	FqX_to_FFX . . . . .	227
FqXn_sqr . . . . .	126	FqX_to_mod . . . . .	125
FqXQ_add . . . . .	127	FqX_to_nfX . . . . .	283
FqXQ_div . . . . .	127	FqX_translate . . . . .	124
FqXQ_inv . . . . .	127	Fq_add . . . . .	120
FqXQ_invsafe . . . . .	127	Fq_div . . . . .	120
FqXQ_matrix_pow . . . . .	128	Fq_ellcard_SEA . . . . .	323
FqXQ_mul . . . . .	127	Fq_elldivpolmod . . . . .	323
FqXQ_pow . . . . .	127	Fq_Fp_mul . . . . .	120
FqXQ_powers . . . . .	127	Fq_halve . . . . .	120
FqXQ_sqr . . . . .	127	Fq_inv . . . . .	120
FqXQ_sub . . . . .	127	Fq_invsafe . . . . .	120
FqXY_eval . . . . .	124	Fq_ispower . . . . .	121
FqXY_evalx . . . . .	124	Fq_issquare . . . . .	121
FqX_add . . . . .	123	Fq_log . . . . .	121

Fq_mul	120
Fq_mulu	120
Fq_neg	120
Fq_neg_inv	120
Fq_pow	121
Fq_powu	121
Fq_red	119
Fq_sqr	120
Fq_sqrt	121
Fq_sqrtn	121
Fq_sub	120
Fq_to_FF	227
Fq_to_FpXQ	119
Fq_to_nf	283
fractor	199
Frobeniusform	173
fromdigitsu	86
fromdigits_2k	86
fujiwara_bound	225
fujiwara_bound_real	225
fun(E, ell)	42
fun(E, H)	42
functions_basic	52
functions_default	52
functions_gp	52
fuse_Z_factor	161
f_PRETTYMAT	239
f_RAW	239, 240
f_TEX	239, 240

## G

gabs[z]	217
gadd	81, 218
gaddgs	15, 218
gaddsg	15, 218
gaddz	15, 26, 81, 219
gadd[z]	81
gaffect	26, 27, 199
gaffsg	27, 199
galoisexport	237
galoisidentify	237
galoisinit	235, 294, 295
galois_group	235
gal_get_den	295
gal_get_e	295
gal_get_gen	295
gal_get_group	295
gal_get_invvdm	295

gal_get_mod	295
gal_get_orders	295
gal_get_p	295
gal_get_pol	295
gal_get_roots	295
gammamellininv	328
gammamellininvinit	328
gammamellininvrt	328
gand	212
garbage collecting	17
gassoc_proto	99
gaussred_from_QR	173
gbezout	216
gboundcf	96
gcdii	96
gceil	209
gclone	27, 67, 68
gcloneref	68
gclone_refc	69
gcmp	210
gcmpgs	211
gcmpsg	211
gcoeff	15, 61, 252
gconj	223
gcopy	27, 68
gcopy_avma	67
gcopy_lg	68
gcvtoi	209
gcvtop	200
gc_needed	23
gdeuc	215
gdiv	218
gdiventgs[z]	215
gdiventres	215
gdiventsg	215
gdivent[z]	215
gdivexact	214
gdivgs	218
gdivmod	215
gdivround	215
gdivsg	218
gdivz	219
gdvd	215
gel	14, 15, 61, 252
GEN	13
GENbinbase	65
gener_F2xq	141
gener_Flxq	135
gener_FpXQ	121

gener_FpXQ_local . . . . .	121	gen_pow_table . . . . .	219
gener_Fq_local . . . . .	121	gen_product . . . . .	219
GENtoGENstr . . . . .	239	gen_RgX_bkeval . . . . .	195
GENtoGENstr_nospace . . . . .	239	gen_search . . . . .	214
GENtostr . . . . .	39, 239	gen_select_order . . . . .	192
GENtostr_raw . . . . .	239	gen_setminus . . . . .	213
GENtostr_unquoted . . . . .	239	gen_Shanks . . . . .	192
GENtoTeXstr . . . . .	39, 239	gen_Shanks_init . . . . .	192
gen_0 . . . . .	13, 32	gen_Shanks_log . . . . .	191
gen_1 . . . . .	13	gen_Shanks_sqrtm . . . . .	192
gen_2 . . . . .	13	gen_sort . . . . .	213
gen_bkeval . . . . .	195	gen_sort_inplace . . . . .	213
gen_bkeval_powers . . . . .	195	gen_sort_uniq . . . . .	213
gen_cmp_RgX . . . . .	214	gen_ZpM_Dixon . . . . .	176
gen_det . . . . .	194	gen_ZpX_Dixon . . . . .	197
gen_digits . . . . .	196	gen_ZpX_Newton . . . . .	197
gen_ellgens . . . . .	193, 315	geq . . . . .	212
gen_ellgroup . . . . .	193	gequal . . . . .	183, 210
gen_factorback . . . . .	220	gequal0 . . . . .	211
gen_factored_order . . . . .	192	gequal1 . . . . .	211
gen_FpM_Wiedemann . . . . .	176	gequalgs . . . . .	211
gen_fromdigits . . . . .	196	gequalm1 . . . . .	211
gen_Gauss . . . . .	194	gequalsg . . . . .	211
gen_Gauss_pivot . . . . .	194	gequalX . . . . .	210
gen_gener . . . . .	192	gerepile . . . . .	18, 20, 26, 27, 65, 91
gen_indexsort . . . . .	213	gerepileall . . . . .	23
gen_indexsort_uniq . . . . .	213	gerepileall . . . . .	20, 23, 66
gen_ker . . . . .	194	gerepileallsp . . . . .	20, 66
gen_m1 . . . . .	13	gerepilecoeffs . . . . .	66
gen_m2 . . . . .	13	gerepilecoeffssp . . . . .	66
gen_matcolinimage . . . . .	194	gerepilecopy . . . . .	20, 23, 66
gen_matcolmul . . . . .	194	gerepilemany . . . . .	66
gen_matid . . . . .	194	gerepilemanysp . . . . .	66
gen_matinimage . . . . .	194	gerepileupto . . . . .	19, 20, 25, 27, 66, 91, 158, 205, 207, 252, 279
gen_matmul . . . . .	194	gerepileuptoint . . . . .	66
gen_order . . . . .	192	gerepileuptoleaf . . . . .	66
gen_PH_log . . . . .	192	getheap . . . . .	69
gen_plog . . . . .	192	getrand . . . . .	96
gen_Pollard_log . . . . .	192	getrealprecision . . . . .	231
gen_pow . . . . .	219	gettime . . . . .	41
gen_powers . . . . .	195, 220	get_arith_Z . . . . .	192
gen_powu . . . . .	219	get_arith_ZZM . . . . .	192
gen_powu_fold . . . . .	219	get_bnf . . . . .	267
gen_powu_fold_i . . . . .	219	get_bnfpol . . . . .	267
gen_powu_i . . . . .	219	get_F2xqE_group . . . . .	193
gen_pow_fold . . . . .	219	get_F2xqX_degree . . . . .	143
gen_pow_fold_i . . . . .	219	get_F2xqX_mod . . . . .	143
gen_pow_i . . . . .	219	get_F2xqX_var . . . . .	143
gen_pow_init . . . . .	219		

get_F2xq_field . . . . .	194	ginvmod . . . . .	215
get_F2x_degree . . . . .	139	gisexactzero . . . . .	211
get_F2x_mod . . . . .	138	glcm . . . . .	216
get_F2x_var . . . . .	139	gle . . . . .	212
get_FlxqE_group . . . . .	193	glt . . . . .	212
get_FlxqXQ_algebra . . . . .	196	gmael . . . . .	15, 61
get_FlxqX_degree . . . . .	136	gmael1 . . . . .	15
get_FlxqX_mod . . . . .	136	gmael2 . . . . .	61
get_FlxqX_var . . . . .	136	gmael3 . . . . .	61
get_Flxq_field . . . . .	194	gmael4 . . . . .	61
get_Flxq_star . . . . .	193	gmael5 . . . . .	61
get_Flx_degree . . . . .	129	gmax . . . . .	210
get_Flx_mod . . . . .	129	gmaxgs . . . . .	211
get_Flx_var . . . . .	129	gmaxsg . . . . .	211
get_Fl_red . . . . .	79	gmax_shallow . . . . .	210
get_FpE_group . . . . .	193	gmin . . . . .	210
get_FpXQE_group . . . . .	193	gmings . . . . .	211
get_FpXQXQ_algebra . . . . .	196	gminsg . . . . .	211
get_FpXQX_algebra . . . . .	196	gmin_shallow . . . . .	210
get_FpXQX_degree . . . . .	126	gmodgs . . . . .	215
get_FpXQX_mod . . . . .	126	gmodsg . . . . .	215
get_FpXQX_var . . . . .	126	gmodulgs . . . . .	201
get_FpXQ_algebra . . . . .	196	gmodulo . . . . .	200
get_FpXQ_star . . . . .	193	gmodulsg . . . . .	201
get_FpX_algebra . . . . .	196	gmodulss . . . . .	201
get_FpX_degree . . . . .	119	gmod[z] . . . . .	215
get_FpX_mod . . . . .	119	gmul . . . . .	218
get_FpX_var . . . . .	119	gmul2n[z] . . . . .	209
get_Fp_field . . . . .	194	gmulgs . . . . .	218
get_Fq_field . . . . .	194	gmulsg . . . . .	218
get_lex . . . . .	260	gmulz . . . . .	219
get_modpr . . . . .	268	gne . . . . .	212
get_nf . . . . .	267	gneg[z] . . . . .	217
get_nfpol . . . . .	267	gneg_i . . . . .	217
get_nf_field . . . . .	194	gnorml1 . . . . .	220
get_prid . . . . .	268	gnorml1_fake . . . . .	220
get_Rg_algebra . . . . .	196	gnorml2 . . . . .	220
gexpo . . . . .	30, 58	gnot . . . . .	212
gexpo_safe . . . . .	58	gor . . . . .	212
gfloor . . . . .	209	<i>GP prototype</i> . . . . .	71
gfrac . . . . .	209	gphelp_keyword_list . . . . .	55
ggamma1m1 . . . . .	232	gpinstall . . . . .	54
ggcd . . . . .	216	gpow . . . . .	218
gge . . . . .	212	gpowers . . . . .	218
ggt . . . . .	212	gpowgs . . . . .	218
ghalf . . . . .	13	gprec . . . . .	200
gidentical . . . . .	174, 210	gprecision . . . . .	59
gimag . . . . .	223	gprec_w . . . . .	200
ginv . . . . .	217	gprec_wensure . . . . .	200



gprec_wtrunc . . . . .	200	group_ident . . . . .	237
gprimepi_lower_bound . . . . .	162	group_ident_trans . . . . .	237
gprimepi_upper_bound . . . . .	162	group_isA4S4 . . . . .	236
gp_alarm_handler . . . . .	55	group_isabelian . . . . .	236
gp_call . . . . .	261	group_leftcoset . . . . .	236
gp_call2 . . . . .	261	group_order . . . . .	236
gp_callbool . . . . .	261	group_perm_normalize . . . . .	236
gp_callprec . . . . .	261	group_quotient . . . . .	236
gp_callvoid . . . . .	261	group_rightcoset . . . . .	236
gp_context_restore . . . . .	55	group_set . . . . .	235
gp_context_save . . . . .	55	group_subgroups . . . . .	236
gp_echo_and_log . . . . .	55	group_subgroup_isnormal . . . . .	236
gp_eval . . . . .	261	group_to_cc . . . . .	236
gp_evalbool . . . . .	261	gshift[z] . . . . .	209
gp_evalprec . . . . .	261	gsigne . . . . .	30, 58
gp_evalupto . . . . .	261	gsincos . . . . .	232
gp_evalvoid . . . . .	261	gsizebyte . . . . .	26
gp_filter . . . . .	54	gsizeword . . . . .	26
gp_format_prompt . . . . .	54	gsmith . . . . .	302
gp_format_time . . . . .	54	gsmithall . . . . .	302
gp_handle_exception . . . . .	53	gsprintf . . . . .	240
gp_help . . . . .	54	gsqr . . . . .	217
gp_load_gprc . . . . .	54	gsqrpowers . . . . .	219
gp_meta . . . . .	54	GSTR . . . . .	34
gp_read_file . . . . .	38, 54	gsub . . . . .	218
gp_read_str . . . . .	36, 37, 54, 74	gsubgs . . . . .	218
gp_read_stream . . . . .	38	gsubsg . . . . .	218
gp_read_str_multiline . . . . .	37	gsubst . . . . .	221
gp_sigint_fun . . . . .	53	gsubz . . . . .	219
Gram matrix . . . . .	171	gsupnorm . . . . .	220
gram_matrix . . . . .	171	gsupnorm_aux . . . . .	220
greal . . . . .	223	gtocol . . . . .	203
gred_rfac2 . . . . .	33	gtodouble . . . . .	28, 199
grem . . . . .	215	gtofp . . . . .	27, 200
grndtoi . . . . .	209	gtolong . . . . .	28, 199
grootsof1 . . . . .	218	gtomat . . . . .	203
ground . . . . .	209	gtomp . . . . .	200
groupelts_abelian_group . . . . .	237	gtopoly . . . . .	201
groupelts_center . . . . .	237	gtopolyrev . . . . .	201
groupelts_conjclasses . . . . .	235	gtos . . . . .	199
groupelts_exponent . . . . .	237	gtoser . . . . .	202
groupelts_set . . . . .	235	gtoser_prec . . . . .	202, 203
group_abelianHNF . . . . .	236	gtovect . . . . .	203
group_abelianSNF . . . . .	236	gtovectsmall . . . . .	203
group_domain . . . . .	235	gtrans . . . . .	253
group_elts . . . . .	235	gtrunc . . . . .	209
group_export . . . . .	237	gtrunc2n . . . . .	85, 209
group_export_GAP . . . . .	237	gunclone . . . . .	27, 69
group_export_MAGMA . . . . .	237	gunclone_deep . . . . .	69

gval . . . . .	210
gvaluation . . . . .	209
gvar . . . . .	32, 35, 58
gvar2 . . . . .	58
gvsprintf . . . . .	240
G_ZGC_mul . . . . .	174
G_ZG_mul . . . . .	174

## H

hamming1 . . . . .	77
hashentry . . . . .	249
hashtable . . . . .	249
hash_create . . . . .	250
hash_create_str . . . . .	250
hash_create_ulong . . . . .	250
hash_dbg . . . . .	251
hash_destroy . . . . .	251
hash_GEN . . . . .	251
hash_haskey_long . . . . .	250
hash_init_GEN . . . . .	250
hash_insert . . . . .	250
hash_insert2 . . . . .	250
hash_insert_long . . . . .	250
hash_keys . . . . .	251
hash_remove . . . . .	250
hash_remove_select . . . . .	250
hash_search . . . . .	250
hash_search2 . . . . .	250
hash_select . . . . .	250
hash_str . . . . .	250, 251
hash_str2 . . . . .	251
hash_values . . . . .	251
hclassno . . . . .	296
hclassno6 . . . . .	296
hclassno6u . . . . .	296
heap . . . . .	14
hexadecimal tree . . . . .	40
HIGHBIT . . . . .	62
HIGHEXPBIT . . . . .	62
HIGHMASK . . . . .	62
HIGHVALPBIT . . . . .	62
HIGHWORD . . . . .	62
hilbertii . . . . .	100
hnf . . . . .	302
hnfall . . . . .	302
hnfdivide . . . . .	301
hnfl11 . . . . .	302
hnfmerge_get_1 . . . . .	280

hnfmod . . . . .	302
hnfmodid . . . . .	302
hnfperm . . . . .	302
hnf_CENTER . . . . .	300
hnf_divscale . . . . .	300
hnf_invimage . . . . .	301
hnf_invscale . . . . .	300
hnf_MODID . . . . .	299
hnf_PART . . . . .	299
hnf_solve . . . . .	301
hqfeval . . . . .	221
hyperell_locally_soluble . . . . .	324
h_APROPOS . . . . .	55
h_LONG . . . . .	54
h_REGULAR . . . . .	54

## I

icopy . . . . .	83
icopyifstack . . . . .	68
icopyspec . . . . .	83
icopy_avma . . . . .	67
idealadd . . . . .	279
idealaddmultoone . . . . .	280
idealaddtoone . . . . .	280
idealaddtoone_i . . . . .	280
idealaddtoone_raw . . . . .	280
idealappr . . . . .	280, 294
idealappr0 . . . . .	294
idealapprfact . . . . .	280
idealchinese . . . . .	280
idealchineseinit . . . . .	280
idealcoprime . . . . .	280
idealcoprimefact . . . . .	280
idealdiv . . . . .	279
idealdivexact . . . . .	279
idealdivpowprime . . . . .	279
idealfactor . . . . .	278, 280
idealfrobenius_aut . . . . .	282
idealhnf . . . . .	278
idealhnf0 . . . . .	278
idealHNF_inv . . . . .	280
idealHNF_inv_Z . . . . .	280
idealHNF_mul . . . . .	279
idealhnf_principal . . . . .	278
idealhnf_shallow . . . . .	278
idealhnf_two . . . . .	279
idealHNF_Z_factor . . . . .	281
idealHNF_Z_factor_i . . . . .	281

idealinv . . . . .	279	indices_to_vec01 . . . . .	284
ideallog . . . . .	278	infinity . . . . .	34
idealmoddivisor . . . . .	290	inf_get_sign . . . . .	34
idealmul . . . . .	279	initprimes . . . . .	64
idealmulpowprime . . . . .	279	initprimetable . . . . .	64
idealmulred . . . . .	279, 287	init_Fq . . . . .	128
idealpow . . . . .	279	init_primepointer_geq . . . . .	64
idealpowred . . . . .	279	init_primepointer_gt . . . . .	64
idealpows . . . . .	279	init_primepointer_leq . . . . .	64
idealprimedec . . . . .	278, 280, 281	init_primepointer_lt . . . . .	64
idealprimedec_degrees . . . . .	281	input . . . . .	37
idealprimedec_galois . . . . .	281	install . . . . .	36, 40, 73, 75
idealprimedec_kummer . . . . .	281	int2n . . . . .	82
idealprimedec_limit_f . . . . .	281	int2u . . . . .	82
idealprimedec_limit_norm . . . . .	281	int2um1 . . . . .	82
idealprincipalunits . . . . .	285	integer . . . . .	29
idealprod . . . . .	279	integser . . . . .	226
idealprodprime . . . . .	279	int_LSW . . . . .	29
idealprodval . . . . .	279	int_MSW . . . . .	29
idealpseudomin . . . . .	289	int_nextW . . . . .	29
idealpseudomin_nonscalar . . . . .	289	int_normalize . . . . .	29
idealpseudored . . . . .	289	int_precW . . . . .	29
idealramfrobenius . . . . .	282	int_W . . . . .	29
idealramfrobenius_aut . . . . .	282	int_W_lg . . . . .	29
idealramgroups_aut . . . . .	282	invmod . . . . .	98
idealred . . . . .	289	invmod2BIL . . . . .	78
idealred0 . . . . .	289	invr . . . . .	91
idealred_elt . . . . .	289	inv_content . . . . .	217
idealsqr . . . . .	279	isclone . . . . .	28
idealstar . . . . .	285	iscomplex . . . . .	212
Idealstarprk . . . . .	290	isexactzero . . . . .	210
idealtyp . . . . .	268	isinexact . . . . .	211
identity_perm . . . . .	234	isinexactreal . . . . .	211
id_MAT . . . . .	268	isint . . . . .	212
id_PRIME . . . . .	268	isint1 . . . . .	211
id_PRINCIPAL . . . . .	268	isintm1 . . . . .	211
ifac_isprime . . . . .	162	isintzero . . . . .	211
ifac_next . . . . .	161	ismpzero . . . . .	211
ifac_read . . . . .	161	isonstack . . . . .	68
ifac_skip . . . . .	161	isprime . . . . .	163
ifac_start . . . . .	161	isprimeAPRCL . . . . .	163
image . . . . .	176	isprimeECPP . . . . .	163
image2 . . . . .	176	isprimepower . . . . .	100
imag_i . . . . .	223	isprincipal . . . . .	287
indefinite binary quadratic form . . . . .	33	isprincipalfact . . . . .	287
indexlexsort . . . . .	213	isprincipalfact_or_fail . . . . .	287
indexpartial . . . . .	286	isprincipalforce . . . . .	294
indexsort . . . . .	213	isprincipalgen . . . . .	294
indexvecsort . . . . .	213	isprincipalgenforce . . . . .	294

isprincipalraygen . . . . .	294
isrationalzero . . . . .	211
isrationalzeroscalar . . . . .	211
isrealappr . . . . .	211
issmall . . . . .	212
is_357_power . . . . .	100, 157, 158
is_bigint . . . . .	83
is_const_t . . . . .	61
is_entry . . . . .	70
is_extscalar_t . . . . .	61
is_intreal_t . . . . .	61
is_linit . . . . .	327
is_matvec_t . . . . .	61
is_nf_extfactor . . . . .	268
is_nf_factor . . . . .	268
is_noncalc_t . . . . .	61
is_pm1 . . . . .	211
is_pth_power . . . . .	158
is_rational_t . . . . .	61
is_real_t . . . . .	61
is_recursive_t . . . . .	60
is_scalar_t . . . . .	61
is_universal_constant . . . . .	199
is_vec_t . . . . .	61
is_Z_factor . . . . .	160
is_Z_factornon0 . . . . .	160
is_Z_factorpos . . . . .	160
itor . . . . .	83
itos . . . . .	27, 83, 199
itostr . . . . .	239
itos_or_0 . . . . .	83
itou . . . . .	83
itou_or_0 . . . . .	83

## K

killblock . . . . .	68
krois . . . . .	98
kroiu . . . . .	98
Kronecker symbol . . . . .	98, 99
kronecker . . . . .	98
Kronecker_to_F2xqX . . . . .	142
Kronecker_to_FlxqX . . . . .	136
Kronecker_to_FpXQX . . . . .	125
Kronecker_to_mod . . . . .	190
Kronecker_to_ZXX . . . . .	180
krosi . . . . .	98
kross . . . . .	98
kroui . . . . .	98

krouu . . . . .	98
-----------------	----

## L

lcmii . . . . .	96
ldata_get_an . . . . .	327
ldata_get_conductor . . . . .	327
ldata_get_degree . . . . .	327
ldata_get_dual . . . . .	327
ldata_get_gammavec . . . . .	327
ldata_get_k . . . . .	327
ldata_get_residue . . . . .	327
ldata_get_rootno . . . . .	327
ldata_get_type . . . . .	327
ldata_isreal . . . . .	327
ldata_vecan . . . . .	327
leading_coeff . . . . .	32, 61
leafcopy . . . . .	83, 253
leafcopy_avma . . . . .	83
Legendre symbol . . . . .	98
lexcmp . . . . .	210
lexsort . . . . .	212
lfun . . . . .	328
lfunellmfpeters . . . . .	328
lfuninit . . . . .	328
lfuninit_make . . . . .	328
lfunlambda . . . . .	328
lfunmisc_to_ldata . . . . .	328
lfunmisc_to_ldata_shallow . . . . .	328
lfunprod_get_fact . . . . .	327
lfunrootno . . . . .	328
lfunrttopoles . . . . .	328
lfunthetacheckinit . . . . .	328
lfunthetacost . . . . .	328
lfunzetakinit . . . . .	328
lfun_get_bitprec . . . . .	327
lfun_get_dom . . . . .	327
lfun_get_domain . . . . .	327
lfun_get_expot . . . . .	327
lfun_get_factgammavec . . . . .	327
lfun_get_k2 . . . . .	327
lfun_get_pol . . . . .	327
lfun_get_Residue . . . . .	327
lfun_get_step . . . . .	327
lfun_get_w2 . . . . .	327
lg . . . . .	28, 57
LGBITS . . . . .	62
lgcols . . . . .	59
lgfint . . . . .	29, 58

LGnumBITS . . . . .	62
lgpol . . . . .	58
lg_increase . . . . .	28
library mode . . . . .	13
lift . . . . .	201
lift0 . . . . .	201
liftall . . . . .	201
liftall_shallow . . . . .	201
liftint . . . . .	201
liftint_shallow . . . . .	201
liftpol . . . . .	201
liftpol_shallow . . . . .	201
lift_shallow . . . . .	201
lincombii . . . . .	91
lindep . . . . .	304
lindep2 . . . . .	304
lindepfull_bit . . . . .	304
lindep_bit . . . . .	304
lindep_padic . . . . .	304
lindep_Xadic . . . . .	304
linit_get_ldata . . . . .	327
linit_get_tech . . . . .	327
linit_get_type . . . . .	327
list . . . . .	34
LLL . . . . .	300, 302
l1l . . . . .	303
l1lfp . . . . .	303
l1lgen . . . . .	303
l1lgram . . . . .	303
l1lgramgen . . . . .	303
l1lgramint . . . . .	303
l1lgramkerim . . . . .	303
l1lgramkerimgen . . . . .	303
l1lint . . . . .	303
l1lintpartial . . . . .	304
l1lintpartial_inplace . . . . .	304
l1lkerim . . . . .	303
l1lkerimgen . . . . .	303
LLL_ALL . . . . .	303
LLL_COMPATIBLE . . . . .	303
LLL_GRAM . . . . .	303
LLL_IM . . . . .	303
LLL_INPLACE . . . . .	303
LLL_KEEP_FIRST . . . . .	303
LLL_KER . . . . .	303
LOG10_2 . . . . .	63
LOG2_10 . . . . .	63
logint . . . . .	95
logintall . . . . .	95

logr_abs . . . . .	231
LONG_IS_64BIT . . . . .	16
LONG_MAX . . . . .	62
long_to_rgb . . . . .	334
loop_break . . . . .	260
LOWMASK . . . . .	62
LOWWORD . . . . .	62

## M

malloc . . . . .	244
mantissa2nr . . . . .	85
mantissa_real . . . . .	31, 85
map_proto_G . . . . .	99
map_proto_GL . . . . .	99
map_proto_lG . . . . .	99
map_proto_lGL . . . . .	99
matbrute . . . . .	242
matdet . . . . .	167
mathnf . . . . .	278
matid . . . . .	204
matid_F2m . . . . .	113
matid_F2xqM . . . . .	142
matid_Flm . . . . .	109
matid_FlxqM . . . . .	114
matpermanent . . . . .	167
matrix . . . . .	34
matrixqz . . . . .	302
matslice . . . . .	253
maxdd . . . . .	88
maxprime . . . . .	13, 63
maxprime_check . . . . .	63
maxss . . . . .	88
maxuu . . . . .	88
MAXVARN . . . . .	62
MEDDEFAULTPREC . . . . .	16, 62
merge_factor . . . . .	214
merge_sort_uniq . . . . .	213
mfcharmodulus . . . . .	331
mfcharorder . . . . .	331
mfcharpol . . . . .	331
mfcuspdim . . . . .	331
MFcusp_get_vMjd . . . . .	330
mfddiv_val . . . . .	331
mfeisensteindim . . . . .	331
mfeisensteinspaceinit . . . . .	331
mfembed . . . . .	331
mffulldim . . . . .	331
mfiscuspidal . . . . .	331

mfmaterembed	331	mkcol4s	205
mfnewdim	331	mkcol5	206
MFnew_get_vj	330	mkcol6	206
mfnumcuspsu	329	mkcolcopy	204
mfnumcuspsu_fact	329	mkcoln	25, 208
mfnumcusps_fact	329	mkcols	205
mfolddim	331	mkcomplex	205
mfsturmNgk	331	mkerr	207
mfsturmNk	331	mkfrac	206
mfsturm_mf	331	mkfraccopy	204
mftobasisES	331	mkfracss	204
mftocol	332	mkintmod	206
mfvecembed	331	mkintmodu	204
mfvectomat	332	mkintn	25, 26, 84, 208
MF_get_basis	329, 330	mkmat	206
MF_get_CHI	329	mkmat2	206
mf_get_CHI	330	mkmat22	206
MF_get_dim	329	mkmat22s	205
MF_get_E	329	mkmat3	206
mf_get_field	330	mkmat4	206
MF_get_fields	330	mkmat5	206
MF_get_gk	329	mkmatcopy	204
mf_get_gk	330	mkmoo	34
MF_get_gN	329	mkoo	34
mf_get_gN	330	mkpolmod	206
MF_get_k	329	mkpoln	25, 208
mf_get_k	330	mkqfi	207
MF_get_M	330	mkquad	205
MF_get_Mindex	330	mkfrac	206
MF_get_Minv	330	mkfraccopy	204
MF_get_N	329	mkvec	206
mf_get_N	330	mkvec2	206
MF_get_newforms	330	mkvec2copy	204
mf_get_NK	330	mkvec2s	205
MF_get_r	329	mkvec3	206
mf_get_r	330	mkvec3s	205
MF_get_S	329, 330	mkvec4	207
MF_get_space	329	mkvec4s	205
mf_get_type	331	mkvec5	207
millerrabin	163	mkveccopy	204
mindd	88	mkvecn	25, 208
minss	88	mkvecs	205
minuu	88	mkvecsmall	205
mkcol	206	mkvecsmall2	205
mkcol2	206	mkvecsmall3	205
mkcol2s	205	mkvecsmall4	205
mkcol3	206	mkvecsmalln	205
mkcol3s	205	Mod16	94
mkcol4	206	mod16	94



nfalgtobasis	275	nfsqr	274
nfarchstar	285	nfsqri	275
nfbasistoalg	275	nfsub	274
nfchecksigns	284	nftrace	274
nfcyclotomicunits	293	nftyp	267
nfC_mutable_mul	275	nfval	274, 284
nfC_nf_mul	275	nfV_to_FqV	283
nfddiv	274	nfX_to_FqX	283
nfdiveuc	274	nf_deg1_prime	282
nfddivrem	274	nf_FORCE	287
nfeltup	292	nf_GEN	287
nfembed	284	nf_GENMAT	287
nffactorback	278	nf_GEN_IF_PRINCIPAL	287
nfgaloisconj	293	nf_get_allroots	269
nfgaloismatrix	295	nf_get_degree	269
nfgaloispermtobasis	295	nf_get_diff	269
nfgcd	182	nf_get_disc	269
nfgcd_all	182	nf_get_G	269
nfgwkummer	292	nf_get_Gtwist	288, 289, 304
nfinit_basic	286	nf_get_Gtwist1	289
nfinit_complete	286	nf_get_index	269
nfinv	274	nf_get_invzk	269
nfinvmodideal	275	nf_get_M	269, 288
nfissquarefree	292	nf_get_pol	268
nfmaxord	285	nf_get_prec	269
nfmaxord_t	285, 286	nf_get_r1	268
nfmaxord_to_nf	286	nf_get_r2	269
nfmod	275	nf_get_ramified_primes	269
nfmodprinit	282, 283	nf_get_roots	269
nfmul	274	nf_get_roundG	269, 289
nfmuli	275	nf_get_sign	269
nfM_det	276	nf_get_Tr	269
nfM_inv	276	nf_get_varn	268
nfM_mul	276	nf_get_zk	269
nfM_nfC_mul	276	nf_get_zkden	269
nfM_to_FqM	283	nf_get_zkprimpart	269
nfnewprec	273	nf_hyperell_locally_soluble	324
nfnewprec_shallow	273	nf_nfzk	292
nfnorm	274	nf_PARTIALFACT	286
nfpoleval	274	nf_pV_to_prV	282
nfpow	274	nf_rnfeq	292
nfpowmodideal	275	nf_rnfeqsimple	292
nfpow_u	274	nf_ROUND2	286
nfrootsof1	293	nf_to_Fp_coprime	290
nfroots_if_split	293	nf_to_Fq	283
nfsign	278, 284	nf_to_Fq_init	282
nfsign_arch	278, 284	nf_to_scalar_or_alg	275
nfsign_from_logarch	285	nf_to_scalar_or_basis	275
nfsign_units	285	nmV_chinese_center	148



nmV_chinese_center_tree	148
nm_Z_mul	156
normalize	203
normalizapol	33, 201
normalizapol_approx	201
normalizapol_lg	201
normalize_frac	60
NO_VARIABLE	32, 35, 58, 62
numberofconjugates	293
numdivu	100
numdivu_fact	100
numerr_name	249
numer_i	216
nv_fromdigits_2k	86
nxCV_chinese_center	148
nxMV_chinese_center	148
nxV_chinese_center	148
nxV_chinese_center_tree	148

## O

<i>obj</i>	265
obj_check	265
obj_checkbuild	265
obj_checkbuild_padicprec	265
obj_checkbuild_prec	265
obj_checkbuild_realprec	265
obj_free	266, 313
obj_init	265
obj_insert	265
obj_insert_shallow	265
obj_reinit	265
odd	78
odd_prime_divisors	159
omega	160
omegau	100
ONLY_DIVIDES	104, 185
ONLY_REM	104, 185
outmat	38
output	38
output	38, 40, 242
out_printf	241
out_putc	241
out_puts	241
out_term_color	242
out_vprintf	241

## P

p-adic number	31
---------------	----

padicprec	152
padicprec_relative	152
padic_to_Fl	154
padic_to_Fp	106
padic_to_Q	152
padic_to_Q_shallow	152
paricfg_buildinfo	76
paricfg_compiledat	76
paricfg_datadir	76
paricfg_gphelp	76
paricfg_mt_engine	76
paricfg_vcsversion	76
paricfg_version	76
paricfg_version_code	76
pariErr	241
PariOUT	240
pariOut	241
paristack_newsize	52
paristack_resize	51
paristack_setsize	51
parivstack_reset	51
parivstack_resize	52
pari_add_defaults_module	52
pari_add_function	52
pari_add_hist	55
pari_add_module	52
pari_alarm	54
pari_ask_confirm	54
pari_calloc	17
pari_CATCH	44
pari_CATCH_reset	45
pari_center	54
pari_close	49
pari_close_opts	51
pari_community	54
pari_compile_str	54
pari_daemon	51
pari_ENDCATCH	45
pari_err	34, 39, 45, 244, 263
pari_err2str	249
pari_errfile	241
pari_err_last	45
pari_err_TYPE	312
pari_fclose	243
pari_flush	38, 241
pari_fopen	243
pari_fopengz	243
pari_fopen_or_fail	243
pari_fprintf	39

pari_fread_chars . . . . .	242	pari_strdup . . . . .	239
pari_free . . . . .	17, 65	pari_strndup . . . . .	239
pari_get_hist . . . . .	55	pari_thread_alloc . . . . .	337
pari_get_histtime . . . . .	55	pari_thread_close . . . . .	338
pari_get_homedir . . . . .	243	pari_thread_free . . . . .	337
pari_hit_return . . . . .	54	pari_thread_init . . . . .	337
pari_infile . . . . .	54	pari_thread_start . . . . .	337
pari_init . . . . .	13, 14, 49	pari_thread_sync . . . . .	337
pari_init_opts . . . . .	49	pari_thread_valloc . . . . .	337
pari_init_primes . . . . .	50, 51	pari_timer . . . . .	41
pari_is_default . . . . .	261	pari_TRY . . . . .	45
pari_is_dir . . . . .	242	pari_unique_dir . . . . .	244
pari_is_file . . . . .	242	pari_unique_filename . . . . .	244
pari_kernel_close . . . . .	50	pari_unique_filename_suffix . . . . .	244
pari_kernel_init . . . . .	50	pari_unlink . . . . .	242
pari_kill_plot_engine . . . . .	333	pari_var_close . . . . .	70
pari_last_was_newline . . . . .	241	pari_var_create . . . . .	70
pari_library_path . . . . .	54	pari_var_init . . . . .	70
pari_malloc . . . . .	17, 65, 247	pari_var_next . . . . .	70
pari_mt_close . . . . .	51	pari_var_next_temp . . . . .	70
pari_mt_init . . . . .	50	PARI_VERSION . . . . .	76
pari_nb_hist . . . . .	55	pari_version . . . . .	76
PARI_OLD_NAMES . . . . .	14	PARI_VERSION_SHIFT . . . . .	76
pari_outfile . . . . .	38, 241	pari_vfprintf . . . . .	39
PARI_plot . . . . .	333	pari_vprintf . . . . .	39
pari_plot_by_file . . . . .	334	pari_vsprintf . . . . .	39
pari_printf . . . . .	38, 39, 40, 71, 241, 242	pari_warn . . . . .	40
pari_print_version . . . . .	54	parser code . . . . .	74
pari_putc . . . . .	38, 71, 241	path_expand . . . . .	243
pari_puts . . . . .	38, 71, 241, 242	perm_commute . . . . .	234
pari_rand . . . . .	96	perm_conj . . . . .	234
pari_realloc . . . . .	17, 247	perm_cycles . . . . .	234
pari_RETRY . . . . .	45	perm_inv . . . . .	234
pari_safefopen . . . . .	243	perm_mul . . . . .	234
pari_set_last_newline . . . . .	241	perm_order . . . . .	234
pari_set_plot_engine . . . . .	333	perm_pow . . . . .	234
pari_sighandler . . . . .	51	perm_sign . . . . .	234
pari_sig_init . . . . .	51	perm_to_Z . . . . .	234
pari_sp . . . . .	17	pgener_Fl . . . . .	79
pari_sprintf . . . . .	39, 239	pgener_Fl_local . . . . .	79
pari_stackcheck_init . . . . .	51	pgener_Fp . . . . .	99
pari_stack_alloc . . . . .	252	pgener_Fp_local . . . . .	99
pari_stack_base . . . . .	252	pgener_Zl . . . . .	79
pari_stack_delete . . . . .	252	pgener_Zp . . . . .	99
pari_stack_init . . . . .	252	Pi2n . . . . .	233
pari_stack_new . . . . .	252	PiI2 . . . . .	233
pari_stack_pushp . . . . .	252	PiI2n . . . . .	234
pari_stdin_isatty . . . . .	243	plotbox . . . . .	333
pari_str . . . . .	240	plotclip . . . . .	333

plotcolor . . . . .	333	polredabsall . . . . .	294
plotcopy . . . . .	333	Polrev . . . . .	203
plotcursor . . . . .	333	polx_F2x . . . . .	139
plotdraw . . . . .	333	polx_F2xX . . . . .	142
ploth . . . . .	333	polx_Flx . . . . .	131
plothraw . . . . .	333	polx_FlxX . . . . .	135
plotsizes . . . . .	333	polx_zx . . . . .	182
plotinit . . . . .	333	polynomial . . . . .	32
plotkill . . . . .	333	pol_0 . . . . .	203
plotline . . . . .	333	pol_1 . . . . .	203
plotlines . . . . .	333	pol_x . . . . .	203
plotlinetype . . . . .	333	pol_xn . . . . .	203
plotmove . . . . .	334	pol_xnall . . . . .	203
plotpoints . . . . .	334	pol_x_powers . . . . .	203
plotpointsize . . . . .	334	pop_lex . . . . .	73, 260
plotpointtype . . . . .	334	power series . . . . .	33
plotrbox . . . . .	334	powersr . . . . .	95
plotrecth . . . . .	333	powgi . . . . .	218, 297
plotrecthraw . . . . .	334	powii . . . . .	94
plotrline . . . . .	334	powis . . . . .	94
plotrmove . . . . .	334	powIs . . . . .	95
plotrpoint . . . . .	334	powiu . . . . .	94
plotscale . . . . .	334	powfrac . . . . .	95
plotstring . . . . .	334	powrs . . . . .	94
point_to_a4a6 . . . . .	313	powrshalf . . . . .	95
point_to_a4a6_F1 . . . . .	313	powru . . . . .	95
pol0_F2x . . . . .	139	powruhalf . . . . .	95
pol0_Flx . . . . .	131	powuu . . . . .	94
pol1_F2x . . . . .	139	ppg . . . . .	160
pol1_F2xX . . . . .	142	ppi . . . . .	160
pol1_Flx . . . . .	131	pple . . . . .	160
pol1_FlxX . . . . .	135	ppo . . . . .	160
polclass . . . . .	324	prec2nbits . . . . .	57
polcoef_i . . . . .	224	prec2nbits_mul . . . . .	57
poldivrem . . . . .	215	prec2ndec . . . . .	56
poleval . . . . .	185, 221	precdbl . . . . .	57
polgalois . . . . .	237	precision . . . . .	59
polhensellift . . . . .	150, 151	precp . . . . .	31, 58
polmod . . . . .	32	PRECPBITS . . . . .	62
polmodular . . . . .	324	PRECPSHIFT . . . . .	62
polmodular_ZM . . . . .	324	preferences file . . . . .	74
polmodular_ZXX . . . . .	324	<i>prid</i> . . . . .	280
polmod_nffix . . . . .	293	prime . . . . .	162
polmod_nffix2 . . . . .	293	primeform . . . . .	298
polmod_to_embed . . . . .	225	primeform_u . . . . .	298
Polred . . . . .	294	primepi_lower_bound . . . . .	162
polred0 . . . . .	294	primepi_upper_bound . . . . .	162
polredabs . . . . .	294	primes . . . . .	162
polredabs2 . . . . .	294	primes_interval . . . . .	162

primes_interval_zv	162
primes_upto_zv	162
primes_zv	162
prime_fact	204
primitive_root	79
primitive_part	216
primpart	216
printf	38, 39, 71
print_fun_list	55
prV_lcm_capZ	282
pr_basis_perm	283
pr_equal	284
pr_get_e	281
pr_get_f	281
pr_get_gen	281
pr_get_p	281
pr_get_tau	281
pr_hnf	281
pr_inv	281
pr_inv_p	281
pr_is_inert	281
pr_norm	281
pr_uniformizer	282
psdraw	334
psploth	334
psplothraw	334
pthread_join	337
push_lex	73, 260
putch	240
puts	240
p_to_FF	226
p_to_FF(p,0)	227

## Q

Qdivii	216
Qevproj_apply	168
Qevproj_apply_vecei	168
Qevproj_down	168
Qevproj_init	168
qfbcompraw	297
qfbforms	224
qfbpowraw	297
qfbsolve	297
qfb_apply_ZM	224
qfb_disc	224
qfb_disc3	224
qfb_equal1	296
qfeval	221

qfevalb	221
qfi	296
qficomp	296
qficompraw	296
qfipowraw	297
qfisolvep	297
qfisqr	296
qfisqrraw	297
qfi_1	296
qfi_log	297
qfi_order	297
qfi_Shanks	297
qflll0	303
qflllgram0	303
qfr	296
qfr3	298
qfr3_comp	298
qfr3_pow	298
qfr3_red	298
qfr3_rho	298
qfr3_to_qfr	299
qfr5	298
qfr5_comp	299
qfr5_dist	299
qfr5_pow	299
qfr5_red	299
qfr5_rho	299
qfr5_to_qfr	299
qfrcomp	296
qfrcompraw	297
qfrpow	297
qfrpowraw	297
qfrsolvep	297
qfrsqr	296
qfrsqrraw	297
qfr_1	296
qfr_data_init	298
qfr_to_qfr5	299
qf_apply_RgM	221
qf_apply_ZM	221
QM_charpoly_ZX	168
QM_charpoly_ZX_bound	168
QM_det	169
QM_gauss	168
QM_ImQ_hnf	302
QM_ImQ_hnfall	302
QM_ImZ_hnf	302
QM_ImZ_hnfall	302
QM_indexrank	168



raw	239	retmkqfi	208
rcopy	83	retmkqfr	208
rdivii	92	retmkquad	208
rdiviiz	92	retmkfrac	208
rdivis	92	retmkvec	207
rdivsi	92	retmkvec2	207
rdivss	92	retmkvec3	207
read	37, 38	retmkvec4	207
readseq	37	retmkvec5	207
real number	30	rfracrecip_to_ser_absolute	202
real2n	82	rfrac_to_ser	202
realprec	57	RgC_add	170
real_0	82	RgC_fpnorml2	172
real_0_bit	82	RgC_gtofp	172
real_1	81	RgC_gtomp	172
real_1_bit	81	RgC_is_ei	172
real_i	223	RgC_is_FFC	227
real_m1	82	RgC_neg	170
real_m2n	82	RgC_RgM_mul	171
rect2ps	334	RgC_RgV_mul	171
rect2ps_i	334	RgC_Rg_add	170
rect2svg	334	RgC_Rg_div	170
reducemodinvertible	305	RgC_Rg_mul	170
reducemodlll	305	RgC_Rg_sub	170
remi2n	91, 177	RgC_sub	170
remlll_pre	79	RgC_to_FpC	106
remll_pre	79	RgC_to_FqC	109
remBIL	62	RgC_to_nfC	275
residual_characteristic	225	RgE_to_F2xqE	321
resultant (reduced)	150	RgE_to_FlxqE	322
resultant	215, 225	RgE_to_FpE	319
resultant2	225	RgE_to_FpXQE	324
retconst_col	207	RgMrow_RgC_mul	171
retconst_vec	207	RgMrow_zc_mul	156
retmkcol	207	RgMs_structelim	175
retmkcol2	207	RgM_add	170
retmkcol3	207	RgM_Babai	174
retmkcol4	207	RgM_check_ZM	166
retmkcol5	207	RgM_det_triangular	173
retmkcol6	207	RgM_diagonal	172
retmkcomplex	208	RgM_diagonal_shallow	172
retmkfrac	208	RgM_dimensions	170
retmkintmod	208	RgM_fpnorml2	172, 220
retmkmat	207	RgM_Fp_init	106
retmkmat2	207	RgM_gram_schmidt	174
retmkmat3	207	RgM_gtofp	172
retmkmat4	207	RgM_gtomp	172, 173
retmkmat5	208	RgM_Hadamard	173
retmkpolmod	208	RgM_hnfall	302

RgM_inv	173	RgV_check_ZV	164
RgM_invimage	173	RgV_dotproduct	171
RgM_inv_upper	173	RgV_dotsquare	171
RgM_isdiagonal	172	RgV_gtofp	172
RgM_isidentity	172	RgV_isin	172
RgM_isscalar	172	RgV_isscalar	172
RgM_is_FFM	227	RgV_is_FpV	106
RgM_is_FpM	106	RgV_is_QV	164
RgM_is_QM	172	RgV_is_ZMV	169
RgM_is_ZM	172	RgV_is_ZV	164
RgM_minor	253	RgV_is_ZVnon0	164
RgM_mul	171	RgV_is_ZVpos	164
RgM_mulreal	171	RgV_kill0	171
RgM_multosym	171	RgV_neg	170
RgM_neg	170	RgV_nffix	293
RgM_powers	171	RgV_polint	171
RgM_QR_init	173	RgV_prod	171
RgM_rescale_to_int	166	RgV_RgC_mul	170
RgM_RgC_invimage	173	RgV_RgM_mul	171
RgM_RgC_mul	171	RgV_Rg_mul	170
RgM_RgC_type	106	RgV_sub	170
RgM_RgV_mul	171	RgV_sum	171
RgM_RgX_mul	171	RgV_sumpart	171
RgM_Rg_add	170	RgV_sumpart2	171
RgM_Rg_add_shallow	170	RgV_to_F2v	113
RgM_Rg_div	170	RgV_to_Flv	154
RgM_Rg_mul	170	RgV_to_FpV	106
RgM_Rg_sub	170	RgV_to_RgM	202
RgM_Rg_sub_shallow	170	RgV_to_RgX	201
RgM_shallowcopy	253	RgV_to_RgX_reverse	201
RgM_solve	173	RgV_to_ser	202
RgM_solve_realimag	173	RgV_to_str	239, 240
RgM_sqr	171	RgV_zc_mul	156
RgM_sub	170	RgV_zm_mul	156
RgM_sumcol	171	RgXnV_red_shallow	188
RgM_to_F2m	113	RgXn_eval	188
RgM_to_Flm	154	RgXn_exp	188
RgM_to_FpM	106	RgXn_inv	188
RgM_to_FqM	109	RgXn_inv_i	188
RgM_to_nfM	275	RgXn_mul	187, 188
RgM_to_RgXV	202	RgXn_powers	188
RgM_to_RgXX	202	RgXn_powu	188
RgM_transmul	171	RgXn_powu_i	188
RgM_transmultosym	171	RgXn_recip_shallow	187
RgM_type	106	RgXn_red_shallow	187
RgM_type2	106	RgXn_reverse	188
RgM_zc_mul	156	RgXn_sqr	187, 188
RgM_zm_mul	156	RgXn_sqrt	188
RgV_add	170	RgXQC_red	189

RgXQM_mul . . . . .	189	RgX_deflate_order . . . . .	184
RgXQM_red . . . . .	189	RgX_degree . . . . .	183
RgXQV_red . . . . .	189	RgX_deriv . . . . .	187
RgXQV_RgXQ_mul . . . . .	189	RgX_digits . . . . .	186
RgXQX_div . . . . .	190	RgX_disc . . . . .	187
RgXQX_divrem . . . . .	190	RgX_div . . . . .	185
RgXQX_mul . . . . .	190	RgX_divrem . . . . .	185
RgXQX_powers . . . . .	190	RgX_divs . . . . .	185
RgXQX_pseudodivrem . . . . .	186	RgX_div_by_X_x . . . . .	185
RgXQX_pseudorem . . . . .	186	RgX_equal . . . . .	183
RgXQX_red . . . . .	189	RgX_equal_var . . . . .	183
RgXQX_rem . . . . .	190	RgX_even_odd . . . . .	140, 183
RgXQX_RgXQ_mul . . . . .	190	RgX_extgcd . . . . .	186, 187
RgXQX_sqr . . . . .	190	RgX_extgcd_simple . . . . .	186
RgXQX_translate . . . . .	190	RgX_fpnorml2 . . . . .	187
RgXQ_charpoly . . . . .	189	RgX_gcd . . . . .	186
RgXQ_inv . . . . .	188	RgX_gcd_simple . . . . .	186
RgXQ_matrix_pow . . . . .	188	RgX_gtofp . . . . .	187
RgXQ_mul . . . . .	188	RgX_inflate . . . . .	184
RgXQ_norm . . . . .	189	RgX_integ . . . . .	187
RgXQ_pow . . . . .	188	RgX_isscalar . . . . .	183
RgXQ_powers . . . . .	188	RgX_is_FpX . . . . .	114
RgXQ_powu . . . . .	188	RgX_is_FpXQX . . . . .	119
RgXQ_ratlift . . . . .	189	RgX_is_monomial . . . . .	183
RgXQ_reverse . . . . .	189	RgX_is_QX . . . . .	183
RgXQ_sqr . . . . .	188	RgX_is_rational . . . . .	183
RgXV_RgV_eval . . . . .	185	RgX_is_ZX . . . . .	183
RgXV_to_RgM . . . . .	202	RgX_mul . . . . .	185
RgXV_unscale . . . . .	187	RgX_mulhigh_i . . . . .	188
RgXX_to_RgM . . . . .	202	RgX_muls . . . . .	185
RgXY_degreeex . . . . .	202	RgX_mulspec . . . . .	186
RgXY_swap . . . . .	202	RgX_mulXn . . . . .	186
RgXY_swapspec . . . . .	202	RgX_mul_i . . . . .	185
RgX_act_G12Q . . . . .	187	RgX_mul_normalized . . . . .	185
RgX_act_ZG12Q . . . . .	187	RgX_neg . . . . .	184
RgX_add . . . . .	184	RgX_nffix . . . . .	293
RgX_addmulXn . . . . .	186	RgX_normalize . . . . .	185
RgX_addmulXn_shallow . . . . .	186	RgX_pseudodivrem . . . . .	186
RgX_addspec . . . . .	186	RgX_pseudorem . . . . .	186
RgX_addspec_shallow . . . . .	186	RgX_recip . . . . .	183
RgX_blocks . . . . .	183	RgX_recip_shallow . . . . .	183
RgX_check_QX . . . . .	181	RgX_rem . . . . .	186
RgX_check_ZX . . . . .	176	RgX_renormalize . . . . .	183
RgX_check_ZXX . . . . .	180	RgX_renormalize_lg . . . . .	183
RgX_coeff . . . . .	183	RgX_rescale . . . . .	187
RgX_copy . . . . .	183	RgX_rescale_to_int . . . . .	184
RgX_cxeval . . . . .	221	RgX_resultant_all . . . . .	187
RgX_deflate . . . . .	183	RgX_RgMV_eval . . . . .	221
RgX_deflate_max . . . . .	184	RgX_RgM_eval . . . . .	221



RgX_RgV_eval . . . . .	185	Rg_is_FF . . . . .	227
RgX_RgXnV_eval . . . . .	188	Rg_is_Fp . . . . .	106
RgX_RgXn_eval . . . . .	188	Rg_is_FpXQ . . . . .	119
RgX_RgXQV_eval . . . . .	189	Rg_nffix . . . . .	292, 293
RgX_RgXQ_eval . . . . .	188, 189	Rg_RgC_sub . . . . .	170
RgX_Rg_add . . . . .	184	Rg_RgX_sub . . . . .	185
RgX_Rg_add_shallow . . . . .	184	Rg_to_F2 . . . . .	154
RgX_Rg_div . . . . .	185	Rg_to_F2xq . . . . .	154
RgX_Rg_divexact . . . . .	185	Rg_to_Fl . . . . .	153, 154
RgX_Rg_eval_bk . . . . .	185	Rg_to_Flxq . . . . .	154
RgX_Rg_mul . . . . .	185	Rg_to_Fp . . . . .	106, 107
RgX_Rg_sub . . . . .	185	Rg_to_FpXQ . . . . .	119
RgX_Rg_type . . . . .	105	Rg_to_Fq . . . . .	119
RgX_rotate_shallow . . . . .	184	Rg_to_RgC . . . . .	202
RgX_shift . . . . .	140, 184	Rg_type . . . . .	104
RgX_shift_inplace . . . . .	184	RM_round_maxrank . . . . .	269, 289, 304
RgX_shift_inplace_init . . . . .	184	rnfcomplete . . . . .	273
RgX_shift_shallow . . . . .	184	rnfdisc_factored . . . . .	292
RgX_splitting . . . . .	132, 183	rnfeltabstorel . . . . .	292
RgX_sqr . . . . .	185	rnfeltreltoabs . . . . .	292
RgX_sqrhigh_i . . . . .	188	rnfeltup . . . . .	292
RgX_sqrspec . . . . .	186	rnfequationall . . . . .	292
RgX_sqr_i . . . . .	185	rnf_build_nfabs . . . . .	273
RgX_sub . . . . .	184	rnf_COND . . . . .	291
RgX_sylvestermatrix . . . . .	208	rnf_get_absdegree . . . . .	270
RgX_to_F2x . . . . .	154	rnf_get_alpha . . . . .	271
RgX_to_Flx . . . . .	154	rnf_get_degree . . . . .	270
RgX_to_FlxqX . . . . .	154	rnf_get_disc . . . . .	271
RgX_to_FpX . . . . .	114	rnf_get_idealdisc . . . . .	271
RgX_to_FpXQX . . . . .	119	rnf_get_index . . . . .	271
RgX_to_FqX . . . . .	119	rnf_get_invzk . . . . .	271
RgX_to_nfX . . . . .	275	rnf_get_k . . . . .	271
RgX_to_RgC . . . . .	201	rnf_get_map . . . . .	271, 292
RgX_to_RgV . . . . .	202	rnf_get_nf . . . . .	271
RgX_to_ser . . . . .	202	rnf_get_nfdegree . . . . .	271
RgX_to_ser_inexact . . . . .	202	rnf_get_nfpol . . . . .	271
RgX_translate . . . . .	187	rnf_get_nfvarn . . . . .	271
RgX_type . . . . .	105	rnf_get_nfzk . . . . .	271, 292
RgX_type2 . . . . .	106	rnf_get_pol . . . . .	271
RgX_type3 . . . . .	106	rnf_get_polabs . . . . .	271, 273
RgX_type_decode . . . . .	105	rnf_get_varn . . . . .	271
RgX_type_is_composite . . . . .	105	rnf_get_zk . . . . .	271
RgX_unscale . . . . .	187	rnf_REL . . . . .	291
RgX_val . . . . .	184	rnf_zkabs . . . . .	273
RgX_valrem . . . . .	184	rootmod . . . . .	145
RgX_valrem_inexact . . . . .	184	rootmod0 . . . . .	145
Rg_col_ei . . . . .	204	rootmod2 . . . . .	145
Rg_get_0 . . . . .	33, 105	rootsof1pow . . . . .	232
Rg_get_1 . . . . .	105	rootsof1powinit . . . . .	232

rootsof1q_cx . . . . .	232
rootsof1u_cx . . . . .	232
rootsof1u_Fp . . . . .	99
rootsof1_cx . . . . .	232
rootsof1_Fl . . . . .	99
rootsof1_Fp . . . . .	99
roots_from_deg1 . . . . .	209
roots_to_pol . . . . .	209
roots_to_pol_r1 . . . . .	209
roundr . . . . .	84
roundr_safe . . . . .	85
row vector . . . . .	33
row . . . . .	253
rowcopy . . . . .	253
rowpermute . . . . .	254
rowslice . . . . .	253
rowslicepermute . . . . .	254
rowsplice . . . . .	254
row_i . . . . .	253
rtodbl . . . . .	28, 199
rtor . . . . .	84
R_abs . . . . .	223
R_abs_shallow . . . . .	223
R_from_QR . . . . .	173

## S

scalarcol . . . . .	204
scalarcol_shallow . . . . .	208
scalarmat . . . . .	204
scalarmat_s . . . . .	204
scalarmat_shallow . . . . .	208
scalarpol . . . . .	203
scalarpol_shallow . . . . .	208
scalarser . . . . .	203
scalar_Flm . . . . .	109
scalar_ZX . . . . .	176
scalar_ZX_shallow . . . . .	176
sdivsi . . . . .	93
sdivsi_rem . . . . .	93
sdivss_rem . . . . .	93
sdomain_isincl . . . . .	328
sd_breakloop . . . . .	262
sd_colors . . . . .	262
sd_compatible . . . . .	262
sd_datadir . . . . .	262
sd_debug . . . . .	262
sd_debugfiles . . . . .	262
sd_debugmem . . . . .	262

sd_echo . . . . .	262
sd_factor_add_primes . . . . .	262
sd_factor_proven . . . . .	262
sd_format . . . . .	262
sd_graphcolormap . . . . .	262
sd_graphcolors . . . . .	262
sd_help . . . . .	262
sd_histfile . . . . .	262
sd_histsize . . . . .	262
sd_intarray . . . . .	264
sd_lines . . . . .	262
sd_linewrap . . . . .	262
sd_log . . . . .	262
sd_logfile . . . . .	262
sd_nbthreads . . . . .	262
sd_new_galois_format . . . . .	262
sd_output . . . . .	263
sd_parisize . . . . .	263
sd_parisizemax . . . . .	263
sd_path . . . . .	263
sd_plothsizes . . . . .	263
sd_prettyprinter . . . . .	263
sd_primelimit . . . . .	263
sd_prompt . . . . .	263
sd_prompt_cont . . . . .	263
sd_psfile . . . . .	263
sd_readline . . . . .	263
sd_realbitprecision . . . . .	263
sd_realprecision . . . . .	263
sd_recover . . . . .	263
sd_secure . . . . .	263
sd_seriesprecision . . . . .	263
sd_simplify . . . . .	263
sd_sopath . . . . .	263
sd_strictargs . . . . .	263
sd_strictmatch . . . . .	263
sd_string . . . . .	264
sd_TeXstyle . . . . .	262
sd_threadsize . . . . .	263
sd_threadsizemax . . . . .	263
sd_timer . . . . .	263
sd_toggle . . . . .	263
sd_ulong . . . . .	264
secure . . . . .	53
serchop0 . . . . .	203
serchop_i . . . . .	203
sertoser . . . . .	225
ser_inv . . . . .	226
ser_isexactzero . . . . .	226



subgroups_tableset	236
subiu	90
subll	77
subllx	77
submulii	91
submuliu	91
submuliu_inplace	91
subresex	216
subui	90
subuu	90
sumdedekind	100
sumdedekind_coprime	100
sumdigitsu	100
switchin	54
switchout	241
szeta	231

## T

tablemul	276
tablemulvec	276
tablemul_ei	276
tablemul_ei_ej	276
tablesearch	214
tableset_find_index	236
tablesqr	276
teichmullerinit	149
term_color	242
term_get_color	242
texe	242
theta_get_an	327
theta_get_bitprec	327
theta_get_K	327
theta_get_m	327
theta_get_R	327
theta_get_sqrtN	328
theta_get_tdom	328
threads	337
timer	41
timer2	41
timer_delay	41
timer_get	41
timer_printf	41
timer_start	41
togglesign	60
togglesign_safe	60, 165, 166
to_famat	174, 277
to_famat_shallow	174, 278
Tp_to_FF	227

trans_eval	232
traverseheap	69
tridiv_bound	159
trivialgroup	235
trivial_fact	204
truecoef	224, 226
truecoeff	61
truedivii	91
truedivis	91
truedivsi	91
truedvmdii	93
truedvmdis	93
truedvmdsi	93
trunc2nr	85
trunc2nr_lg	85
truncr	84
trunc_safe	84
TWOPOTBITS_IN_LONG	62
typ	28, 57
TYPBITS	62
type number	28
type	14
type_name	57
TYPnumBITS	62
TYPSHIFT	62
typ_BNF	267
typ_BNR	267
typ_NF	267
t_CLOSURE	34
t_COL	33
t_COMPLEX	31
t_ELL_Fp	311
t_ELL_Fq	311, 317
t_ELL_NF	311
t_ELL_Q	311
t_ELL_Qp	311
t_ELL_Rg	312
t_ERROR	34
t_FFELT	31
t_FF_F2xq	31
t_FF_Flxq	31
t_FF_FpXQ	31
t_FRAC	31
t_INFINITY	34
t_INT	29
t_INTMOD	31
t_LIST	34
t_MAT	34
t_PADIC	31

t_POL	32
t_POLMOD	32
t_QFI	33
t_QFR	33
t_QUAD	31
t_REAL	30
t_RFRAC	33
t_SER	33
t_STR	34
t_VEC	33
t_VECSMALL	34

## U

uabsdiviu_rem	93
uabsdivui_rem	93
udivuu_rem	93
ugcd	95
ugcdiu	95
ugcdui	96
uislucaspsp	100
uisprime	163
uisprimepower	100
uisprime_101	163
uisprime_661	163
uissquare	157
uissquareall	157
uissquarefree	100
uissquarefree_fact	100
uis_357_power	100
uis_357_powermod	100
ulcm	96
ulogint	95
ulogintall	95
ulong	49
ULONG_MAX	62
umodi2n	94
umodiu	94
umodsu	94
umodui	94
umuluu_le	91
umuluu_or_0	91
unegisfundamental	100
unextprime	162
unsetisclone	28
uordinal	239
uposisfundamental	100
upowuu	95
upper_to_cx	199

uprecprime	162
uprime	162
uprimepi	162
upr_norm	281
usqrt	157
usqrtn	157
usumdivk_fact	100
usumdiv_fact	100
utoi	83
utoineg	83
utoipos	83
utor	83
uu32toi	26, 84
uu32toineg	84
uutoi	84
uutoineg	84
u_chinese_coprime	145
u_forprime_arith_init	164
u_forprime_init	43, 164
u_forprime_next	43, 164
u_forprime_restrict	164
u_lval	86
u_lvalrem	86
u_lvalrem_stop	86
u_ppo	160
u_pval	86
u_pvalrem	86
u_sumdedekind_coprime	100

## V

vali	85
valp	31, 33, 58
VALPBITS	62
VALPnumBITS	62
vals	85
varargs	25
varhigher	35
variable (priority)	34
variable (temporary)	36
variable (user)	35
variable number	32, 35, 72
varlower	35
varn	32, 33, 35, 58
VARNBITS	62
varncmp	35
varnmax	37
varnmin	37
VARNnumBITS	62



ZabM_inv_ratlift . . . . .	277	zero_zv . . . . .	169
ZabM_ker . . . . .	276	zero_zx . . . . .	182
ZabM_pseudoinv . . . . .	276	ZGCs_add . . . . .	175
zCs_to_ZC . . . . .	175	ZGC_G_mul . . . . .	174
ZC_add . . . . .	164	ZGC_G_mul_inplace . . . . .	174
ZC_copy . . . . .	165	ZGC_Z_mul . . . . .	174
ZC_hnfrem . . . . .	305	ZG_add . . . . .	174
ZC_hnfremdiv . . . . .	304	ZG_G_mul . . . . .	174
ZC_is_ei . . . . .	168	ZG_mul . . . . .	174
ZC_lincomb . . . . .	165	ZG_neg . . . . .	174
ZC_lincomb1_inplace . . . . .	165	ZG_normalize . . . . .	174
ZC_lincomb1_inplace_i . . . . .	165	ZG_sub . . . . .	174
ZC_neg . . . . .	165	ZG_Z_mul . . . . .	174
ZC_nfval . . . . .	284	Zideallog . . . . .	307
ZC_nfvalrem . . . . .	284	zidealstar . . . . .	294
ZC_prdvd . . . . .	284	zidealstarinit . . . . .	294
ZC_Q_mul . . . . .	167	zidealstarinitgen . . . . .	294
ZC_reducemodlll . . . . .	305	zkchinese . . . . .	280
ZC_reducemodmatrix . . . . .	305	zkchinese1 . . . . .	280
ZC_sub . . . . .	164	zkchineseinit . . . . .	280
zc_to_ZC . . . . .	155	zkC_multable_mul . . . . .	275
ZC_union_shallow . . . . .	166	zkmodprinit . . . . .	283
ZC_ZV_mul . . . . .	165	zkmultable_capZ . . . . .	276
ZC_Z_add . . . . .	165	zkmultable_inv . . . . .	276
ZC_Z_div . . . . .	165	zk_inv . . . . .	276
ZC_Z_divexact . . . . .	165	zk_multable . . . . .	275, 279
ZC_z_mul . . . . .	156	zk_scalar_or_multable . . . . .	276, 281
ZC_Z_mul . . . . .	165	zk_to_Fq . . . . .	283
ZC_Z_sub . . . . .	165	zk_to_Fq_init . . . . .	283
zerocol . . . . .	203	zlm_echelon . . . . .	152
zeromat . . . . .	204	ZlM_gauss . . . . .	152
zeromatcopy . . . . .	204	ZMrow_ZC_mul . . . . .	167
zeropadic . . . . .	203	zMs_to_ZM . . . . .	175
zeropadic_shallow . . . . .	208	zMs_ZC_mul . . . . .	175
zeropol . . . . .	203	ZMV_to_FlmV . . . . .	169
zeroser . . . . .	203	ZMV_to_zmV . . . . .	169
zerovec . . . . .	203	zmV_to_ZMV . . . . .	169
zerovec_block . . . . .	204	ZM_add . . . . .	166
zero_F2m . . . . .	113	ZM_charpoly . . . . .	167
zero_F2m_copy . . . . .	113	ZM_copy . . . . .	166
zero_F2v . . . . .	113	zm_copy . . . . .	169
zero_F2x . . . . .	139	ZM_det . . . . .	167
zero_Flm . . . . .	111	ZM_detmult . . . . .	167
zero_Flm_copy . . . . .	111	ZM_det_triangular . . . . .	168
zero_Flv . . . . .	111	ZM_diag_mul . . . . .	166
zero_Flx . . . . .	131	ZM_equal . . . . .	166
zero_FlxC . . . . .	133	ZM_equal0 . . . . .	166
zero_FlxM . . . . .	133	ZM_famat_limit . . . . .	278
zero_zm . . . . .	169	ZM_gauss . . . . .	168

ZM_hnf . . . . .	299, 302	ZM_snfclean . . . . .	301
ZM_hnfall . . . . .	299, 300, 302	ZM_snf_group . . . . .	301, 302
ZM_hnfall_i . . . . .	300	ZM_sqr . . . . .	166
ZM_hnfcenter . . . . .	300	ZM_sub . . . . .	166
ZM_hnfdivrem . . . . .	305	ZM_supnorm . . . . .	167, 220
ZM_hnflll . . . . .	300	ZM_togglesign . . . . .	166
ZM_hnfmod . . . . .	299, 302	ZM_to_F2m . . . . .	113
ZM_hnfmodall . . . . .	299, 300	ZM_to_Flm . . . . .	154
ZM_hnfmodall_i . . . . .	300	zm_to_Flm . . . . .	155
ZM_hnfmodid . . . . .	299, 302	ZM_to_zm . . . . .	154
ZM_hnfmodprime . . . . .	299	zm_to_ZM . . . . .	155
ZM_hnfperm . . . . .	300	zm_to_zxV . . . . .	156
ZM_hnfrem . . . . .	305	ZM_transmul . . . . .	167
ZM_hnf_knapsack . . . . .	300	ZM_transmultosym . . . . .	166
ZM_imagecompl . . . . .	167	zm_transpose . . . . .	169
ZM_incremental_CRT . . . . .	146	ZM_zc_mul . . . . .	156
ZM_indeximage . . . . .	167	ZM_ZC_mul . . . . .	166
ZM_indexrank . . . . .	167	zm_zc_mul . . . . .	169
ZM_init_CRT . . . . .	146	ZM_zm_mul . . . . .	156
ZM_inv . . . . .	167	ZM_ZX_mul . . . . .	166
ZM_inv_ratlift . . . . .	167	ZM_Z_div . . . . .	167
ZM_isdiagonal . . . . .	168	ZM_Z_divexact . . . . .	167
ZM_ishnf . . . . .	168	ZM_Z_mul . . . . .	166
ZM_isidentity . . . . .	168	zncharcheck . . . . .	306
ZM_isscalar . . . . .	168	zncharconj . . . . .	306
ZM_ker . . . . .	167	znchardiv . . . . .	306
ZM_lll . . . . .	303	znchareval . . . . .	307
ZM_lll_norms . . . . .	304	zncharker . . . . .	307
ZM_max_lg . . . . .	167	zncharmul . . . . .	307
ZM_mul . . . . .	166	zncharorder . . . . .	307
zm_mul . . . . .	169	zncharpow . . . . .	307
ZM_multosym . . . . .	166	znchar_quad . . . . .	307
ZM_mul_diag . . . . .	166	znconreyfromchar . . . . .	307
ZM_neg . . . . .	166	znconreyfromchar_normalized . . . . .	307
ZM_nm_mul . . . . .	156	znconreylog_normalize . . . . .	307
ZM_nv_mod_tree . . . . .	147	znconrey_check . . . . .	307
ZM_permanent . . . . .	167	znconrey_normalized . . . . .	307
zm_permanent . . . . .	169	znstar_get_conreycyc . . . . .	272
ZM_pow . . . . .	167	znstar_get_conreygen . . . . .	272
ZM_powu . . . . .	167	znstar_get_cyc . . . . .	272
ZM_pseudoinv . . . . .	167	znstar_get_faN . . . . .	272
ZM_Q_mul . . . . .	167	znstar_get_gen . . . . .	272
ZM_rank . . . . .	167	znstar_get_N . . . . .	272
ZM_reducemodlll . . . . .	305	znstar_get_no . . . . .	272
ZM_reducemodmatrix . . . . .	305	znstar_get_pe . . . . .	272
zm_row . . . . .	169	znstar_get_U . . . . .	272
ZM_snf . . . . .	301	znstar_get_Ui . . . . .	272
ZM_snfall . . . . .	301	Zn_ispower . . . . .	99
ZM_snfall_i . . . . .	301	Zn_issquare . . . . .	99



Zn_sqrt . . . . .	98	ZV_content . . . . .	165
ZpMs_ZpCs_solve . . . . .	176	zv_content . . . . .	169
ZpM_echelon . . . . .	152	zv_copy . . . . .	169
ZpXQM_prodFrobenius . . . . .	151	ZV_dotproduct . . . . .	165
ZpXQX_digits . . . . .	152	zv_dotproduct . . . . .	169
ZpXQX_divrem . . . . .	152	ZV_dotsquare . . . . .	165
ZpXQX_liftfact . . . . .	151	ZV_dvd . . . . .	165
ZpXQX_liftroot . . . . .	151, 152	ZV_equal . . . . .	164
ZpXQX_liftroot_vald . . . . .	152	zv_equal . . . . .	169
ZpXQX_roots . . . . .	152	ZV_equal0 . . . . .	164
ZpXQ_div . . . . .	151	zv_equal0 . . . . .	169
ZpXQ_inv . . . . .	151	ZV_extgcd . . . . .	96, 165
ZpXQ_invlift . . . . .	151	ZV_indexsort . . . . .	166
ZpXQ_log . . . . .	151	ZV_isscalar . . . . .	172
ZpXQ_sqrt . . . . .	151	ZV_lval . . . . .	87
ZpXQ_sqrtnlift . . . . .	151	ZV_lvalrem . . . . .	87
ZpX_disc_val . . . . .	150	ZV_max_lg . . . . .	165
ZpX_Frobenius . . . . .	150	zv_neg . . . . .	168
ZpX_gcd . . . . .	150	ZV_neg_inplace . . . . .	165
ZpX_liftfact . . . . .	150	zv_neg_inplace . . . . .	168
ZpX_liftroot . . . . .	149, 150, 152	ZV_nv_mod_tree . . . . .	147
ZpX_liftroots . . . . .	150	ZV_prod . . . . .	165
ZpX_monic_factor . . . . .	151	zv_prod . . . . .	169
ZpX_reduced_resultant . . . . .	150	ZV_producttree . . . . .	147, 148
ZpX_reduced_resultant_fast . . . . .	150	zv_prod_Z . . . . .	169
ZpX_resultant_val . . . . .	150	ZV_pval . . . . .	87
ZpX_roots . . . . .	150	ZV_pvalrem . . . . .	87
ZpX_ZpXQ_liftroot . . . . .	151	ZV_search . . . . .	166
ZpX_ZpXQ_liftroot_ea . . . . .	151	zv_search . . . . .	169
Zp_issquare . . . . .	99	ZV_snfall . . . . .	301
Zp_sqrt . . . . .	149	ZV_snf_group . . . . .	301
Zp_sqrtlift . . . . .	149	ZV_snf_trunc . . . . .	301
Zp_sqrtnlift . . . . .	149	ZV_sort . . . . .	165, 166
Zp_treichmuller . . . . .	149	ZV_sort_inplace . . . . .	165
ZqX_liftfact . . . . .	152	ZV_sort_uniq . . . . .	165
ZqX_liftroot . . . . .	152	ZV_sum . . . . .	165
ZqX_roots . . . . .	152	zv_sum . . . . .	169
Zq_sqrtnlift . . . . .	151	ZV_togglesign . . . . .	165
zvV_equal . . . . .	169	ZV_to_F2v . . . . .	113
ZV_abscmp . . . . .	164	ZV_to_Flv . . . . .	154
ZV_allpnqn . . . . .	96	zv_to_Flv . . . . .	155
ZV_cba . . . . .	160	ZV_to_nv . . . . .	154
ZV_cba_extend . . . . .	160	ZV_to_zv . . . . .	154
ZV_chinese . . . . .	147	zv_to_ZV . . . . .	155
ZV_chinesetree . . . . .	148	zv_to_zx . . . . .	156
ZV_chinese_center . . . . .	147	ZV_union_shallow . . . . .	166
ZV_chinese_tree . . . . .	148	ZV_zc_mul . . . . .	156
ZV_cmp . . . . .	164, 214	ZV_zMs_mul . . . . .	175
zv_cmp0 . . . . .	169	zv_ZM_mul . . . . .	156

ZV_ZM_mul . . . . .	167	ZX_equal . . . . .	176, 179
ZV_Z_dvd . . . . .	87	ZX_equal1 . . . . .	176
zv_z_mul . . . . .	169	ZX_eval1 . . . . .	178
ZXC_nv_mod_tree . . . . .	147	ZX_factor . . . . .	178
ZXM_incremental_CRT . . . . .	146	ZX_gcd . . . . .	177
ZXM_init_CRT . . . . .	146	ZX_gcd_all . . . . .	177
ZXM_nv_mod_tree . . . . .	147	ZX_graeffe . . . . .	178
ZXn_mul . . . . .	180	ZX_incremental_CRT . . . . .	146
ZXn_sqr . . . . .	180	ZX_init_CRT . . . . .	146
ZXQM_mul . . . . .	180	ZX_is_irred . . . . .	178
ZXQM_sqr . . . . .	180	ZX_is_monic . . . . .	176
ZXQX_dvd . . . . .	186	ZX_is_squarefree . . . . .	178
ZXQX_mul . . . . .	180	ZX_lval . . . . .	87
ZXQX_sqr . . . . .	180	ZX_lvalrem . . . . .	87
ZXQ_charpoly . . . . .	179	ZX_max_lg . . . . .	176
ZXQ_mul . . . . .	179	ZX_mod_Xnm1 . . . . .	177
ZXQ_sqr . . . . .	179	ZX_mul . . . . .	177, 181
ZXT_remi2n . . . . .	179	ZX_mulspec . . . . .	177
ZXT_to_FlxT . . . . .	154	ZX_mulu . . . . .	177
ZXV_dotproduct . . . . .	179	ZX_neg . . . . .	176
ZXV_equal . . . . .	179	ZX_nv_mod_tree . . . . .	147
ZXV_remi2n . . . . .	179	ZX_primitive_to_monic . . . . .	177
ZXV_to_FlxV . . . . .	154	ZX_pval . . . . .	87
ZXV_Z_mul . . . . .	179	ZX_pvalrem . . . . .	87
ZXXT_to_FlxXT . . . . .	154	ZX_Q_mul . . . . .	178
ZXXV_to_FlxXV . . . . .	154	ZX_Q_normalize . . . . .	178, 285
ZXX_max_lg . . . . .	180	ZX_radical . . . . .	177
ZXX_mul_Kronecker . . . . .	181	ZX_rem . . . . .	177
ZXX_nv_mod_tree . . . . .	147	ZX_remi2n . . . . .	177
ZXX_renormalize . . . . .	180	ZX_renormalize . . . . .	176
ZXX_sqr_Kronecker . . . . .	181	zx_renormalize . . . . .	182
ZXX_to_F2xX . . . . .	142	ZX_rescale . . . . .	178
ZXX_to_FlxX . . . . .	154	ZX_rescale2n . . . . .	178
zxX_to_FlxX . . . . .	154	ZX_rescale_lt . . . . .	178
zxX_to_Kronecker . . . . .	136	ZX_resultant . . . . .	178
ZXX_to_Kronecker . . . . .	180	zx_shift . . . . .	182
ZXX_to_Kronecker_spec . . . . .	180	ZX_shifti . . . . .	177
ZXX_Z_add_shallow . . . . .	180	ZX_sqr . . . . .	177, 181
ZXX_Z_divexact . . . . .	180	ZX_sqrspec . . . . .	177
ZXX_Z_mul . . . . .	180	ZX_squff . . . . .	178
ZX_add . . . . .	176	ZX_sturm . . . . .	179
ZX_compositum_disjoint . . . . .	179	ZX_sturmpart . . . . .	179
ZX_content . . . . .	177	ZX_sub . . . . .	176
ZX_copy . . . . .	176	ZX_to_F2x . . . . .	139
ZX_deflate_max . . . . .	178	ZX_to_Flx . . . . .	154
ZX_deflate_order . . . . .	178	zx_to_Flx . . . . .	155
ZX_deriv . . . . .	178	ZX_to_monic . . . . .	177
ZX_disc . . . . .	178	zx_to_zv . . . . .	156
ZX_div_by_X_1 . . . . .	177	zx_to_ZX . . . . .	155

ZX_translate . . . . .	178	Z_ppgle . . . . .	160
ZX_unscale . . . . .	178	Z_ppio . . . . .	160
ZX_unscale2n . . . . .	178	Z_ppo . . . . .	160
ZX_unscale_div . . . . .	178	Z_pval . . . . .	86
ZX_Uspensky . . . . .	178	z_pval . . . . .	86
ZX_val . . . . .	177	Z_pvalrem . . . . .	86
ZX_valrem . . . . .	177	z_pvalrem . . . . .	86
ZX_Zp_root . . . . .	150	Z_smoother . . . . .	158
ZX_ZXY_resultant . . . . .	179	Z_to_F2x . . . . .	139
ZX_ZXY_rnfequation . . . . .	179	Z_to_Flx . . . . .	156
ZX_Z_add . . . . .	176	Z_to_FpX . . . . .	116
ZX_Z_add_shallow . . . . .	176	Z_to_perm . . . . .	234
ZX_Z_divexact . . . . .	177	Z_ZC_sub . . . . .	165
ZX_Z_mul . . . . .	177	Z_ZV_mod . . . . .	146
ZX_Z_normalize . . . . .	177, 178	Z_ZV_mod_tree . . . . .	147
ZX_Z_sub . . . . .	176	Z_ZX_sub . . . . .	177
ZX_z_unscale . . . . .	178		
Z_cba . . . . .	160		
Z_chinese . . . . .	145	_evalexpo . . . . .	59
Z_chinese_all . . . . .	145	_evallg . . . . .	59
Z_chinese_coprime . . . . .	145	_evalprecp . . . . .	59
Z_chinese_post . . . . .	145	_evalvalp . . . . .	59
Z_chinese_pre . . . . .	145		
Z_content . . . . .	216		
Z_ECM . . . . .	159		
Z_factor . . . . .	158, 159, 160		
Z_factor_limit . . . . .	158, 160		
Z_factor_listP . . . . .	158		
Z_factor_until . . . . .	158		
Z_FF_div . . . . .	228		
Z_incremental_CRT . . . . .	146		
Z_init_CRT . . . . .	146		
Z_isanypower . . . . .	157, 160		
Z_isfundamental . . . . .	162		
Z_ispow2 . . . . .	157		
Z_ispower . . . . .	157		
Z_ispowerall . . . . .	157		
Z_issmooth . . . . .	158		
Z_issmooth_fact . . . . .	158		
Z_issquare . . . . .	157		
Z_issquareall . . . . .	157		
Z_issquarefree . . . . .	162		
Z_lval . . . . .	86		
z_lval . . . . .	86		
Z_lvalrem . . . . .	86		
z_lvalrem . . . . .	86		
Z_lvalrem_stop . . . . .	86		
Z_nv_mod . . . . .	147		
Z_pollardbrent . . . . .	159		