

# Algebraic Number Theory

(PARI-GP version 2.11.0)

## Binary Quadratic Forms

create  $ax^2 + bxy + cy^2$  (distance  $d$ )      `Qfb( $a, b, c, \{d\}$ )`  
reduce  $x$  ( $s = \sqrt{D}$ ,  $l = \lfloor s \rfloor$ )      `qfbred( $x, \{flag\}, \{D\}, \{l\}, \{s\}$ )`  
return  $[y, g]$ ,  $g \in \text{SL}_2(\mathbf{Z})$ ,  $y = g \cdot x$  reduced      `qfbreds12( $x$ )`  
composition of forms       $x*y$  or `qfbnucomp( $x, y, l$ )`  
 $n$ -th power of form       $x^n$  or `qfbnupow( $x, n$ )`  
composition without reduction      `qfbcomprow( $x, y$ )`  
 $n$ -th power without reduction      `qfbpowrow( $x, n$ )`  
prime form of disc.  $x$  above prime  $p$       `qfbprimeform( $x, p$ )`  
class number of disc.  $x$       `qfbclassno( $x$ )`  
Hurwitz class number of disc.  $x$       `qfbhclassno( $x$ )`  
solve  $Q(x, y) = p$  in integers,  $p$  prime      `qfbsolve( $Q, p$ )`

## Quadratic Fields

quadratic number  $\omega = \sqrt{x}$  or  $(1 + \sqrt{x})/2$       `quadgen( $x$ )`  
minimal polynomial of  $\omega$       `quadpoly( $x$ )`  
discriminant of  $\mathbf{Q}(\sqrt{x})$       `quaddisc( $x$ )`  
regulator of real quadratic field      `quadregulator( $x$ )`  
fundamental unit in real  $\mathbf{Q}(\sqrt{D})$       `quadunit( $D, \{w\}$ )`  
class group of  $\mathbf{Q}(\sqrt{D})$       `quadclassunit( $D, \{flag\}, \{t\}$ )`  
Hilbert class field of  $\mathbf{Q}(\sqrt{D})$       `quadhilbert( $D, \{flag\}$ )`  
... using specific class invariant ( $D < 0$ )      `polclass( $D, \{inv\}$ )`  
ray class field modulo  $f$  of  $\mathbf{Q}(\sqrt{D})$       `quadray( $D, f, \{flag\}$ )`

## General Number Fields: Initializations

The number field  $K = \mathbf{Q}[X]/(f)$  is given by irreducible  $f \in \mathbf{Q}[X]$ . We denote  $\theta = \bar{X}$  the canonical root of  $f$  in  $K$ . A  $nf$  structure contains a maximal order and allows operations on elements and ideals. A  $bnf$  adds class group and units. A  $bnr$  is attached to ray class groups and class field theory. A  $rnf$  is attached to relative extensions  $L/K$ .

init number field structure  $nf$       `nfinit( $f, \{flag\}$ )`  
known integer basis  $B$       `nfinit( $[f, B]$ )`  
order maximal at  $vp = [p_1, \dots, p_k]$       `nfinit( $[f, vp]$ )`  
order maximal at all  $p \leq P$       `nfinit( $[f, P]$ )`  
certify maximal order      `nfcertify( $nf$ )`

### nf members:

a monic  $F \in \mathbf{Z}[X]$  defining  $K$        $nf.pol$   
number of real/complex places       $nf.r1/r2/sign$   
discriminant of  $nf$        $nf.disc$   
 $T_2$  matrix       $nf.t2$   
complex roots of  $F$        $nf.roots$   
integral basis of  $\mathbf{Z}_K$  as powers of  $\theta$        $nf.zk$   
different/codifferent       $nf.diff, nf.codiff$   
index  $[\mathbf{Z}_K : \mathbf{Z}[X]/(F)]$        $nf.index$   
recompute  $nf$  using current precision      `nfnewprec( $nf$ )`  
init relative  $rnf$   $L = K[Y]/(g)$       `rnfinit( $nf, g$ )`  
init  $bnf$  structure      `bnfinit( $f, \{flag\}$ )`  
**bnf members:** same as  $nf$ , plus  
underlying  $nf$        $bnf.nf$   
classgroup       $bnf.clgp$   
regulator       $bnf.reg$   
fundamental/torsion units       $bnf.fu, bnf.tu$

compress a  $bnf$  for storage      `bnfcompress( $bnf$ )`  
recover a  $bnf$  from compressed  $bnfz$       `bnfinit( $bnfz$ )`  
add  $S$ -class group and units, yield  $bnfS$       `bnfsunit( $bnf, S$ )`  
init class field structure  $bnr$       `bnrinit( $bnf, m, \{flag\}$ )`  
**bnr members:** same as  $bnf$ , plus  
underlying  $bnf$        $bnr.bnf$   
big ideal structure       $bnr.bid$   
modulus       $bnr.mod$   
structure of  $(\mathbf{Z}_K/m)^*$        $bnr.zkst$

## Fields, subfields, embeddings

**Defining polynomials, embeddings**  
smallest poly defining  $f = 0$  (slow)      `polredabs( $f, \{flag\}$ )`  
small poly defining  $f = 0$  (fast)      `polredbest( $f, \{flag\}$ )`  
random Tschirnhausen transform of  $f$       `poltschirnhaus( $f$ )`  
 $\mathbf{Q}[t]/(f) \subset \mathbf{Q}[t]/(g)$  ? Isomorphic?      `nfisincl( $f, g$ ), nfisom`  
reverse polmod  $a = A(t) \bmod T(t)$       `modreverse( $a$ )`  
compositum of  $\mathbf{Q}[t]/(f)$ ,  $\mathbf{Q}[t]/(g)$       `polcompositum( $f, g, \{flag\}$ )`  
compositum of  $K[t]/(f)$ ,  $K[t]/(g)$       `nfcompositum( $nf, f, g, \{flag\}$ )`  
splitting field of  $K$  (degree divides  $d$ )      `nfsplitting( $nf, \{d\}$ )`  
signs of real embeddings of  $x$       `nfeltsign( $nf, x, \{pl\}$ )`  
complex embeddings of  $x$       `nfeltembed( $nf, x, \{pl\}$ )`  
 $T \in K[t]$ , # of real roots of  $\sigma(T) \in R[t]$       `nfpolsturm( $nf, T, \{pl\}$ )`

### Subfields, polynomial factorization

subfields (of degree  $d$ ) of  $nf$       `nfsubfields( $nf, \{d\}$ )`  
 $d$ -th degree subfield of  $\mathbf{Q}(\zeta_n)$       `polsubcyclo( $n, d, \{v\}$ )`  
roots of unity in  $nf$       `nfrootsof1( $nf$ )`  
roots of  $g$  belonging to  $nf$       `nfroots( $nf, g$ )`  
factor  $g$  in  $nf$       `nfactor( $nf, g$ )`  
factor  $g \bmod$  prime  $pr$  in  $nf$       `nfactormod( $nf, g, pr$ )`

### Linear and algebraic relations

poly of degree  $\leq k$  with root  $x \in \mathbf{C}$       `algdep( $x, k$ )`  
alg. dep. with pol. coeffs for series  $s$       `seralgdep( $s, x, y$ )`  
small linear rel. on coords of vector  $x$       `lindep( $x$ )`

## Basic Number Field Arithmetic (nf)

Number field elements are `t_INT`, `t_FRAC`, `t_POL`, `t_POLMOD`, or `t_COL` (on integral basis  $nf.zk$ ).

### Basic operations

$x + y$       `nfeltadd( $nf, x, y$ )`  
 $x \times y$       `nfeltmul( $nf, x, y$ )`  
 $x^n$ ,  $n \in \mathbf{Z}$       `nfeltpow( $nf, x, n$ )`  
 $x/y$       `nfeltdiv( $nf, x, y$ )`  
 $q = x \backslash y := \text{round}(x/y)$       `nfeltdiveuc( $nf, x, y$ )`  
 $r = x \% y := x - (x \backslash y)y$       `nfeltmod( $nf, x, y$ )`  
...  $[q, r]$  as above      `nfeltdivrem( $nf, x, y$ )`  
reduce  $x$  modulo ideal  $A$       `nfeltreduce( $nf, x, A$ )`  
absolute trace  $\text{Tr}_{K/\mathbf{Q}}(x)$       `nfelttrace( $nf, x$ )`  
absolute norm  $N_{K/\mathbf{Q}}(x)$       `nfeltnorm( $nf, x$ )`

### Multiplicative structure of $K^*$ ; $K^*/(K^*)^n$

valuation  $vp_{\mathfrak{p}}(x)$       `nfeltval( $nf, x, \mathfrak{p}$ )`  
... write  $x = \pi^{vp_{\mathfrak{p}}(x)}y$       `nfeltval( $nf, x, \mathfrak{p}, \&y$ )`  
quadratic Hilbert symbol (at  $\mathfrak{p}$ )      `nfhilbert( $nf, a, b, \{\mathfrak{p}\}$ )`  
 $b$  such that  $xb^n = v$  is small      `idealredmodpower( $nf, x, n$ )`

### Maximal order and discriminant

integral basis of field  $\mathbf{Q}[x]/(f)$       `nfbasis( $f$ )`  
field discriminant of field  $f = 0$       `nfdisc( $f$ )`  
express  $x$  on integer basis      `nfalgtobasis( $nf, x$ )`  
express element  $x$  as a polmod      `nfbasistoalg( $nf, x$ )`

### Dedekind Zeta Function $\zeta_K$ , Hecke $L$ series

$R = [c, w, h]$  in initialization means we restrict  $s \in \mathbf{C}$  to domain  $|\Re(s) - c| < w$ ,  $|\Im(s)| < h$ ;  $R = [w, h]$  encodes  $[1/2, w, h]$  and  $[h]$  encodes  $R = [1/2, 0, h]$  (critical line up to height  $h$ ).  
 $\zeta_K$  as Dirichlet series,  $N(I) < b$       `dirzetak( $nf, b$ )`  
init  $\zeta_K^{(k)}(s)$  for  $k \leq n$       `L = lfunitinit( $bnf, R, \{n = 0\}$ )`  
compute  $\zeta_K(s)$  ( $n$ -th derivative)      `lfun( $L, s, \{n = 0\}$ )`  
compute  $\Lambda_K(s)$  ( $n$ -th derivative)      `lfunlambda( $L, s, \{n = 0\}$ )`

init  $L_K^{(k)}(s, \chi)$  for  $k \leq n$       `L = lfunitinit( $[bnr, chi], R, \{n = 0\}$ )`  
compute  $L_K(s, \chi)$  ( $n$ -th derivative)      `lfun( $L, s, \{n\}$ )`  
Artin root number of  $K$       `bnrrootnumber( $bnr, chi, \{flag\}$ )`  
 $L(1, \chi)$ , for all  $\chi$  trivial on  $H$       `bnrL1( $bnr, \{H\}, \{flag\}$ )`

## Class Groups & Units (bnf, bnr)

Class field theory data  $a_1, \{a_2\}$  is usually  $bnr$  (ray class field),  $bnr, H$  (congruence subgroup) or  $bnr, \chi$  (character on `bnr.clgp`). Any of these define a unique abelian extension of  $K$ .

remove GRH assumption from  $bnf$       `bnfcertify( $bnf$ )`  
expo. of ideal  $x$  on class gp      `bnfisprincipal( $bnf, x, \{flag\}$ )`  
expo. of ideal  $x$  on ray class gp      `bnrisprincipal( $bnr, x, \{flag\}$ )`  
expo. of  $x$  on fund. units      `bnfisunit( $bnf, x$ )`  
as above for  $S$ -units      `bnfissunit( $bnfs, x$ )`  
signs of real embeddings of  $bnf.fu$       `bnfsignunit( $bnf$ )`  
narrow class group      `bnfnarrow( $bnf$ )`

### Class Field Theory

ray class number for modulus  $m$       `bnrclassno( $bnf, m$ )`  
discriminant of class field      `bnrdisc( $a_1, \{a_2\}$ )`  
ray class numbers,  $l$  list of moduli      `bnrclassnolist( $bnf, l$ )`  
discriminants of class fields      `bnrdisclist( $bnf, l, \{arch\}, \{flag\}$ )`  
decode output from `bnrdisclist`      `bnfdecodemodule( $nf, fa$ )`  
is modulus the conductor?      `bnrconductor( $a_1, \{a_2\}$ )`  
is class field  $(bnr, H)$  Galois over  $K^G$       `bnrisgalois( $bnr, G, H$ )`  
action of automorphism on  $bnr.gen$       `bnrgaloismatrix( $bnr, aut$ )`  
apply `bnrgaloismatrix`  $M$  to  $H$       `bnrgaloisapply( $bnr, M, H$ )`  
characters on `bnr.clgp` s.t.  $\chi(g_i) = e(v_i)$       `bnrchar( $bnr, g, \{v\}$ )`  
conductor of character  $\chi$       `bnrconductor( $bnr, chi$ )`  
conductor of extension      `bnrconductor( $a_1, \{a_2\}, \{flag\}$ )`  
conductor of extension  $K[Y]/(g)$       `rnfconductor( $bnf, g$ )`  
Artin group of extension  $K[Y]/(g)$       `rnfnormgroup( $bnr, g$ )`  
subgroups of  $bnr$ , index  $\leq b$       `subgrouplist( $bnr, b, \{flag\}$ )`  
rel. eq. for class field def'd by  $sub$       `rnfkummer( $bnr, sub, \{d\}$ )`  
same, using Stark units (real field)      `bnrstark( $bnr, sub, \{flag\}$ )`  
is  $a$  an  $n$ -th power in  $K_v$  ?      `nfislocalpower( $nf, v, a, n$ )`  
cyclic  $L/K$  satisf. local conditions      `nfgrunwaldwang( $nf, P, D, pl$ )`

### Logarithmic class group

logarithmic  $\ell$ -class group      `bnflog( $bnf, \ell$ )`  
 $[\bar{e}(F_v/Q_p), \bar{f}(F_v/Q_p)]$       `bnflogef( $bnf, pr$ )`  
 $\exp \deg_F(A)$       `bnflogdegree( $bnf, A, \ell$ )`  
is  $\ell$ -extension  $L/K$  locally cyclotomic      `rnfislocalcyclo( $rnf$ )`

**Ideals:** elements, primes, or matrix of generators in HNF

is  $id$  an ideal in  $nf$  ? nfsideal( $nf, id$ )  
is  $x$  principal in  $bnf$  ? bnfisprincipal( $bnf, x$ )  
give  $[a, b]$ , s.t.  $a\mathbf{Z}_K + b\mathbf{Z}_K = x$  idealtwoelt( $nf, x, \{a\}$ )  
put ideal  $a$  ( $a\mathbf{Z}_K + b\mathbf{Z}_K$ ) in HNF form idealhnf( $nf, a, \{b\}$ )  
norm of ideal  $x$  idealnrm( $nf, x$ )  
minimum of ideal  $x$  (direction  $v$ ) idealmin( $nf, x, v$ )  
LLL-reduce the ideal  $x$  (direction  $v$ ) idealred( $nf, x, \{v\}$ )

**Ideal Operations**

add ideals  $x$  and  $y$  idealadd( $nf, x, y$ )  
multiply ideals  $x$  and  $y$  idealmul( $nf, x, y, \{flag\}$ )  
intersection of ideals  $x$  and  $y$  idealintersect( $nf, x, y, \{flag\}$ )  
 $n$ -th power of ideal  $x$  idealpow( $nf, x, n, \{flag\}$ )  
inverse of ideal  $x$  idealinv( $nf, x$ )  
divide ideal  $x$  by  $y$  idealdiv( $nf, x, y, \{flag\}$ )  
Find  $(a, b) \in x \times y, a + b = 1$  idealaddtoone( $nf, x, \{y\}$ )  
coprime integral  $A, B$  such that  $x = A/B$  idealnumden( $nf, x$ )

**Primes and Multiplicative Structure**

factor ideal  $x$  in  $\mathbf{Z}_K$  idealfactor( $nf, x$ )  
expand ideal factorization in  $K$  idealfactorback( $nf, f, \{e\}$ )  
is ideal  $A$  an  $n$ -th power ? idealispower( $nf, A, n$ )  
expand elt factorization in  $K$  nffactorback( $nf, f, \{e\}$ )  
decomposition of prime  $p$  in  $\mathbf{Z}_K$  idealprimedec( $nf, p$ )  
valuation of  $x$  at prime ideal  $pr$  idealval( $nf, x, pr$ )  
weak approximation theorem in  $nf$  idealchinese( $nf, x, y$ )  
 $a \in K$ , s.t.  $v_{\mathfrak{p}}(a) = v_{\mathfrak{p}}(x)$  if  $v_{\mathfrak{p}}(x) \neq 0$  idealappr( $nf, x$ )  
 $a \in K$  such that  $(a \cdot x, y) = 1$  idealcoprime( $nf, x, y$ )  
give  $bid$  =structure of  $(\mathbf{Z}_K/id)^*$  idealstar( $nf, id, \{flag\}$ )  
structure of  $(1 + \mathfrak{p})/(1 + \mathfrak{p}^k)$  idealprincipalunits( $nf, pr, k$ )  
discrete log of  $x$  in  $(\mathbf{Z}_K/bid)^*$  ideallog( $nf, x, bid$ )  
**idealstar** of all ideals of norm  $\leq b$  ideallist( $nf, b, \{flag\}$ )  
add Archimedean places ideallistarch( $nf, b, \{ar\}, \{flag\}$ )  
init **modpr** structure nfmodprinit( $nf, pr$ )  
project  $t$  to  $\mathbf{Z}_K/pr$  nfmodpr( $nf, t, modpr$ )  
lift from  $\mathbf{Z}_K/pr$  nfmodprlift( $nf, t, modpr$ )

**Galois theory over  $\mathbf{Q}$**

conjugates of a root  $\theta$  of  $nf$  nfgaloisconj( $nf, \{flag\}$ )  
apply Galois automorphism  $s$  to  $x$  nfgaloisapply( $nf, s, x$ )  
Galois group of field  $\mathbf{Q}[x]/(f)$  polgalois( $f$ )  
initializes a Galois group structure  $G$  galoisinit( $pol, \{den\}$ )  
character table of  $G$  galoischartable( $G$ )  
conjugacy classes of  $G$  galoisconjclasses( $G$ )  
 $\det(1 - \rho(g)T)$ ,  $\chi$  character of  $\rho$  galoischarpoly( $G, \chi, \{o\}$ )  
 $\det(\rho(g))$ ,  $\chi$  character of  $\rho$  galoischarDET( $G, \chi, \{o\}$ )  
action of  $p$  in nfgaloisconj form galoispermtopol( $G, \{p\}$ )  
identify as abstract group galoisidentify( $G$ )  
export a group for GAP/MAGMA galoisexport( $G, \{flag\}$ )  
subgroups of the Galois group  $G$  galoissubgroups( $G$ )  
is subgroup  $H$  normal? galoisisnormal( $G, H$ )  
subfields from subgroups galoissubfields( $G, \{flag\}, \{v\}$ )  
fixed field galoisfixedfield( $G, perm, \{flag\}, \{v\}$ )  
Frobenius at maximal ideal  $P$  idealfrobenius( $nf, G, P$ )  
ramification groups at  $P$  idealramgroups( $nf, G, P$ )  
is  $G$  abelian? galoisisabelian( $G, \{flag\}$ )  
abelian number fields/ $\mathbf{Q}$  galoissubcyclo( $N, H, \{flag\}, \{v\}$ )

**Algebraic Number Theory**

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**The galpol package**

query the package: polynomial galoisgetpol(a,b,{s})  
... : permutation group galoisgetgroup(a,b)  
... : group description galoisgetname(a,b)

**Relative Number Fields (rnf)**

Extension  $L/K$  is defined by  $T \in K[x]$ . rnfequation( $nf, T, \{flag\}$ )  
absolute equation of  $L$  rnfisabelian( $nf, T$ )  
is  $L/K$  abelian? rnfalgtobasis( $rnf, x$ )  
relative nfalgtobasis rnfbasistoalg( $rnf, x$ )  
relative nfbasistoalg rnfidealhnf( $rnf, x$ )  
relative idealhnf rnfidealmul( $rnf, x, y$ )  
relative idealmul rnfidealtwoelt( $rnf, x$ )  
relative idealtwoelt

**Lifts and Push-downs**

absolute  $\rightarrow$  relative representation for  $x$  rnfeltabstorel( $rnf, x$ )  
relative  $\rightarrow$  absolute representation for  $x$  rnfeltreltoabs( $rnf, x$ )  
lift  $x$  to the relative field rnfeltup( $rnf, x$ )  
push  $x$  down to the base field rnfeltdown( $rnf, x$ )  
idem for  $x$  ideal: (rnfideal)reltoabs, abstorel, up, down

**Norms and Trace**

relative norm of element  $x \in L$  rnfeltnrm( $rnf, x$ )  
relative trace of element  $x \in L$  rnfelttrace( $rnf, x$ )  
absolute norm of ideal  $x$  rnfidealnrmabs( $rnf, x$ )  
relative norm of ideal  $x$  rnfidealnrmrel( $rnf, x$ )  
solutions of  $N_{K/\mathbf{Q}}(y) = x \in \mathbf{Z}$  bnfisintnorm( $bnf, x$ )  
is  $x \in \mathbf{Q}$  a norm from  $K$ ? bnfisnorm( $bnf, x, \{flag\}$ )  
initialize  $T$  for norm eq. solver rnfisnorminit( $K, pol, \{flag\}$ )  
is  $a \in K$  a norm from  $L$ ? rnfisnorm( $T, a, \{flag\}$ )  
initialize  $t$  for Thue equation solver thueinit( $f$ )  
solve Thue equation  $f(x, y) = a$  thue( $t, a, \{sol\}$ )  
characteristic poly. of  $a \bmod T$  rnfcharpoly( $nf, T, a, \{v\}$ )

**Factorization**

factor ideal  $x$  in  $L$  rnfidealfactor( $rnf, x$ )  
 $[S, T]: T_{i,j} \mid S_i; S$  primes of  $K$  above  $p$  rnfidealprimedec( $rnf, p$ )

**Maximal order  $\mathbf{Z}_L$  as a  $\mathbf{Z}_K$ -module**

relative polredbest rnfpolredbest( $nf, T$ )  
relative polredabs rnfpolredabs( $nf, T$ )  
relative Dedekind criterion, prime  $pr$  rnfdedekind( $nf, T, pr$ )  
discriminant of relative extension rnfdisc( $nf, T$ )  
pseudo-basis of  $\mathbf{Z}_L$  rnfpsedobasis( $nf, T$ )

**General  $\mathbf{Z}_K$ -modules:**  $M = [\text{matrix, vec. of ideals}] \subset L$   
relative HNF / SNF nfhnf( $nf, M$ ), nfsnf  
multiple of  $\det M$  nfdetint( $nf, M$ )  
HNF of  $M$  where  $d = nfdetint(M)$  nfhnfmod( $x, d$ )  
reduced basis for  $M$  rnflllgram( $nf, T, M$ )  
determinant of pseudo-matrix  $M$  rnfDET( $nf, M$ )  
Steinitz class of  $M$  rnfsteinitz( $nf, M$ )  
 $\mathbf{Z}_K$ -basis of  $M$  if  $\mathbf{Z}_K$ -free, or 0 rnfhnfbasis( $bnf, M$ )  
 $n$ -basis of  $M$ , or  $(n + 1)$ -generating set rnfBasis( $bnf, M$ )  
is  $M$  a free  $\mathbf{Z}_K$ -module? rnfisfree( $bnf, M$ )

**Associative Algebras**

$A$  is a general associative algebra given by a multiplication table  $mt$  (over  $\mathbf{Q}$  or  $\mathbf{F}_p$ ); represented by  $al$  from **algtabinit**.  
create  $al$  from  $mt$  (over  $\mathbf{F}_p$ ) algtabinit( $mt, \{p = 0\}$ )  
group algebra  $\mathbf{Q}[G]$  (or  $\mathbf{F}_p[G]$ ) alggroup( $G, \{p = 0\}$ )  
center of group algebra alggroupcenter( $G, \{p = 0\}$ )

**Properties**

is  $(mt, p)$  OK for **algtabinit**? algisassociative( $mt, \{p = 0\}$ )  
multiplication table  $mt$  algmtable( $al$ )  
dimension of  $A$  over prime subfield algdim( $al$ )  
characteristic of  $A$  algchar( $al$ )  
is  $A$  commutative? algiscommutative( $al$ )  
is  $A$  simple? algissimple( $al$ )  
is  $A$  semi-simple? algissemisimple( $al$ )  
center of  $A$  algcenter( $al$ )  
Jacobson radical of  $A$  algradical( $al$ )  
radical  $J$  and simple factors of  $A/J$  algsimpledec( $al$ )

**Operations on algebras**

create  $A/I, I$  two-sided ideal algquotient( $al, I$ )  
create  $A_1 \otimes A_2$  algtensor( $al1, al2$ )  
create subalgebra from basis  $B$  algsubalg( $al, B$ )  
quotients by ortho. central idempotents  $e$  algcentralproj( $al, e$ )  
isomorphic alg. with integral mult. table algmakeintegral( $mt$ )  
prime subalgebra of semi-simple  $A$  over  $\mathbf{F}_p$  algprimesubalg( $al$ )  
find isomorphism  $A \cong M_d(\mathbf{F}_q)$  algsplit( $al$ )

**Operations on lattices in algebras**

lattice generated by cols. of  $M$  alglathnf( $al, M$ )  
... by the products  $xy, x \in lat1, y \in lat2$  alglatmul( $al, lat1, lat2$ )  
sum  $lat1 + lat2$  of the lattices alglatadd( $al, lat1, lat2$ )  
intersection  $lat1 \cap lat2$  alglatinter( $al, lat1, lat2$ )  
test  $lat1 \subset lat2$  alglatsubset( $al, lat1, lat2$ )  
generalized index  $(lat2 : lat1)$  alglatindex( $al, lat1, lat2$ )  
 $\{x \in al \mid x \cdot lat1 \subset lat2\}$  alglatlefttransporter( $al, lat1, lat2$ )  
 $\{x \in al \mid lat1 \cdot x \subset lat2\}$  alglatrightrighttransporter( $al, lat1, lat2$ )  
test  $x \in lat$  (set  $c = \text{coord. of } x$ ) alglatcontains( $al, lat, x, \{&c\}$ )  
element of  $lat$  with coordinates  $c$  alglatelement( $al, lat, c$ )

**Operations on elements**

$a + b, a - b, -a$  algadd( $al, a, b$ ), algsub, algneg  
 $a \times b, a^2$  algmul( $al, a, b$ ), algsqr  
 $a^n, a^{-1}$  algpow( $al, a, n$ ), alginv  
is  $x$  invertible ? (then set  $z = x^{-1}$ ) algisinv( $al, x, \{&z\}$ )  
find  $z$  such that  $x \times z = y$  algdivl( $al, x, y$ )  
find  $z$  such that  $z \times x = y$  algdivr( $al, x, y$ )  
does  $z$  s.t.  $x \times z = y$  exist? (set it) algisdivl( $al, x, y, \{&z\}$ )  
matrix of  $v \mapsto x \cdot v$  algtomatrix( $al, x$ )  
absolute norm algnorm( $al, x$ )  
absolute trace algtrace( $al, x$ )  
absolute char. polynomial algcharpoly( $al, x$ )  
given  $a \in A$  and polynomial  $T$ , return  $T(a)$  algpoleval( $al, T, a$ )  
random element in a box algrandom( $al, b$ )

Based on an earlier version by Joseph H. Silverman

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Central Simple Algebras

$A$  is a central simple algebra over a number field  $K$ ; represented by  $al$  from **alginit**;  $K$  is given by a  $nf$  structure.

create CSA from data	<b>alginit</b> ( $B, C, \{v\}, \{maxord = 1\}$ )
multiplication table over $K$	$B = K, C = mt$
cyclic algebra ( $L/K, \sigma, b$ )	$B = rnf, C = [sigma, b]$
quaternion algebra $(a, b)_K$	$B = K, C = [a, b]$
matrix algebra $M_d(K)$	$B = K, C = d$
local Hasse invariants over $K$	$B = K, C = [d, [PR, HF], HI]$

Properties

type of $al$ ( $mt, CSA$ )	<b>algtype</b> ( $al$ )
dimension of $A$ over $\mathbf{Q}$	<b>algdim</b> ( $al, 1$ )
dimension of $al$ over its center $K$	<b>algdim</b> ( $al$ )
degree of $A$ ( $= \sqrt{\dim_K A}$ )	<b>algdegree</b> ( $al$ )
$al$ a cyclic algebra ( $L/K, \sigma, b$ ); return $\sigma$	<b>algaut</b> ( $al$ )
...return $b$	<b>algb</b> ( $al$ )
...return $L/K$ , as an $rnf$	<b>algsplittingfield</b> ( $al$ )
split $A$ over an extension of $K$	<b>algsplittingdata</b> ( $al$ )
splitting field of $A$ as an $rnf$ over center	<b>algsplittingfield</b> ( $al$ )
multiplication table over center	<b>algrelmultable</b> ( $al$ )
places of $K$ at which $A$ ramifies	<b>algramifiedplaces</b> ( $al$ )
Hasse invariants at finite places of $K$	<b>alghassef</b> ( $al$ )
Hasse invariants at infinite places of $K$	<b>alghassei</b> ( $al$ )
Hasse invariant at place $v$	<b>alghasse</b> ( $al, v$ )
index of $A$ over $K$ (at place $v$ )	<b>algindex</b> ( $al, \{v\}$ )
is $al$ a division algebra? (at place $v$ )	<b>algisdivision</b> ( $al, \{v\}$ )
is $A$ ramified? (at place $v$ )	<b>algisramified</b> ( $al, \{v\}$ )
is $A$ split? (at place $v$ )	<b>algissplit</b> ( $al, \{v\}$ )

Operations on elements

reduced norm	<b>algnorm</b> ( $al, x$ )
reduced trace	<b>algtrace</b> ( $al, x$ )
reduced char. polynomial	<b>algcharpoly</b> ( $al, x$ )
express $x$ on integral basis	<b>algalgtobasis</b> ( $al, x$ )
convert $x$ to algebraic form	<b>algbasistoalg</b> ( $al, x$ )
map $x \in A$ to $M_d(L)$ , $L$ split. field	<b>algtomatrix</b> ( $al, x$ )

Orders

<b>Z</b> -basis of order $\mathcal{O}_0$	<b>algbasis</b> ( $al$ )
discriminant of order $\mathcal{O}_0$	<b>algdisc</b> ( $al$ )
<b>Z</b> -basis of natural order in terms $\mathcal{O}_0$ 's basis	<b>alginvbasis</b> ( $al$ )