

# Algebraic Number Theory

(PARI-GP version 2.13.3)

## Binary Quadratic Forms

create  $ax^2 + bxy + cy^2$  (distance  $d$ )      **qfb**( $a, b, c, \{d\}$ )  
reduce  $x$  ( $s = \sqrt{D}$ ,  $l = \lfloor s \rfloor$ )      **qfbred**( $x, \{flag\}, \{D\}, \{l\}, \{s\}$ )  
return  $[y, g]$ ,  $g \in \text{SL}_2(\mathbf{Z})$ ,  $y = g \cdot x$  reduced      **qfbreds12**( $x$ )  
composition of forms       $x*y$  or **qfbnucomp**( $x, y, l$ )  
 $n$ -th power of form       $x^n$  or **qfbnupow**( $x, n$ )  
composition without reduction      **qfbcomprow**( $x, y$ )  
 $n$ -th power without reduction      **qfbpowrow**( $x, n$ )  
prime form of disc.  $x$  above prime  $p$       **qfbprimeform**( $x, p$ )  
class number of disc.  $x$       **qfbclassno**( $x$ )  
Hurwitz class number of disc.  $x$       **qfbhclassno**( $x$ )  
solve  $Q(x, y) = n$  in integers      **qfbsolve**( $Q, n$ )

## Quadratic Fields

quadratic number  $\omega = \sqrt{x}$  or  $(1 + \sqrt{x})/2$       **quadgen**( $x$ )  
minimal polynomial of  $\omega$       **quadpoly**( $x$ )  
discriminant of  $\mathbf{Q}(\sqrt{x})$       **quaddisc**( $x$ )  
regulator of real quadratic field      **quadregulator**( $x$ )  
fundamental unit in real  $\mathbf{Q}(\sqrt{D})$       **quadunit**( $D, \{w\}$ )  
class group of  $\mathbf{Q}(\sqrt{D})$       **quadclassunit**( $D, \{flag\}, \{t\}$ )  
Hilbert class field of  $\mathbf{Q}(\sqrt{D})$       **quadhilbert**( $D, \{flag\}$ )  
... using specific class invariant ( $D < 0$ )      **polclass**( $D, \{inv\}$ )  
ray class field modulo  $f$  of  $\mathbf{Q}(\sqrt{D})$       **quadrays**( $D, f, \{flag\}$ )

## General Number Fields: Initializations

The number field  $K = \mathbf{Q}[X]/(f)$  is given by irreducible  $f \in \mathbf{Q}[X]$ .  
We denote  $\theta = \bar{X}$  the canonical root of  $f$  in  $K$ . A  $nf$  structure  
contains a maximal order and allows operations on elements and  
ideals. A  $bnf$  adds class group and units. A  $bnr$  is attached to ray  
class groups and class field theory. A  $rmf$  is attached to relative  
extensions  $L/K$ .

init number field structure $nf$	<b>nfinit</b> ( $f, \{flag\}$ )
known integer basis $B$	<b>nfinit</b> ( $[f, B]$ )
order maximal at $vp = [p_1, \dots, p_k]$	<b>nfinit</b> ( $[f, vp]$ )
order maximal at all $p \leq P$	<b>nfinit</b> ( $[f, P]$ )
certify maximal order	<b>nfcertify</b> ( $nf$ )

### nf members:

a monic $F \in \mathbf{Z}[X]$ defining $K$	$nf.pol$
number of real/complex places	$nf.r1/r2/sign$
discriminant of $nf$	$nf.disc$
primes ramified in $nf$	$nf.p$
$T_2$ matrix	$nf.t2$
complex roots of $F$	$nf.roots$
integral basis of $\mathbf{Z}_K$ as powers of $\theta$	$nf.zk$
different/codifferent	$nf.diff, nf.codiff$
index $[\mathbf{Z}_K : \mathbf{Z}[X]/(F)]$	$nf.index$
recompute $nf$ using current precision	<b>nfnewprec</b> ( $nf$ )
init relative $rmf$ $L = K[Y]/(g)$	<b>rnfin</b> ( $nf, g$ )
init $bnf$ structure	<b>bnfinit</b> ( $f, l$ )

**bnf members:** same as  $nf$ , plus

underlying $nf$	$bnf.nf$
class group, regulator	$bnf.clgp, bnf.reg$
fundamental/torsion units	$bnf.fu, bnf.tu$
add $S$ -class group and units, yield $bnfS$	<b>bnfsunit</b> ( $bnf, S$ )

init class field structure $bnr$	<b>bnrinit</b> ( $bnf, m, \{flag\}$ )
<b>bnr members:</b> same as $bnf$ , plus	
underlying $bnf$	$bnr.bnf$
big ideal structure	$bnr.bid$
modulus $m$	$bnr.mod$
structure of $(\mathbf{Z}_K/m)^*$	$bnr.zkst$

## Fields, subfields, embeddings

**Defining polynomials, embeddings**  
smallest poly defining  $f = 0$  (slow)      **polredabs**( $f, \{flag\}$ )  
small poly defining  $f = 0$  (fast)      **polredbest**( $f, \{flag\}$ )  
random Tschirnhausen transform of  $f$       **poltschirnhaus**( $f$ )  
 $\mathbf{Q}[t]/(f) \subset \mathbf{Q}[t]/(g)$  ? Isomorphic?      **nfisincl**( $f, g$ ), **nfisisom**  
reverse polmod  $a = A(t) \bmod T(t)$       **modreverse**( $a$ )  
compositum of  $\mathbf{Q}[t]/(f)$ ,  $\mathbf{Q}[t]/(g)$       **polcompositum**( $f, g, \{flag\}$ )  
compositum of  $K[t]/(f)$ ,  $K[t]/(g)$       **nfcompositum**( $nf, f, g, \{flag\}$ )  
splitting field of  $K$  (degree divides  $d$ )      **nfsplitting**( $nf, \{d\}$ )  
signs of real embeddings of  $x$       **nfeltsign**( $nf, x, \{pl\}$ )  
complex embeddings of  $x$       **nfeltembed**( $nf, x, \{pl\}$ )  
 $T \in K[t]$ , # of real roots of  $\sigma(T) \in R[t]$       **nfpolsturm**( $nf, T, \{pl\}$ )

## Subfields, polynomial factorization

subfields (of degree  $d$ ) of  $nf$       **nfsubfields**( $nf, \{d\}$ )  
maximal subfields of  $nf$       **nfsubfieldsmax**( $nf$ )  
maximal CM subfield of  $nf$       **nfsubfieldscm**( $nf$ )  
 $d$ -th degree subfield of  $\mathbf{Q}(\zeta_n)$       **polsubcyclo**( $n, d, \{v\}$ )  
roots of unity in  $nf$       **nfrootsof1**( $nf$ )  
roots of  $g$  belonging to  $nf$       **nfroots**( $nf, g$ )  
factor  $g$  in  $nf$       **nffactor**( $nf, g$ )

## Linear and algebraic relations

poly of degree  $\leq k$  with root  $x \in \mathbf{C}$       **algdep**( $x, k$ )  
alg. dep. with pol. coeffs for series  $s$       **seralgdep**( $s, x, y$ )  
small linear rel. on coords of vector  $x$       **lindep**( $x$ )

## Basic Number Field Arithmetic (nf)

Number field elements are **t\_INT**, **t\_FRAC**, **t\_POL**, **t\_POLMOD**, or **t\_COL**  
(on integral basis  $nf.zk$ ).

### Basic operations

$x + y$	<b>nfeltadd</b> ( $nf, x, y$ )
$x \times y$	<b>nfeltmul</b> ( $nf, x, y$ )
$x^n$ , $n \in \mathbf{Z}$	<b>nfeltpow</b> ( $nf, x, n$ )
$x/y$	<b>nfeltdiv</b> ( $nf, x, y$ )
$q = x \setminus y := \text{round}(x/y)$	<b>nfeltdivu</b> ( $nf, x, y$ )
$r = x \% y := x - (x \setminus y)y$	<b>nfeltmod</b> ( $nf, x, y$ )
... $[q, r]$ as above	<b>nfeltdivrem</b> ( $nf, x, y$ )
reduce $x$ modulo ideal $A$	<b>nfeltreduce</b> ( $nf, x, A$ )
absolute trace $\text{Tr}_{K/\mathbf{Q}}(x)$	<b>nfelttrace</b> ( $nf, x$ )
absolute norm $N_{K/\mathbf{Q}}(x)$	<b>nfeltnorm</b> ( $nf, x$ )

## Multiplicative structure of $K^*$ ; $K^*/(K^*)^n$

valuation $v_p(x)$	<b>nfeltval</b> ( $nf, x, p$ )
... write $x = \pi^{v_p(x)}y$	<b>nfeltval</b> ( $nf, x, p, \&y$ )
quadratic Hilbert symbol (at $p$ )	<b>nfhilbert</b> ( $nf, a, b, \{p\}$ )
$b$ such that $xb^n = v$ is small	<b>idealredmodpower</b> ( $nf, x, n$ )

## Maximal order and discriminant

integral basis of field $\mathbf{Q}[x]/(f)$	<b>nfbasis</b> ( $f$ )
field discriminant of $\mathbf{Q}[x]/(f)$	<b>nfdisc</b> ( $f$ )
... and factorization	<b>nfdiscfactors</b> ( $f$ )
express $x$ on integer basis	<b>nfalgtobasis</b> ( $nf, x$ )
express element $x$ as a polmod	<b>nfbasistoalg</b> ( $nf, x$ )

## Dedekind Zeta Function $\zeta_K$ , Hecke $L$ series

$R = [c, w, h]$  in initialization means we restrict  $s \in \mathbf{C}$  to domain  
 $|\Re(s) - c| < w$ ,  $|\Im(s)| < h$ ;  $R = [w, h]$  encodes  $[1/2, w, h]$  and  $[h]$   
encodes  $R = [1/2, 0, h]$  (critical line up to height  $h$ ).

$\zeta_K$ as Dirichlet series, $N(I) < b$	<b>dirzetak</b> ( $nf, b$ )
init $\zeta_K^{(k)}(s)$ for $k \leq n$	<b>L = lfunit</b> ( $bnf, R, \{n = 0\}$ )
compute $\zeta_K(s)$ ( $n$ -th derivative)	<b>lfun</b> ( $L, s, \{n = 0\}$ )
compute $\Lambda_K(s)$ ( $n$ -th derivative)	<b>lfunlambda</b> ( $L, s, \{n = 0\}$ )

init $L_K^{(k)}(s, \chi)$ for $k \leq n$	<b>L = lfunit</b> ( $[bnr, chi], R, \{n = 0\}$ )
compute $L_K(s, \chi)$ ( $n$ -th derivative)	<b>lfun</b> ( $L, s, \{n\}$ )
Artin root number of $K$	<b>bnrrootnumber</b> ( $bnr, chi, \{flag\}$ )
$L(1, \chi)$ , for all $\chi$ trivial on $H$	<b>bnrL1</b> ( $bnr, \{H\}, \{flag\}$ )

## Class Groups & Units (bnf, bnr)

Class field theory data  $a_1, \{a_2\}$  is usually  $bnr$  (ray class field),  
 $bnr, H$  (congruence subgroup) or  $bnr, \chi$  (character on **bnr.clgp**).  
Any of these define a unique abelian extension of  $K$ .

units / $S$ -units	<b>bnfunits</b> ( $bnf, \{S\}$ )
remove GRH assumption from $bnf$	<b>bnfcertify</b> ( $bnf$ )
expo. of ideal $x$ on class gp	<b>bnfisprincipal</b> ( $bnf, x, \{flag\}$ )
expo. of ideal $x$ on ray class gp	<b>bnrisprincipal</b> ( $bnr, x, \{flag\}$ )
expo. of $x$ on fund. units	<b>bnfisunit</b> ( $bnf, x$ )
... on $S$ -units, $U$ is <b>bnfunits</b> ( $bnf, S$ )	<b>bnfisunit</b> ( $bnfs, U$ )
signs of real embeddings of $bnf.fu$	<b>bnfsignunit</b> ( $bnf$ )
narrow class group	<b>bnfnarrow</b> ( $bnf$ )

## Class Field Theory

ray class number for modulus $m$	<b>bnrclassno</b> ( $bnf, m$ )
discriminant of class field	<b>bnrdisc</b> ( $a_1, \{a_2\}$ )
ray class numbers, $l$ list of moduli	<b>bnrclassnolist</b> ( $bnf, l$ )
discriminants of class fields	<b>bnrdisc</b> ( $bnf, l, \{arch\}, \{flag\}$ )
decode output from <b>bnrdisc</b>	<b>bnfdecodemodule</b> ( $nf, fa$ )
is modulus the conductor?	<b>bnrisconductor</b> ( $a_1, \{a_2\}$ )
is class field ( $bnr, H$ ) Galois over $K^G$	<b>bnrisgalois</b> ( $bnr, G, H$ )
action of automorphism on <b>bnr.gen</b>	<b>bnrgaloismatrix</b> ( $bnr, aut$ )
apply <b>bnrgaloismatrix</b> $M$ to $H$	<b>bnrgaloisapply</b> ( $bnr, M, H$ )
characters on <b>bnr.clgp</b> s.t. $\chi(g_i) = e(v_i)$	<b>bnrchar</b> ( $bnr, g, \{v\}$ )
conductor of character $\chi$	<b>bnrconductor</b> ( $bnr, chi$ )
conductor of extension	<b>bnrconductor</b> ( $a_1, \{a_2\}, \{flag\}$ )
conductor of extension $K[Y]/(g)$	<b>rnfconductor</b> ( $bnf, g$ )
canonical projection $\text{Cl}_F \rightarrow \text{Cl}_f, f \mid F$	<b>bnrmmap</b>
Artin group of extension $K[Y]/(g)$	<b>rnfnormgroup</b> ( $bnr, g$ )
subgroups of $bnr$ , index $\leq b$	<b>subgroup</b> ( $bnr, b, \{flag\}$ )
class field defined by $H \subset \text{Cl}_f$	<b>bnrclassfield</b> ( $bnr, H$ )
... low level equivalent, prime degree	<b>rnfkummer</b> ( $bnr, H$ )
same, using Stark units (real field)	<b>bnrstark</b> ( $bnr, sub, \{flag\}$ )
is a an $n$ -th power in $K_v$ ?	<b>nfislocalpower</b> ( $nf, v, a, n$ )
cyclic $L/K$ satisf. local conditions	<b>nfgrunwaldwang</b> ( $nf, P, D, pl$ )

<b>Logarithmic class group</b>	
logarithmic $\ell$ -class group	<code>bnflog(<i>bnf</i>, <math>\ell</math>)</code>
$[\tilde{e}(F_v/Q_p), \tilde{f}(F_v/Q_p)]$	<code>bnflog<sub>ef</sub>(<i>bnf</i>, <i>pr</i>)</code>
$\exp \deg_F(A)$	<code>bnflogdegree(<i>bnf</i>, <i>A</i>, <math>\ell</math>)</code>
is $\ell$ -extension $L/K$ locally cyclotomic	<code>rnfislocalcyclo(<i>rnf</i>)</code>

**Ideals:** elements, primes, or matrix of generators in HNF

is $id$ an ideal in $nf$ ?	<code>nfisideal(<i>nf</i>, <i>id</i>)</code>
is $x$ principal in $bnf$ ?	<code>bnfisprincipal(<i>bnf</i>, <i>x</i>)</code>
give $[a, b]$ , s.t. $a\mathbf{Z}_K + b\mathbf{Z}_K = x$	<code>idealtwoelt(<i>nf</i>, <i>x</i>, {<i>a</i>})</code>
put ideal $a$ ( $a\mathbf{Z}_K + b\mathbf{Z}_K$ ) in HNF form	<code>idealhnf(<i>nf</i>, <i>a</i>, {<i>b</i>})</code>
norm of ideal $x$	<code>idealn<sub>orm</sub>(<i>nf</i>, <i>x</i>)</code>
minimum of ideal $x$ (direction $v$ )	<code>ideal<sub>min</sub>(<i>nf</i>, <i>x</i>, <i>v</i>)</code>
LLL-reduce the ideal $x$ (direction $v$ )	<code>ideal<sub>red</sub>(<i>nf</i>, <i>x</i>, {<i>v</i>})</code>

### Ideal Operations

add ideals $x$ and $y$	<code>ideal<sub>add</sub>(<i>nf</i>, <i>x</i>, <i>y</i>)</code>
multiply ideals $x$ and $y$	<code>idealmul(<i>nf</i>, <i>x</i>, <i>y</i>, {<i>flag</i>})</code>
intersection of ideal $x$ with $Q$	<code>ideal<sub>down</sub>(<i>nf</i>, <i>x</i>)</code>
intersection of ideals $x$ and $y$	<code>ideal<sub>intersect</sub>(<i>nf</i>, <i>x</i>, <i>y</i>, {<i>flag</i>})</code>
$n$ -th power of ideal $x$	<code>ideal<sub>pow</sub>(<i>nf</i>, <i>x</i>, <i>n</i>, {<i>flag</i>})</code>
inverse of ideal $x$	<code>ideal<sub>inv</sub>(<i>nf</i>, <i>x</i>)</code>
divide ideal $x$ by $y$	<code>ideal<sub>div</sub>(<i>nf</i>, <i>x</i>, <i>y</i>, {<i>flag</i>})</code>
Find $(a, b) \in x \times y$ , $a + b = 1$	<code>ideal<sub>addtoone</sub>(<i>nf</i>, <i>x</i>, {<i>y</i>})</code>
coprime integral $A, B$ such that $x = A/B$	<code>ideal<sub>numden</sub>(<i>nf</i>, <i>x</i>)</code>

### Primes and Multiplicative Structure

check whether $x$ is a maximal ideal	<code>ideal<sub>ismaximal</sub>(<i>nf</i>, <i>x</i>)</code>
factor ideal $x$ in $\mathbf{Z}_K$	<code>ideal<sub>factor</sub>(<i>nf</i>, <i>x</i>)</code>
expand ideal factorization in $K$	<code>ideal<sub>factorback</sub>(<i>nf</i>, <i>f</i>, {<i>e</i>})</code>
is ideal $A$ an $n$ -th power ?	<code>ideal<sub>ispower</sub>(<i>nf</i>, <i>A</i>, <i>n</i>)</code>
expand elt factorization in $K$	<code>nffactor<sub>back</sub>(<i>nf</i>, <i>f</i>, {<i>e</i>})</code>
decomposition of prime $p$ in $\mathbf{Z}_K$	<code>ideal<sub>primedec</sub>(<i>nf</i>, <i>p</i>)</code>
valuation of $x$ at prime ideal $pr$	<code>ideal<sub>val</sub>(<i>nf</i>, <i>x</i>, <i>pr</i>)</code>
weak approximation theorem in $nf$	<code>ideal<sub>chinese</sub>(<i>nf</i>, <i>x</i>, <i>y</i>)</code>
$a \in K$ , s.t. $v_{\mathfrak{p}}(a) = v_{\mathfrak{p}}(x)$ if $v_{\mathfrak{p}}(x) \neq 0$	<code>ideal<sub>appr</sub>(<i>nf</i>, <i>x</i>)</code>
$a \in K$ such that $(a \cdot x, y) = 1$	<code>ideal<sub>coprime</sub>(<i>nf</i>, <i>x</i>, <i>y</i>)</code>
give $bid$ =structure of $(\mathbf{Z}_K/id)^*$	<code>ideal<sub>star</sub>(<i>nf</i>, <i>id</i>, {<i>flag</i>})</code>
structure of $(1 + \mathfrak{p})/(1 + \mathfrak{p}^k)$	<code>ideal<sub>principalunits</sub>(<i>nf</i>, <i>pr</i>, <i>k</i>)</code>
discrete log of $x$ in $(\mathbf{Z}_K/bid)^*$	<code>ideal<sub>log</sub>(<i>nf</i>, <i>x</i>, <i>bid</i>)</code>
<code>ideal<sub>star</sub></code> of all ideals of norm $\leq b$	<code>ideal<sub>list</sub>(<i>nf</i>, <i>b</i>, {<i>flag</i>})</code>
add Archimedean places	<code>ideal<sub>listarch</sub>(<i>nf</i>, <i>b</i>, {<i>ar</i>}, {<i>flag</i>})</code>
init <code>modpr</code> structure	<code>nfm<sub>odprinit</sub>(<i>nf</i>, <i>pr</i>, {<i>v</i>})</code>
project $t$ to $\mathbf{Z}_K/pr$	<code>nfm<sub>odpr</sub>(<i>nf</i>, <i>t</i>, <i>modpr</i>)</code>
lift from $\mathbf{Z}_K/pr$	<code>nfm<sub>odprlift</sub>(<i>nf</i>, <i>t</i>, <i>modpr</i>)</code>

### Galois theory over $\mathbf{Q}$

conjugates of a root $\theta$ of $nf$	<code>nfgaloisconj(<i>nf</i>, {<i>flag</i>})</code>
apply Galois automorphism $s$ to $x$	<code>nfgaloisapply(<i>nf</i>, <i>s</i>, <i>x</i>)</code>
Galois group of field $\mathbf{Q}[x]/(f)$	<code>polgalois(<i>f</i>)</code>
initializes a Galois group structure $G$	<code>galois<sub>init</sub>(<i>pol</i>, {<i>den</i>})</code>
character table of $G$	<code>galois<sub>chartable</sub>(<i>G</i>)</code>
conjugacy classes of $G$	<code>galois<sub>conjclasses</sub>(<i>G</i>)</code>
$\det(1 - \rho(g)T)$ , $\chi$ character of $\rho$	<code>galois<sub>charpoly</sub>(<i>G</i>, <math>\chi</math>, {<i>o</i>})</code>
$\det(\rho(g))$ , $\chi$ character of $\rho$	<code>galois<sub>chardet</sub>(<i>G</i>, <math>\chi</math>, {<i>o</i>})</code>
action of $p$ in nfgaloisconj form	<code>galois<sub>permtopol</sub>(<i>G</i>, {<i>p</i>})</code>
identify as abstract group	<code>galois<sub>identify</sub>(<i>G</i>)</code>
export a group for GAP/MAGMA	<code>galois<sub>export</sub>(<i>G</i>, {<i>flag</i>})</code>
subgroups of the Galois group $G$	<code>galois<sub>subgroups</sub>(<i>G</i>)</code>
is subgroup $H$ normal?	<code>galois<sub>isnormal</sub>(<i>G</i>, <i>H</i>)</code>

# Algebraic Number Theory

(PARI-GP version 2.13.3)

subfields from subgroups	<code>galois<sub>subfields</sub>(<i>G</i>, {<i>flag</i>}, {<i>v</i>})</code>
fixed field	<code>galois<sub>fixedfield</sub>(<i>G</i>, <i>perm</i>, {<i>flag</i>}, {<i>v</i>})</code>
Frobenius at maximal ideal $P$	<code>ideal<sub>frobenius</sub>(<i>nf</i>, <i>G</i>, <i>P</i>)</code>
ramification groups at $P$	<code>ideal<sub>ramgroups</sub>(<i>nf</i>, <i>G</i>, <i>P</i>)</code>
is $G$ abelian?	<code>galois<sub>isabelian</sub>(<i>G</i>, {<i>flag</i>})</code>
abelian number fields/ $\mathbf{Q}$	<code>galois<sub>subcyclo</sub>(<i>N</i>, <i>H</i>, {<i>flag</i>}, {<i>v</i>})</code>

#### The galpol package

query the package: polynomial	<code>galois<sub>getpol</sub>(<i>a</i>, <i>b</i>, {<i>s</i>})</code>
...: permutation group	<code>galois<sub>getgroup</sub>(<i>a</i>, <i>b</i>)</code>
...: group description	<code>galois<sub>getname</sub>(<i>a</i>, <i>b</i>)</code>

### Relative Number Fields (rnf)

Extension  $L/K$  is defined by  $T \in K[x]$ .

absolute equation of $L$	<code>rnfequation(<i>nf</i>, <i>T</i>, {<i>flag</i>})</code>
is $L/K$ abelian?	<code>rnfisabelian(<i>nf</i>, <i>T</i>)</code>
relative nfalgtobasis	<code>rnfalgtobasis(<i>rnf</i>, <i>x</i>)</code>
relative nf <sub>basistoalg</sub>	<code>rnfbasistoalg(<i>rnf</i>, <i>x</i>)</code>
relative idealhnf	<code>rnfidealhnf(<i>rnf</i>, <i>x</i>)</code>
relative idealmul	<code>rnfidealmul(<i>rnf</i>, <i>x</i>, <i>y</i>)</code>
relative idealtwoelt	<code>rnfidealtwoelt(<i>rnf</i>, <i>x</i>)</code>

#### Lifts and Push-downs

absolute $\rightarrow$ relative representation for $x$	<code>rnfeltabstorel(<i>rnf</i>, <i>x</i>)</code>
relative $\rightarrow$ absolute representation for $x$	<code>rnfeltreltoabs(<i>rnf</i>, <i>x</i>)</code>
lift $x$ to the relative field	<code>rnfeltup(<i>rnf</i>, <i>x</i>)</code>
push $x$ down to the base field	<code>rnfeltdown(<i>rnf</i>, <i>x</i>)</code>
idem for $x$ ideal: (rnfideal)reltoabs, abstorel, up, down	

### Norms and Trace

relative norm of element $x \in L$	<code>rnfeltnorm(<i>rnf</i>, <i>x</i>)</code>
relative trace of element $x \in L$	<code>rnfelttrace(<i>rnf</i>, <i>x</i>)</code>
absolute norm of ideal $x$	<code>rnfidealnormabs(<i>rnf</i>, <i>x</i>)</code>
relative norm of ideal $x$	<code>rnfidealnormrel(<i>rnf</i>, <i>x</i>)</code>
solutions of $N_{K/\mathbf{Q}}(y) = x \in \mathbf{Z}$	<code>bnfisintnorm(<i>bnf</i>, <i>x</i>)</code>
is $x \in \mathbf{Q}$ a norm from $K$ ?	<code>bnfisnorm(<i>bnf</i>, <i>x</i>, {<i>flag</i>})</code>
initialize $T$ for norm eq. solver	<code>rnfisnorminit(<i>K</i>, <i>pol</i>, {<i>flag</i>})</code>
is $a \in K$ a norm from $L$ ?	<code>rnfisnorm(<i>T</i>, <i>a</i>, {<i>flag</i>})</code>
initialize $t$ for Thue equation solver	<code>thue<sub>init</sub>(<i>f</i>)</code>
solve Thue equation $f(x, y) = a$	<code>thue(<i>t</i>, <i>a</i>, {<i>sol</i>})</code>
characteristic poly. of $a$ mod $T$	<code>rnfcharpoly(<i>nf</i>, <i>T</i>, <i>a</i>, {<i>v</i>})</code>

#### Factorization

factor ideal $x$ in $L$	<code>rnfidealfactor(<i>rnf</i>, <i>x</i>)</code>
$[S, T]: T_{i,j} \mid S_i$ ; $S$ primes of $K$ above $p$	<code>rnfidealprimedec(<i>rnf</i>, <i>p</i>)</code>

#### Maximal order $\mathbf{Z}_L$ as a $\mathbf{Z}_K$ -module

relative polredbest	<code>rnfpolredbest(<i>nf</i>, <i>T</i>)</code>
relative polredabs	<code>rnfpolredabs(<i>nf</i>, <i>T</i>)</code>
relative Dedekind criterion, prime $pr$	<code>rnfdedekind(<i>nf</i>, <i>T</i>, <i>pr</i>)</code>
discriminant of relative extension	<code>rnfdisc(<i>nf</i>, <i>T</i>)</code>
pseudo-basis of $\mathbf{Z}_L$	<code>rnfpseudobasis(<i>nf</i>, <i>T</i>)</code>

#### General $\mathbf{Z}_K$ -modules: $M = [\text{matrix, vec. of ideals}] \subset L$

relative HNF / SNF	<code>nfhnf(<i>nf</i>, <i>M</i>), nfsnf</code>
multiple of det $M$	<code>nfdetint(<i>nf</i>, <i>M</i>)</code>
HNF of $M$ where $d = nfdetint(M)$	<code>nfhnfmod(<i>x</i>, <i>d</i>)</code>
reduced basis for $M$	<code>rnfilllgram(<i>nf</i>, <i>T</i>, <i>M</i>)</code>
determinant of pseudo-matrix $M$	<code>rnfdet(<i>nf</i>, <i>M</i>)</code>
Steinitz class of $M$	<code>rnfst<sub>Steinitz</sub>(<i>nf</i>, <i>M</i>)</code>

$\mathbf{Z}_K$ -basis of $M$ if $\mathbf{Z}_K$ -free, or 0	<code>rnfhnbasis(<i>bnf</i>, <i>M</i>)</code>
$n$ -basis of $M$ , or $(n + 1)$ -generating set	<code>rnfbasis(<i>bnf</i>, <i>M</i>)</code>
is $M$ a free $\mathbf{Z}_K$ -module?	<code>rnfisfree(<i>bnf</i>, <i>M</i>)</code>

### Associative Algebras

$A$  is a general associative algebra given by a multiplication table  $mt$  (over  $\mathbf{Q}$  or  $\mathbf{F}_p$ ); represented by  $al$  from algtableinit.

create $al$ from $mt$ (over $\mathbf{F}_p$ )	<code>algtableinit(<i>mt</i>, {<i>p</i> = 0})</code>
group algebra $\mathbf{Q}[G]$ (or $\mathbf{F}_p[G]$ )	<code>alggroup(<i>G</i>, {<i>p</i> = 0})</code>
center of group algebra	<code>alggroupcenter(<i>G</i>, {<i>p</i> = 0})</code>

#### Properties

is $(mt, p)$ OK for algtableinit?	<code>algisassociative(<i>mt</i>, {<i>p</i> = 0})</code>
multiplication table $mt$	<code>algmultable(<i>al</i>)</code>
dimension of $A$ over prime subfield	<code>algdim(<i>al</i>)</code>
characteristic of $A$	<code>algchar(<i>al</i>)</code>
is $A$ commutative?	<code>algiscommutative(<i>al</i>)</code>
is $A$ simple?	<code>algissimple(<i>al</i>)</code>
is $A$ semi-simple?	<code>algissemisimple(<i>al</i>)</code>
center of $A$	<code>algcenter(<i>al</i>)</code>
Jacobson radical of $A$	<code>algradical(<i>al</i>)</code>
radical $J$ and simple factors of $A/J$	<code>algsimpledec(<i>al</i>)</code>

#### Operations on algebras

create $A/I$ , $I$ two-sided ideal	<code>algquotient(<i>al</i>, <i>I</i>)</code>
create $A_1 \otimes A_2$	<code>algtensor(<i>al</i><sub>1</sub>, <i>al</i><sub>2</sub>)</code>
create subalgebra from basis $B$	<code>algsubalg(<i>al</i>, <i>B</i>)</code>
quotients by ortho. central idempotents $e$	<code>algcentralproj(<i>al</i>, <i>e</i>)</code>
isomorphic alg. with integral mult. table	<code>algmakeintegral(<i>mt</i>)</code>
prime subalgebra of semi-simple $A$ over $\mathbf{F}_p$	<code>algprimesubalg(<i>al</i>)</code>
find isomorphism $A \cong M_d(\mathbf{F}_q)$	<code>algsplit(<i>al</i>)</code>

### Operations on lattices in algebras

lattice generated by cols. of $M$	<code>alglathnf(<i>al</i>, <i>M</i>)</code>
... by the products $xy$ , $x \in lat_1$ , $y \in lat_2$	<code>alglatmul(<i>al</i>, <i>lat</i><sub>1</sub>, <i>lat</i><sub>2</sub>)</code>
sum $lat_1 + lat_2$ of the lattices	<code>alglatadd(<i>al</i>, <i>lat</i><sub>1</sub>, <i>lat</i><sub>2</sub>)</code>
intersection $lat_1 \cap lat_2$	<code>alglatinter(<i>al</i>, <i>lat</i><sub>1</sub>, <i>lat</i><sub>2</sub>)</code>
test $lat_1 \subset lat_2$	<code>alglatsubset(<i>al</i>, <i>lat</i><sub>1</sub>, <i>lat</i><sub>2</sub>)</code>
generalized index $(lat_2 : lat_1)$	<code>alglatindex(<i>al</i>, <i>lat</i><sub>1</sub>, <i>lat</i><sub>2</sub>)</code>
$\{x \in al \mid x \cdot lat_1 \subset lat_2\}$	<code>alglatlefttransporter(<i>al</i>, <i>lat</i><sub>1</sub>, <i>lat</i><sub>2</sub>)</code>
$\{x \in al \mid lat_1 \cdot x \subset lat_2\}$	<code>alglatrighttransporter(<i>al</i>, <i>lat</i><sub>1</sub>, <i>lat</i><sub>2</sub>)</code>
test $x \in lat$ (set $c =$ coord. of $x$ )	<code>alglatcontains(<i>al</i>, <i>lat</i>, <i>x</i>, {\&amp;c})</code>
element of $lat$ with coordinates $c$	<code>alglatelement(<i>al</i>, <i>lat</i>, <i>c</i>)</code>

#### Operations on elements

$a + b$ , $a - b$ , $-a$	<code>algadd(<i>al</i>, <i>a</i>, <i>b</i>)</code> , <code>algsub</code> , <code>algneg</code>
$a \times b$ , $a^2$	<code>algmul(<i>al</i>, <i>a</i>, <i>b</i>)</code> , <code>algsqr</code>
$a^n$ , $a^{-1}$	<code>algpow(<i>al</i>, <i>a</i>, <i>n</i>)</code> , <code>alginv</code>
is $x$ invertible ? (then set $z = x^{-1}$ )	<code>algisinv(<i>al</i>, <i>x</i>, {\&amp;z})</code>
find $z$ such that $x \times z = y$	<code>algdivl(<i>al</i>, <i>x</i>, <i>y</i>)</code>
find $z$ such that $z \times x = y$	<code>algdivr(<i>al</i>, <i>x</i>, <i>y</i>)</code>
does $z$ s.t. $x \times z = y$ exist? (set it)	<code>algisdivl(<i>al</i>, <i>x</i>, <i>y</i>, {\&amp;z})</code>
matrix of $v \mapsto x \cdot v$	<code>alg<sub>tomatrix</sub>(<i>al</i>, <i>x</i>)</code>
absolute norm	<code>algnorm(<i>al</i>, <i>x</i>)</code>
absolute trace	<code>algtrace(<i>al</i>, <i>x</i>)</code>
absolute char. polynomial	<code>algcharpoly(<i>al</i>, <i>x</i>)</code>
given $a \in A$ and polynomial $T$ , return $T(a)$	<code>algpoleval(<i>al</i>, <i>T</i>, <i>a</i>)</code>
random element in a box	<code>algrandom(<i>al</i>, <i>b</i>)</code>

Based on an earlier version by Joseph H. Silverman

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Central Simple Algebras

$A$  is a central simple algebra over a number field  $K$ ; represented by  $al$  from **alginit**;  $K$  is given by a  $nf$  structure.  
create CSA from data           **alginit**( $B, C, \{v\}, \{maxord = 1\}$ )  
multiplication table over  $K$             $B = K, C = mt$   
cyclic algebra ( $L/K, \sigma, b$ )            $B = rnf, C = [sigma, b]$   
quaternion algebra  $(a, b)_K$             $B = K, C = [a, b]$   
matrix algebra  $M_d(K)$             $B = K, C = d$   
local Hasse invariants over  $K$     $B = K, C = [d, [PR, HF], HI]$

Properties

type of  $al$  ( $mt, CSA$ )           **algtype**( $al$ )  
dimension of  $A$  over  $\mathbf{Q}$            **algdim**( $al, 1$ )  
dimension of  $al$  over its center  $K$    **algdim**( $al$ )  
degree of  $A$  ( $= \sqrt{\dim_K A}$ )       **algdegree**( $al$ )  
 $al$  a cyclic algebra ( $L/K, \sigma, b$ ); return  $\sigma$    **algaut**( $al$ )  
...return  $b$                        **algb**( $al$ )  
...return  $L/K$ , as an  $rnf$        **algsplittingfield**( $al$ )  
split  $A$  over an extension of  $K$        **algsplittingdata**( $al$ )  
splitting field of  $A$  as an  $rnf$  over center   **algsplittingfield**( $al$ )  
multiplication table over center       **algrelmultable**( $al$ )  
places of  $K$  at which  $A$  ramifies       **algramifiedplaces**( $al$ )  
Hasse invariants at finite places of  $K$    **alghassef**( $al$ )  
Hasse invariants at infinite places of  $K$    **alghassei**( $al$ )  
Hasse invariant at place  $v$            **alghasse**( $al, v$ )  
index of  $A$  over  $K$  (at place  $v$ )       **algindex**( $al, \{v\}$ )  
is  $al$  a division algebra? (at place  $v$ )   **algisdivision**( $al, \{v\}$ )  
is  $A$  ramified? (at place  $v$ )       **algisramified**( $al, \{v\}$ )  
is  $A$  split? (at place  $v$ )           **algissplit**( $al, \{v\}$ )

Operations on elements

reduced norm                   **algnorm**( $al, x$ )  
reduced trace                  **algtrace**( $al, x$ )  
reduced char. polynomial       **algcharpoly**( $al, x$ )  
express  $x$  on integral basis   **algalgtobasis**( $al, x$ )  
convert  $x$  to algebraic form   **algbasistoalg**( $al, x$ )  
map  $x \in A$  to  $M_d(L)$ ,  $L$  split. field   **algtomatrix**( $al, x$ )

Orders

**Z**-basis of order  $\mathcal{O}_0$            **algbasis**( $al$ )  
discriminant of order  $\mathcal{O}_0$        **algdisc**( $al$ )  
**Z**-basis of natural order in terms  $\mathcal{O}_0$ 's basis   **alginvbasis**( $al$ )